

MATRIX CHARACTERIZATION FOR THE MANIFOLD OF EUCLIDEAN INNER PRODUCTS OF 2-DIMENSIONAL TORUS

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ABSTRACT. In this paper we study the manifold of all Euclidean inner products of the 2-dimensional torus and we find a representation of its elements by matrices theory.

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1. INTRODUCTION

The manifold $\mathcal{S}(M)$ of all Euclidean inner products given on all tangent spaces of an n -dimensional real differentiable manifold M has been processed by *Turtoi A.* and some important beautiful geometric properties of this manifold have been studied [1]. This is the total space of the fiber bundle over the base M , with standard fiber $GL(n, \mathbb{R})/O(n)$ and structure group $GL(n, \mathbb{R})$ which is associated with $L(M)$, the principal bundle of linear frames over M . Therefore, it is a real differentiable manifold of dimension $n+n(n+1)/2$. The manifold $\mathcal{S}(M)$ differs from the manifold $\mathcal{M}(M)$ of all Riemannian metrics on M which is an infinite dimensional differentiable manifold [3]. Moreover, *Petrosanu D. M.* gives an important example of this manifold by constructing the manifold $\mathcal{S}(S^n)$ of all Euclidean inner products for the sphere S^n in [2] and obtains a representation of points of this manifold by matrices.

In this paper we construct the manifold $\mathcal{S}(T)$ of all Euclidean inner products for the 2-dimensional torus $T = S^1 \times S^1$. Then we obtain a representation of points of this manifold by matrices.

In the first section we give a matrix representation of the principal bundle of linear frames of 2-dimensional torus.

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Section two is devoted to construct the manifold of all Euclidean inner products of 2-dimensional torus. Also we give a matrix exhibition for the points of this manifold.

Note that in this paper by M we mean a real differentiable manifold of dimension n .

Now we recall the definition of linear frame.

Definition 1.1. ([4]) *A linear frame u at a point $x \in M$ is an ordered basis $X_1, \dots, X_n$ of the tangent space $T_x(M)$.*

Let $L(M)$ be the set of all linear frames u at all points of M . Every element $u \in L(M)$ may be considered either as a couple $u = (x, X)$ with $x \in M$ and $X = (X_1, \dots, X_n)$ a frame of the tangent space $T_x(M)$ or as a non-singular linear mapping $u : \mathbb{R}^n \rightarrow T_x(M)$ defined by $u(e_i) = X_i$, for $i = 1, \dots, n$. Let $\pi' : L(M) \rightarrow M$ be the projection which maps a linear frame $u = (x, X)$ into x . According to [4], $L(M)$ is the total space of a principal fiber bundle over the base M with structure group the general linear group $GL(n, \mathbb{R})$. $L(M)$ is called the bundle of linear frames over M .

Let $\mathcal{S}(\mathbb{R}^n)$ be the manifold of all Euclidean inner products on $\mathbb{R}^n$, then $\mathcal{S}(\mathbb{R}^n)$ is the homogeneous space $GL(n, \mathbb{R})/O(n)$. [1]

Here, we present the definition of the manifold of all Euclidean inner products of a differentiable manifold M .

Definition 1.2. ([1]) *The manifold of all Euclidean inner products of a differentiable manifold M is the total space $\mathcal{S}(M)$ of the fiber bundle over the base space M , with standard fiber the homogeneous space $\mathcal{S}(\mathbb{R}^n) = GL(n, \mathbb{R})/O(n)$ and structure group $GL(n, \mathbb{R})$ which is associated with $L(M)$. This associated bundle is also denoted by $L(M) \times_{GL(n, \mathbb{R})} \mathcal{S}(\mathbb{R}^n)$.*

We denote by $\pi : \mathcal{S}(M) \rightarrow M$ the canonical projection of the associated bundle $\mathcal{S}(M)$. The fiber $\pi^{-1}(x)$ over $x \in M$ may be considered as $\mathcal{S}(T_x(M))$ which is the space of all Euclidean inner products on the tangent space $T_x(M)$.

Example 1.3. ([2]) *The manifold of all Euclidean inner products of the n -dimensional sphere S^n is*

$$\mathcal{S}(S^n) = \left\{ A^t A \mid A \in GL(n+1, \mathbb{R}), A^t A \in \widetilde{GL(n, \mathbb{R})} \right\},$$

where A^t is the transposed matrix of A and $\widetilde{GL(n, \mathbb{R})}$ is the Lie subgroup of $GL(n+1, \mathbb{R})$ containing of the elements

$$\widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \quad M \in GL(n, \mathbb{R}).$$

2. Bundle of Linear Frames over 2-dimensional Torus

Let $(\mathbb{R}^3, \langle, \rangle)$ be the standard Euclidean space of dimension 3 and $\|x\| = \langle x, x \rangle^{1/2}$, for $x \in \mathbb{R}^3$. Let

$$S^1 = \{x \in \mathbb{R}^3 \mid x = (x_1, x_2, 0), \langle x, x \rangle = 1\}$$

be the 1-dimensional sphere in $\mathbb{R}^3$ and $T = S^1 \times S^1$ be the 2-dimensional torus, which is a real differentiable manifold.

Remark 2.1. Each $x \in S^1$ is in the form $(\cos 2\pi s, \sin 2\pi s, 0)$, for a unique s , $0 < s \leq 1$. We define $x^\perp$ to be the vector $(-\sin 2\pi s, \cos 2\pi s, 0)$. By this notation, we can consider each point $p \in T$ as a pair (x, y) where x and y are vectors in $\mathbb{R}^3$ such that $x \in S^1$, $\|y\| = 1$ and $\langle x^\perp, y \rangle = 0$.

A linear frame at a point $(x, y) \in T$ is given by two independent vectors in $\mathbb{R}^3$ which are orthogonal to the vector y . Indeed, a linear frame at a point $(x, y) \in T$ is in the form $((x, y), X_1, X_2)$ where X_1 and X_2 are independent vectors in $\mathbb{R}^3$ which are orthogonal to the vector y .

For $x = (\cos 2\pi s, \sin 2\pi s, 0)$ for a unique s , $0 < s \leq 1$, $y = (y_1, y_2, y_3)$ and $X_i = (x_{1i}, x_{2i}, x_{3i})$, $i = 1, 2$, let $u = (X_1, X_2)$ be a linear frame in $(x, y) \in T$. We correspond the matrix

$$(2.1) \quad X = \begin{pmatrix} x_{11} & x_{12} & sy_1 \\ x_{21} & x_{22} & sy_2 \\ x_{31} & x_{32} & sy_3 \end{pmatrix} \in GL(3, \mathbb{R}),$$

to this linear frame.

We see that $X^t X \in \widetilde{GL(2, \mathbb{R})}_r$ for $r = s^2$, where $\widetilde{GL(2, \mathbb{R})}_r$, $0 < r \leq 1$, is the Lie subgroup of the Lie group $GL(3, \mathbb{R})$ containing of all non-singular matrices

$$\widetilde{M}_r = \begin{pmatrix} M & 0 \\ 0 & r \end{pmatrix}, \quad M \in GL(2, \mathbb{R}).$$

To give another property of the matrix (2.1), we require the following notation:

Remark 2.2. For a vector $z \in \mathbb{R}^3$ we define $\widehat{z}$ to be the vector

$$(-\sin 2\pi \|z\|, \cos 2\pi \|z\|, 0).$$

Therefore, for the third column $X_3 = sy$ of the matrix (2.1), $\widehat{X_3} = x^\perp$ and so $\langle \widehat{X_3}, X_3 \rangle = 0$.

Finally, we have:

Proposition 2.3. The manifold of linear frames over the 2-dimensional torus $T = S^1 \times S^1$ is

$$(2.2) \quad L(T) = \left\{ X \in GL(3, \mathbb{R}) \left| \begin{array}{l} X^t X \in \widetilde{GL(2, \mathbb{R})}_r \text{ for some } r, 0 < r \leq 1 \\ \& \langle \widehat{X_3}, X_3 \rangle = 0, \\ \text{where } X_3 \text{ is the 3th column of } X \end{array} \right. \right\}.$$

Proof. Let

$$X = (X_1 \quad X_2 \quad X_3) \in GL(3, \mathbb{R}),$$

where X_i is the i th column of X ,

$$X^t X \in \widetilde{GL(2, \mathbb{R})}_r,$$

for some $0 < r \leq 1$ and $\langle \widehat{X_3}, X_3 \rangle = 0$. We take $s = \sqrt{r}$, $x = (\cos 2\pi s, \sin 2\pi s, 0)$ and $y = (1/s)X_3$. $\square$

The Lie group $GL(2, \mathbb{R})$, identified with $\widetilde{GL(2, \mathbb{R})}$, acts freely on the manifold $L(T)$ on the right as follows:

$$(X, M) \rightarrow X\widetilde{M} \text{ where } \widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\pi' : L(T) \rightarrow T$ be the projection defined by

$$\pi'(X) = (x, y), \quad (x, y) \in T,$$

where $X \in L(T)$ is given by (2.1). We see that for every $\widetilde{M} \in \widetilde{GL(2, \mathbb{R})}$,

$$\pi'(X\widetilde{M}) = \pi'(X).$$

Moreover, we have for all $X, Y \in L(T)$, $\pi'(X) = \pi'(Y)$ if and only if there is an element $\widetilde{M} \in \widetilde{GL(2, \mathbb{R})}$ such that $Y = X\widetilde{M}$. It follows that $L(T)$ is the total space of the principle fiber bundle of linear frames over torus T .

3. Manifold of Euclidean Inner Products of 2-dimensional Torus

The manifold of all Euclidean inner products of the torus $T = S^1 \times S^1$ is the total space $\mathcal{S}(T)$ of the fiber bundle over the base space T , with standard fiber $\mathcal{S}(\mathbb{R}^2)$ and structure group $GL(2, \mathbb{R})$ which is associated with $L(T)$. This fiber bundle is obtained as follows:

$GL(2, \mathbb{R})$, identified with $GL(2, \mathbb{R})$, acts on the product manifold $L(T) \times \mathcal{S}(\mathbb{R}^2)$ on the right as follows: An element $\widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in \widetilde{GL(2, \mathbb{R})}$ maps $(X, g) \in L(T) \times \mathcal{S}(\mathbb{R}^2)$ into the element

$$(X, g)\widetilde{M} = (X\widetilde{M}, \widetilde{M}^{-1}\widetilde{g}(\widetilde{M}^{-1})^t) \in L(T) \times \mathcal{S}(\mathbb{R}^2),$$

where g is the associated matrix of the inner product g on $\mathbb{R}^2$ with respect to the standard frame of $\mathbb{R}^2$ and $\widetilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$.

The quotient space of this action on $L(T) \times \mathcal{S}(\mathbb{R}^2)$ is the set $\mathcal{S}(T)$ of all Euclidean inner products of the torus T .

The mapping $L(T) \times \mathcal{S}(\mathbb{R}^2) \rightarrow T$ which maps (X, g) into $\pi'(X)$, induces the projection $\pi : \mathcal{S}(T) \rightarrow T$. There is a differentiable structure in $\mathcal{S}(T)$ such that for any coordinate neighborhood U_α of T , the set $\pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha \times \mathcal{S}(\mathbb{R}^2)$. By this differentiable structure in $\mathcal{S}(T)$, the projection π is a differentiable map of $\mathcal{S}(T)$ onto T .

Now, we obtain a representation of $\mathcal{S}(T)$ by matrices.

According to [2], every element $X \in L(T)$ given by (2.1), induces an inner product on the tangent space of T at $(x, y) \in T$ whose associated matrix with respect to the frame $u = (X_1, X_2)$ is

$$M = \begin{pmatrix} \langle X_1, X_1 \rangle & \langle X_1, X_2 \rangle \\ \langle X_2, X_1 \rangle & \langle X_2, X_2 \rangle \end{pmatrix}$$

and we have

$$X^t X = \begin{pmatrix} M & 0 \\ 0 & s^2 \end{pmatrix}.$$

Moreover, if $X, Y \in L(T)$ have the same last column $sy = (sy_1, sy_2, sy_3)$, then $X^t X = Y^t Y$ if and only if there is an element $O \in \widetilde{O(2)}$ such that $Y = OX$ where

$$\widetilde{O(2)} = O(3) \cap \widetilde{GL(2, \mathbb{R})}$$

and so, $\widetilde{O(2)}$ is identified with $O(2)$. Therefore, two matrices $X, Y \in L(T)$ with the same last column sy , induce the same inner product on the tangent space of T at (x, y) where $x = (\cos 2\pi s, \sin 2\pi s, 0)$, if and only if $Y = OX$, for some $O \in \widetilde{O(2)}$.

Therefore, we conclude that:

Proposition 3.1. *The manifold of Euclidean inner products of the 2-dimensional torus $T = S^1 \times S^1$ is*

$$(3.1) \quad \mathcal{S}(T) = \left\{ X^t X \left| \begin{array}{l} X \in GL(3, \mathbb{R}), X^t X \in \widetilde{GL(2, \mathbb{R})}_r \\ \text{for some } r, 0 < r \leq 1 \text{ \& } \langle \widehat{X}_3, X_3 \rangle = 0, \\ \text{where } X_3 \text{ is the 3th column of } X \end{array} \right. \right\}.$$

Proof. It is clearly proved by the discussion before the proposition and the equality (2.2). $\square$

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