

LIFTING THEOREM CONCERNING DISTALITY ON A FIBRE BUNDLE

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ABSTRACT. In this note we have proved the lifting properties on fibre bundle W under skew product flow π and established that if bounded solutions are positively uniformly stable then a π -invariant subset of W is minimal.

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1. INTRODUCTION

Skew product flow was originated in the study of ordinary differential equation, introduced by Sell and Sacker in [3], on the product space $W = X \times Y$ of two topological spaces X and Y . It was further studied by Miller [2], Sell and Sacker [5], [4] and Egawa [1]. In [5] authors defined distality of projection map and they obtained equivalent criteria of distal property of projection map. In this paper we have defined skew product flow on fibre bundle over a Banach manifold and also obtained local representation of such flow. Then we have defined distality of projection map on compact π -invariant subset of the fibre bundle and have obtained equivalent criteria of distality of projection map on fibre bundle. In this paper we have defined

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distality on a compact π -invariant subset M of a fibre bundle (p, W, B) and on its base B , π being a skew product flow on the fibre bundle. It has been shown that distality can be lifted from the base B to the subset M . This result generalizes the first part of theorem-2 of Sell and Sacker [5] concerning lifting of distality from the base Y to an π -invariant subset M of the product space $X \times Y$, π being a skew-product flow on $X \times Y$. At last we have established that if bounded solutions are positively uniformly stable then a π -invariant subset of W is minimal.

2. PRELIMINARIES

Let W denotes a topological space and T denotes a set of real numbers R or the set of integers Z . A mapping π is defined by $\pi : W \times T \rightarrow W$ is called a flow [5] if i) π is continuous, ii) $\pi(\omega, 0) = \omega$, for $\omega \in W$, iii) $\pi(\pi(\omega, s), t) = \pi(\omega, s + t)$, $\omega \in W$ and $s, t \in T$.

If $T = R$, π is called a continuous flow.

If $T = Z$, π is called a discrete flow.

A set $M \subseteq W$ is said to be π -invariant if $\gamma(\omega) \subseteq M$ whenever $\omega \in M$ where $\gamma(\omega) = \{\pi(\omega, t), t \in R\}$, M is positively or negatively π -invariant if $\gamma^+(\omega) \subseteq M$ whenever $\omega \in M$ where $\gamma^+(\omega) = \{\pi(\omega, t) : t \in R, t \geq 0\}$ or $\gamma^-(\omega) \subseteq M$ where $\gamma^-(\omega) = \{\pi(\omega, t) : t \in R, t \leq 0\}$.

Let X and Y be topological spaces and let $T = R$ or Z . A flow, $\pi : X \times Y \times T \rightarrow X \times Y$ is called a skew product flow on $X \times Y$ [5] if there exists a continuous mapping ϕ on $X \times Y \times T \rightarrow X$ and a flow σ on Y where $\sigma : Y \times T \rightarrow Y$ such that π has the form (ϕ, σ) where $\pi(x, y, t) = (\phi(x, y, t), \sigma(y, t))$ $\forall x \in X, y \in Y, t \in T$.

Here we shall consider $T = R$ throughout our discussion.

In [4] the authors have also defined distal property of the projective map in the following way.

Let $\pi = (\phi, \sigma)$ be a skew product flow on $X \times Y$ and $M \subseteq X \times Y$ be a compact π -invariant set with $p : X \times Y \rightarrow Y$ be the natural projection and let $\mu(y) = \text{card}(p^{-1}(y) \cap M)$, for $y \in Y$ and X be also metrizable, then p/M is said to be of distal type (or M is said to have the fibre distal property), if for any two points (x_1, y) and (x_2, y) in $p^{-1}(y) \cap M$ with $x_1 \neq x_2$, there is an $\alpha = \alpha(x_1, x_2, y) > 0$ such that

$$(2.1) \quad d(\phi(x_1, y, t), \phi(x_2, y, t)) > \alpha, \quad t \in R,$$

where d is the metric of the topological space X . If (2.1) holds for $t \geq 0$ then p/M is said to be of positive distal type and if (2.1) holds for $t \leq 0$ then p/M is said to be of negative distal type.

A set $M \subseteq X \times Y$ is said to be an N -fold covering space of Y with the covering projection p , if $\text{Card } p^{-1}(Y) = N$, $0 < N < \infty$ and for each $y \in Y$, there is an open neighbourhood V of y such that $p^{-1}(V)$ consist of N disjoint open sets U_i and $p/U_i : U_i \rightarrow V$ is a homeomorphism for each $i = 1, 2, \dots, N$. Sell and Sacker have proved the structure theorem in [5] which is given below.

Theorem 2.1: *Suppose π be a skew-product flow on $X \times Y$ where Y is a compact minimal set (where minimal set is a closed invariant set which contains no proper subset with the same properties) in the flow σ and X is metrizable. Let M be a compact π -invariant subset of $X \times Y$, then the following statements are equivalent.*

- (A) p/M is of distal type and there is a $y_0 \in Y$ such that $\mu(y_0) = N$ is finite.
- (B) p/M is of positive(or negative) distal type and there is a $y_0 \in Y$ such that $\mu(y_0) = N$ is finite.
- (C) $\mu(y) = N < \infty$ for all $y \in Y$.
- (D) M is a N -fold covering space of Y with covering map p/M .

Also if any one of the above conditions is satisfied then M can be expressed as a finite union of minimal sets $M_1, \dots, M_k$, where M_i is a n_i -fold covering of Y , $i = 1, 2, \dots, k$ and $n_1 + n_2 + \dots + n_k = N$.

While proving the theorem 2.1 the authors have obtained some observations, which have important role in our paper.

Observation 2.1: If $\mu(y) = N < \infty$ and (C) holds for all $y \in Y$, then there is a $\beta > 0$ such that for all $y \in Y$ and any pair $\omega_1, \omega_2 \in p^{-1}(y)$ with $\omega_1 \neq \omega_2$.

$$d.\rho(\omega_1.t, \omega_2.t) = d(q(\omega_1.t), q(\omega_2.t)) \geq 3\beta,$$

for every pseudometric ρ and all $t \in R$ where $q : X \times Y \rightarrow X$ such that

$q(x, y) = x$. In other words, if (C) holds, then p/M is of distal type and then $\alpha = \alpha(x_1, x_2, y)$ given in the definition of distal type can be chosen uniformly equal to 3β .

Observation 2.2: When (C) holds, p/M becomes an open mapping. In fact, for every convergent net $\{y_n\}$ in Y with $y_n \rightarrow y$, $\limsup p^{-1}(y_n) = p^{-1}(y)$ and p/M is open.

3. SKEW PRODUCT FLOW ON A FIBRE-BUNDLE OVER A BANACH MANIFOLD

Firstly we define Banach manifold, fibre bundle over a Banach manifold and skew product flow on it. In this section we have obtained local representation of skew-product flow on W .

Next such local representation of skew product flow on a fibre bundle has been used to define distality of the projection map. At last by applying equivalent local criteria of distality corresponding equivalent criteria of distal map on a fibre bundle over a Banach manifold has been obtained.

Definition 3.1: A Banach manifold B is a topological space such that every point has a neighbourhood, homeomorphic to an open set of a Banach space possessing c^∞ structure.

Example: We can consider C^n, R^n as Banach manifold.

Definition 3.2: A fibre bundle whose base space is a Banach manifold is called a fibre bundle over a Banach manifold.

Definition 3.3: If W is a fibre bundle with a compact base space B and projection p then a flow π on W is called a skew product flow, if there exists a flow σ on B such that

$$(3.1) \quad p(\pi(w, t)) = \sigma(p(w), t); \quad w \in W, \quad t \in R.$$

Now we state and prove a theorem as follows:

Theorem 3.1: Let π be a skew product flow on a fibre bundle (p, W, B) with base space B (a real Banach manifold) and projection p and let the corresponding flow on B be σ . Let $b_1 \in B_1$ and $b_1.t = \sigma(b_1, t) \in B_1.t$; $B_1, B_1.t$ being open in B and let $p^{-1}(b_1)$ and $p^{-1}(b_1.t)$ be diffeomorphic to local bundles $\{u_1\} \times E_1$ and $\{u_1.t\} \times F_1$ respectively, $u_1 \in U_1$ (E_1, F_1 being real Banach spaces) under ϕ and ψ , where $\phi : p^{-1}(B_1) \rightarrow U_1 \times E_1$ and $\psi^t : p^{-1}(B_1.t) \rightarrow U_1.t \times F_1$. Then π has a local representation of the form $\pi_1 = (\sigma', \phi')$ such that $\pi_1(u_1, e_1, t) = (\sigma'(u_1.0, t), \phi'(u_1, e_1, t))$; $e_1 \in E_1$ where $\phi' = \psi^t \circ \pi_1 \circ \phi^{-1}$ is a c^∞ map from E_1 to F_1 for each $u_1 \in U_1'$, U_1' varying over small neighbourhoods of u_1 in U_1 for some t and σ' is such that $\sigma'(\phi(b_1).0, t) = \psi^t \circ \sigma(b_1, t)$ for each $b_1 \in B_1$ (i.e., σ' is a local flow on $U_1 \times 0$).

Proof: For each $(u_1, e_1) \in U_1 \times E_1$, $\omega_1 = \phi^{-1}(u_1, e_1) \in p^{-1}(B_1) \subset W_1 \subset W$. Also $\pi(\omega_1, t) \in p^{-1}(B_1.t) \subset W$. So $\psi^t(\pi(\omega_1, t)) \in U_1.t \times F_1$. As π is skew product flow, p commutes with π and σ .

i.e., $p(\pi(\omega_1, t)) = \sigma(p(\omega_1), t)$. Hence $\pi(\omega_1, t) = p^{-1}(\sigma(p(\omega_1), t))$.

So $\psi^t(\pi(\omega_1, t)) = \psi^t(p^{-1}(b_1.t)) \in U_1.t \times F_1$ i.e., local representation of π has the form $\pi_1 = (\sigma', \phi')$ given by $\pi_1(u_1, e_1, t) = (\sigma'(u_1.0, t), \phi'(u_1, e_1, t))$,

$e_1 \in E_1$, where $\phi'(u_1, e_1, t) = \psi^t \circ \pi_1 \circ \phi^{-1}(u_1, e_1)$ is a c^∞ map from E_1 to F_1 for each $u_1 \in U_1'$, U_1' varying over all small neighbourhoods of u_1 in U_1 and for some t .

Definition 3.4: Let W be a fibre bundle with projection p and base B which is a Banach manifold. Let σ be a flow on B and let (σ'_i, φ'_i) be the local representations of a skew product flow π on local bundles $U_i \times E_i$, $i = 1, 2, 3, \dots$ (E_i being Banach spaces). Let $M \subset W$ be a compact π -invariant subset and $B' \subset B$ be a totally bounded subset which is minimal with respect to σ . Then p/M is defined to be of distal type (or M is said to have the fibre distal property) if for any two points $\omega_i \neq \omega'_i$ on W , $\omega_i, \omega'_i \in p^{-1}(b_i) \cap M$, $b_i \in B'$, there is an $\alpha_i = \alpha_i(e_i, e'_i, u_i) > 0$, $i = 1, 2, 3, \dots$ such that

$$(3.2) \quad \|\phi'_i(u_i, e_i, t) - \phi'_i(u_i, e'_i, t)\|_{F_i} \geq \alpha_i, \quad t \in R,$$

where $p^{-1}(b_i)$ is diffeomorphic to $\{u_i\} \times E_i$ and $p^{-1}(b_i.t)$ is diffeomorphic to $\{u_i.t\} \times F_i$, $e_i, e'_i \in E_i$, ϕ'_i is a c^∞ map from E_i to F_i for each $u_i \in U'_i$, U'_i is varying over all small neighbourhoods of u_i in U_i and for $t \in R$. Here p/M is said to be of positively

(or negatively) distal type according as equation (3.2) holds only for $t \geq 0$ (or $t \leq 0$).

Definition 3.5: Let (p, W, B) be a fibre bundle over a Banach manifold B . A set $M \subseteq W$ is said to be a finite fold covering space of B if,

- (i) for each $b \in B$, $\text{Card}\{p^{-1}(b) \cap M\}$ is finite.
- (ii) $\text{Card}\{p^{-1}(b) \cap M\} = N$, $0 < N < \infty$, then there is an open neighbourhood V of b such that $p^{-1}(V) \cap M$ consists of N disjoint open sets U_i , where $p_i : U_i \rightarrow V$ is a homeomorphism, for each $i = 1, 2, 3, \dots, N$.

Next we state and prove the structure theorem as follows:

Theorem 3.2: Let π be a skew product flow on a fibre bundle W , with projection p and base space B , which is a Banach manifold. Let σ be the flow on B and let $B' \subset B$ be a totally bounded subset which is minimal with respect to the flow σ . Then the following statements are equivalent:

- (A) p/M_0 is of distal type and there is a $b_0 \in B'$ such that $\mu_0(b_0) = N_0 < \infty$, for some π -invariant subset M_0 of W where $\mu_0(b_0) = \text{Card}\{p^{-1}(b_0) \cap M_0\}$.
- (B) p/M_0 is of positive(negative) distal type and there is a $b_0 \in B'$ such that $\mu_0(b_0) = N_0$ is finite.
- (C) For each $b \in B'$, $\mu'(b)$ is finite, where $\mu'(b) = \text{Card}\{p^{-1}(b) \cap M'\}$, for some π -invariant subset M' of W .
- (D) There is a finite fold covering space of B' .

Proof: Simply $(A) \Rightarrow (B)$.

To show $(C) \Rightarrow (A)$, let us take any two points $\omega_1, \omega_1' \in W$, $\omega_1 \neq \omega_1'$, such that $\omega_1, \omega_1' \in p^{-1}(B_1)$, B_1 being open in B' .

Let $\phi_1 : p^{-1}(B_1) \rightarrow U_1 \times E_1$ and $\psi_1^t : p^{-1}(B_1.t) \rightarrow U_1.t \times F_1$ be diffeomorphisms, where $U_1.t = \sigma_1'(U_1.0, t)$, σ_1' being given by $\sigma_1'(\phi_1(b_1).0, t) = \psi_1^t \circ \sigma(b_1, t)$, $b_1 \in$

B_1 ; E_1, F_1 being Banach spaces.

Let M' be some compact π -invariant subset of W .

We consider $M_1 \subseteq p^{-1}(b_1) \cap M'$, is also π -invariant.

Let $\phi_1(M_1) = M_1' \subset U_1 \times E_1$.

Now by (C), $\mu'(b_1) = N' < \infty$, for some $N' > 0$, where

$$\mu'(b_1) = \text{Card}\{p^{-1}(b_1) \cap M'\}.$$

So M_1' also contains finite number of elements.

Let $q_1 : U_1 \times E_1 \rightarrow U_1 \times \{0\}$ be the natural projection, then $q_1^{-1}(u_1, 0) \cap M_1'$; $(u_1, 0) \in U_1 \times \{0\}$, contains finite number of elements. Again we know that π has a local representation of the form (σ_1', ϕ_1') on $U_1 \times E_1$, where σ_1' is the induced flow on U_1' and $\phi_1' = \psi_1^t \circ \pi_1 \circ \phi_1^{-1}$ is a c^∞ map from E_1 to F_1 for each $u_1 \in U_1$ and for some $t \in R$. So, M_1' remains (σ_1', ϕ_1') invariant in $U_1 \times E_1$.

Thus condition (C) of theorem-3.2 holds, hence condition (A) of theorem-2.1 follows. So q_1/M_1 is of distal type. Hence there exists an $\alpha_1 = \alpha_1(u_1, e_1, e_1') > 0$ such that,

$$\|\phi_1'(u_1, e_1, t) - \phi_1'(u_1, e_1', t)\|_{F_1} \geq \alpha_1,$$

for each $u_1 \in U_1$, $e_1, e_1' \in E_1$ and for some $t \in R$.

Similarly, for $B_2 \subset B'$, B_2 being open, if $p^{-1}(B_2)$ is diffeomorphic to $U_2 \times E_2$, $p^{-1}(B_2.t)$ is diffeomorphic to $U_2.t \times F_2$ and if $M_2 \subseteq p^{-1}(b_2) \cap M'$, $b_2 \in B_2$ be π -invariant, then we have,

$$\|\phi_2'(u_2, e_2, t) - \phi_2'(u_2, e_2', t)\|_{F_2} \geq \alpha_2 > 0,$$

for each $u_2 \in U_2$, $e_2, e_2' \in E_2$ and for some $t \in R$, $\alpha_2 = \alpha_2(u_2, e_2, e_2')$.

Now if we consider $\alpha' = \min(\alpha_1, \alpha_2) > 0$ and $M_0' = M_1 \cup M_2$ then

$$\| \phi_i'(u_i, e_i, t) - \phi_i'(u_i, e_i', t) \|_{F_i} \geq \alpha',$$

for each $u_i \in U$, $U = U_1 \cup U_2$, $e_i, e_i' \in E_i$ and for some $t \in R$, $i = 1, 2$.

Continuing this process and assuming $M = \bigcup_i M_i$, M becomes a compact π -invariant subset of W .

Also for any $b_i \in B' = \bigcup_i B_i$ and $\omega_1, \omega_2 \in p^{-1}(b_i) \cap M$,

there is an $\alpha = \inf_i \{\alpha_i\}$ such that

$$\| \phi_i'(u_i, e_i, t) - \phi_i'(u_i, e_i', t) \|_{F_i} \geq \alpha,$$

for some i , $i = 1, 2, \dots, u_i \in U = \bigcup_i U_i$; $e_i, e_i' \in E_i$ and for some $t \in R$. Taking $M = M_0$, it follows that p/M_0 is of distal type.

The 2nd part follows from (C) by taking $b_0 = b$. Thus proof of (C) $\Rightarrow$ (A) is complete.

Now we have to show (B) $\Rightarrow$ (C), and so let $b_0 \in B' \subset B$ be given with $\mu_0(b_0) = N_0 < \infty$ and let b be any other point in B' .

Since B' is minimal, i.e, B' is non-empty, closed and σ -invariant,

the set $\gamma^+(b_0) = \{\sigma(b_0, t), t \in I_{b_0}, t \geq 0\}$ is dense in B' .

Therefore, there is a sequence $\{t_n\}$ in R such that

$$b_0.t_n \rightarrow b, \text{ where } b_0.t_n = \sigma(b_0, t_n).$$

Since $\mu_0(b_0) = N_0 < \infty$, let $p^{-1}(b_0) \cap M = \{\omega_1, \omega_2, \dots, \omega_{N_0}\}$, M being compact π -invariant in W .

Now let $p^{-1}(b_0)$ be diffeomorphic to $\{u_0\} \times E_0$ and $p^{-1}(b_0.t)$ be diffeomorphic to $\{u_0.t\} \times F_0$ under ϕ_0 and ψ_0^t , and let (σ_0', ϕ_0')

be a local representation of π . By positive distal property of the projection map p , there is an

$\alpha = \alpha(u_0, e_1, e_2)$ such that

$$(3.3) \quad \|\phi'_0(u_0, e_0, t) - \phi'_0(u_0, e'_0, t)\|_{F_0} \geq \alpha,$$

for each $u_0 \in U_0$; $\omega_i, \omega_j \in p^{-1}(b_0) \cap M$, $\omega_i \neq \omega_j$, $i, j = 1, 2, \dots, N_0$,

$e_0, e'_0 \in E_0$ and for some $t \geq 0$. We choose subsequences

$\{\omega_1.t_n\}, \dots, \{\omega_{N_0}.t_n\}$ where $\pi(\omega_i, t_n) = \omega_i.t_n$, $i = 1, \dots, N_0$. Then they will converge to limits, say $\widehat{\omega_1}, \dots, \widehat{\omega_{N_0}}$ (as p commutes with the flow and $b_0.t_n \rightarrow b$ in B'). So, $\widehat{\omega_1}, \widehat{\omega_2}, \dots, \widehat{\omega_{N_0}} \in p^{-1}(b)$.

Now from (3.3),

$$\|\phi'_0(u_0, e_0, t_n) - \phi'_0(u_0, e'_0, t_n)\|_{F_0} \geq \alpha, \quad t_n \geq 0.$$

Taking limits as $n \rightarrow \infty$ and considering continuity of $\|\cdot\|_{F_0}$, we have

$$\|\phi'(u, e, t) - \phi'(u, e', t)\|_F \geq \alpha, \quad u \in U, \quad e, e' \in E,$$

where $\phi : p^{-1}(b) \rightarrow U \times E$, $\psi^t : p^{-1}(b.t) \rightarrow U.t \times F$ are diffeomorphisms, $\phi' = \psi^t \circ \pi_t \circ \phi^{-1}$ is c^∞ map from E to F for each $u \in U$, $t \in R$. So, taking $q : U \times E \rightarrow U \times \{0\}$, as the natural projection, $\text{Card}(q^{-1}(u, 0)) \geq N_0$. Now we show that, $\text{Card}(q^{-1}(u, 0)) = N_0$.

If possible, let $\text{Card}(q^{-1}(u, 0)) = N_0 + 1$. Then considering $b.t_n \rightarrow b_0$ in B' and arguing similarly as above, we shall get

$$\text{Card}(q^{-1}(u_0, 0)) \geq N_0 + 1$$

$$\text{and hence } \mu(b_0) = \text{Card}\{\phi^{-1}(q^{-1}(u_0, 0))\} \geq N_0 + 1,$$

which is a contradiction to the fact that $\mu(b_0) = N_0$.

So $\mu(b) = N_0 < \infty$ for $b \in B'$.

Thus we see that (C) follows from (B) where $N' = N$ and $M' = M$.

Next we have to show (C) $\Rightarrow$ (D).

Let us fix an element $b \in B_1 \subset B'$, B_1 being open.

Let $\phi : p^{-1}(B_1) \rightarrow U \times E$ and $\psi^t : p^{-1}(B_1.t) \rightarrow U.t \times F$

be diffeomorphisms, where E, F being Banach spaces.

As $\mu'(b)$ is finite i.e, $\text{Card } \{p^{-1}(b) \cap M'\}$ is finite, for some π -invariant subset M' of W , so taking $M_1 \subseteq p^{-1}(b) \cap M'$, as π -invariant in W , $\phi(M_1)$ also contains finite number of elements. Let $q : U \times E \rightarrow U \times \{0\}$ be the natural projection, then $\text{Card}(q^{-1}(U \times \{0\}) \cap \phi(M))$ is finite and these elements be $e_1, e_2, \dots, e_N$. So by theorem-2.1, $q/\phi(M_1)$ is of distal type.

Thus condition (C) of theorem-2.1 holds. Now by observation-2.2 of theorem-2.1, q becomes open. Further from observation-2.1 of the same theorem, there is a $\beta > 0$ such that for each $u \in U$, and for any pair of elements $e_i, e_j \in \phi(M_1)$, $i \neq j$.

$$\| \phi'(u, e_i, t) - \phi'(u, e_j, t) \|_F \geq 3\beta \text{ for some } t \in R.$$

So, proceeding as in theorem 2.1, it is found that there exists some $0 < N' < \infty$ such that $\phi(M_1)$ is an N' -fold covering space of U with covering map $q/\phi(M_1)$ i.e, for each $(u, 0) \in U \times \{0\}$ there is an open neighbourhood $(u, 0) \in U \times \{0\}$ such that $q^{-1}(U \times \{0\}) \cap \phi(M_1)$ consists of N' disjoint open sets $U_i \times E_i$ where $q_i : U_i \times E_i \rightarrow U \times \{0\}$ is a homeomorphism for each $i = 1, 2, \dots, N'$. Then $\phi^{-1}(U_i \times E_i) = \overline{U_i}$ are disjoint open sets in $p^{-1}(B_1) \cap M'$ for $i = 1, 2, \dots, N'$.

As p commutes with the flows, so $p_i = \phi^{-1} \circ q_i \circ \phi$ becomes homeomorphisms from $\overline{U_i}$ to open neighbourhood B_1 of b .

Thus from definition-3.5, M' is a finite fold covering space of B_1 .

This completes the proof of the theorem 3.2.

4. DISTAL PROPERTY ON A FIBRE BUNDLE OVER A BANACH MANIFOLD

Definition 4.1: Let (p, W, B) be a fibre bundle over a Banach manifold B and let σ be a flow on B . Let $b_{1i}, b_{2i} \in B_i$, $b_{1i} \neq b_{2i}$ such that $\sigma(b_{ji}, t) = b_{ji}.t \in \sigma(B_i, t) = B_i.t$; $j = 1, 2$,

where $b_{ji} = \phi_i^{-1}(u_{ji}, 0)$, $b_{ji}.t = (\psi_i^t)^{-1}(u_{ji}.t, 0)$

$(u_{ji}.t, 0) = \sigma_i'(u_{ji}, 0, t)$, σ_i' being the flow on the local bundle $U_i \times E_i$ defined as $\sigma_i'(u_{ji}, 0, t) = \psi_i^t(\sigma(b_{ji}, t))$, $b_{ji} \in B_i, j = 1, 2, 3, \dots$, σ is said to be distal on B , if

corresponding to each $b_{1i}, b_{2i} \in B_i$, their is an $\alpha_i > 0, i \in I$ such that

$$(4.1) \quad \|\psi_i^t(\sigma(b_{1i}, t)) - \psi_i^t(\sigma(b_{2i}, t))\|_{U_i} \geq \alpha_i, \quad t \in R.$$

In terms of local flow, definition-4.1 may be written as

$$(4.2) \quad \|(\sigma_i'(u_{1i}, 0, t) - \sigma_i'(u_{2i}, 0, t))\|_{U_i} \geq \alpha_i, \quad t \in R, \quad i \in I.$$

Definition 4.2: Let (p, W, B) be a fibre bundle over a Banach manifold B with π as a skew-product flow on W and σ be a flow on B . Let M be a compact π invariant subset of W where $\omega_{1i} \in W_i, \omega_{1k} \in W_k$. Considering $\pi(\omega_{1i}, t) = \omega_{1i}.t \in \pi(\omega_i, t) = \omega_i.t$ and $\pi(\omega_{1k}, t) = \omega_{1k}.t \in \pi(\omega_k, t) = \omega_k.t$, so that $\omega_{1i} = \phi_i^{-1}(u_{1i}, e_{1i})$ and $\omega_{1k} = \phi_k^{-1}(u_{1k}, e_{1k})$ where $(u_{1i}, e_{1i}) \in U_i \times E_i$ and $(u_{1k}, e_{1k}) \in U_k \times E_k$. Again $\omega_{1i}.t = (\psi_i^t)^{-1}(u_{1i}.t, f_i)$ and $\omega_{1k}.t = (\psi_k^t)^{-1}(u_{1k}.t, f_k)$ where $(u_{1i}.t, f_i) \in U_i.t \times F_i$ and $(u_{1k}.t, f_k) \in U_k.t \times F_k$.

Let $\tau_{ik} = (\tau_{ik}^1, \tau_{ik}^2) : U_i \times E_i \longrightarrow U_k \times E_k$ and $\tau_{ik}^t = (\tau_{ik}^{t1}, \tau_{ik}^{t2}) : U_i.t \times F_i \longrightarrow U_k.t \times F_k$ be the local bundle isomorphisms where $\tau_{ik}^t(U_i \times E_i) = (\sigma_k', \phi_k')(\tau_{ik}(U_i \times E_i))$, (σ_k', ϕ_k') being the local representation π_{1k} of π on the local bundle $U_k \times E_k$. If $\|\cdot\|_{U \times E}$, $U \times E$ being a local bundle be defined by

$$\|u, e\|_{U \times E} = \max(\|u\|_U, \|e\|_E), \quad (u, e) \in U \times E$$

then flow π on M is said to be distal, if corresponding to each $\omega_{1i} \in W_i, \omega_{1k} \in W_k$, there is an $\alpha_{ik} > 0$, such that

$$(4.3) \quad \|\tau_{ik}^t(\psi_i^t(\omega_{1i}.t)) - \psi_k^t(\omega_{1k}.t)\|_{U_k.t \times F_k} > \alpha_{ik}, \quad t \in R, \quad i, k \in I.$$

In terms of the local representations π_{1i} and π_{1k} on the local bundles $U_i \times E_i$ and $U_k \times E_k$ respectively, (4.3) may be alternatively expressed as

$$(4.4) \quad \|\tau_{ik}^t((\sigma_i'(u_{1i}, 0, t), \phi_i'(u_{1i}, e_{1i}, t)) - (\sigma_k'(u_{1k}, 0, t), \phi_k'(u_{1k}, e_{1k}, t)))\|_{U_k.t \times E_k} \geq \alpha_{ik}, \quad \forall t \in R, \quad i, k \in I,$$

and finally by

$$(4.5) \quad \| (\sigma_k'(\tau_{ik}(u_{1i}, 0), t), \phi_k'(\tau_{ik}(u_{1i}, e_{1i}, t))) \\ - (\sigma_k'(u_{1k}, 0, t), \phi_k'(u_{1k}, e_{1k}, t))) \|_{U_k.t \times F_k} \geq \alpha_{ik}, \quad \forall t \in R, \quad i, k \in I.$$

Definition 4.3: Let W be a fibre bundle with projection p and base B which is a Banach manifold, σ be a flow on B and (σ'_i, φ'_i) be the local representations of a skew product flow π on local bundles $U_i \times E_i$, $i = 1, 2, 3, \dots$ (E_i being Banach spaces). Also let $M \subset W$ be a compact π -invariant metrizable subset and $B' \subset B$ be a totally bounded subset which is minimal with respect to σ'_i . Then solutions φ'_i is said to be *positively uniformly stable* if for each $\epsilon > 0 \exists \delta > 0$ such that $d_W(\pi(\omega_1, y_1, t_1), \pi(\omega_2, y_2, t_2)) < \epsilon$, in M where $\|\sigma'_i(y_1, t_1) - \sigma'_i(y_2, t_2)\| < \delta$, in B' and $t_2 = t_1 + \tau$ for some $\tau > 0$.

Now we discuss lifting theorem as an application of the distality of projection map p/M on a π -invariant subset of the fibre bundle (p, W, B) . We now prove the following lifting theorem.

Theorem 4.1: *Let π be a skew-product flow on a fibre bundle (p, W, B) over a Banach manifold B , σ be a flow on B and $B' \subset B$ be a totally bounded subset which is minimal with respect to the flow σ and let any one of the conditions (A) – (D) of theorem-3.2 holds. If M be a compact π -invariant subset of W and σ be distal on B' , then π is distal on M .*

Proof: We consider the following cases:

(a) $\omega_{1i}, \omega_{2i} \in W_i$, $\omega_{1i} \neq \omega_{2i}$ (i.e, we consider in particular two distinct elements of the same local bundle chart (ω_1, ϕ_i)). In this case, there are two subcases

(i) $\omega_{1i} = \phi_i^{-1}(u_{1i}, e_{1i})$, $\omega_{2i} = \phi_i^{-1}(u_{1i}, e_{2i})$

(ii) $\omega_{1i} = \phi_i^{-1}(u_{1i}, e_{1i})$, $\omega_{2i} = \phi_i^{-1}(u_{2i}, e_{2i})$.

In both these subcases τ_{ik} of definition-4.2 is the identity map.

For (i) considering equation-4.5 and as the projection map is distal, we have from observation 2.1

$$\begin{aligned}
& \| (\sigma_i'(u_{1i}, 0, t), \phi_i'(u_{1i}, e_{1i}, t)) - (\sigma_i'(u_{1i}, 0, t), \phi_i'(u_{1i}, e_{2i}, t)) \|_{U_i \cdot t \times F_i} \\
&= \max(\| \sigma_i'(u_{1i}, 0, t) - \sigma_i'(u_{1i}, 0, t) \|_{U_i}, \| \phi_i'(u_{1i}, e_{1i}, t) - \phi_i'(u_{1i}, e_{2i}, t) \|_{F_i}) \\
&\geq \| \phi_i'(u_{1i}, e_{1i}, t) - \phi_i'(u_{1i}, e_{2i}, t) \|_{F_i} \geq 3\beta, \quad \forall t \in R,
\end{aligned}$$

For (ii), we have

$$\begin{aligned}
& \| (\sigma_i'(u_{1i}.0, t), \phi_i'(u_{1i}, e_{1i}, t)) - \sigma_i'(u_{2i}.0, t), \phi_i'(u_{2i}, e_{2i}, t) \|_{U_i \cdot t \times F_i} \\
&= \max(\| \sigma_i'(u_{1i}.0, t) - \sigma_i'(u_{2i}.0, t) \|_{U_i}, \| \phi_i'(u_{1i}, e_{1i}, t) - \phi_i'(u_{2i}, e_{2i}, t) \|_{F_i}) \\
&\geq \| \sigma_i'(u_{1i}.0, t) - \sigma_i'(u_{2i}.0, t) \| \geq \alpha_i, \quad \forall t \in R,
\end{aligned}$$

by distality of the flow σ .

(b) $\omega_{1i} \in W_i$, $\omega_{1k} \in W_k$.

We consider equation-4.5 and observe that there may be two subcases

- (i) $\tau_{ik}(u_{1i}, 0) = (u_{1k}, 0)$,
- (ii) $\tau_{ik}(u_{1i}, 0) = (u_{2k}, 0)$, $u_{1k} \neq u_{2k}$.

Similarly subcase b(i) can be shown as subcase a(i) and b(ii) as subcase a(ii).

Thus in all cases equation-4.5 is satisfied for some $\alpha_{ik} > 0$, $\forall t \in R$;

$i, k \in I$.

This completes the proof of theorem-4.1.

Theorem 4.2: *Let σ be a minimal flow on a Banach manifold B and let $\pi = (\sigma'_i, \phi'_i)$ be a skew product flow on fibre bundle W . If bounded solutions ϕ'_i are positively uniformly stable, then M is a minimal set of π .*

Proof: Let ϕ'_i are positively uniformly stable. By definition M is compact π -invariant metrizable subset of W . We have to prove M is minimal. By previous theorem p/M is of distal type, so it is also negatively distal type for $t \leq 0$. Let $U_r(\omega) = \{M_\omega/y \in B', d_W(M_\omega, y) < r\}$ for $r > 0$ and $U_r = \bigcup_{\omega \in M} U_r(\omega)$. Then U_r is an open subset of W which contain M . If M is not minimal then we chose a neighbourhood V of M such that ω_0 does not contain in $\bar{V}$ and $\exists \epsilon > 0$ such that $U_\epsilon \subset V$ (as ϕ' is positively stable we can get an ϵ) and also $\exists \omega' \in U_\epsilon$ where $\delta(\epsilon) > 0$ so that ϕ' is positively uniformly stable, where $\omega_0, \omega' \in M$ and $y_0, y' \in B'$.

We take $\omega^* \in M$ and from definition of stability

$d_W((\omega', y', t_1), (\omega^*, y^*, t_2)) < \epsilon$, whenever

$$||\sigma'_i(y', t_1) - \sigma'_i(y^*, t_2)|| < \delta(\epsilon).$$

Since M is invariant set of π , $(\sigma'_i, \phi'_i) \in M$ for $t \geq 0$

Hence $\phi'_i \in U_\epsilon$ for $t \geq 0$, i.e., $\phi'_i \in V$.

Since $M \subset V \Leftrightarrow M \subset U_\epsilon \subset V$

$\Rightarrow (\sigma'_i, \phi'_i) \in U_\epsilon$, $\pi \in U_\epsilon$, for $t \geq 0$.

$\pi((\omega, \tau), t) = \pi(\omega, \tau + t) \in U_\epsilon$, for $t \geq 0$

$\Rightarrow \overline{U}_\epsilon \subset \overline{V}$.

$\Rightarrow \omega_0 \in \overline{V}$.

So ω_0 does not contain in $\overline{V}$ is a contradiction.

Therefore M is minimal.

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