

HOMOLOGY OF A TYPE OF PENTAHEDRON

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ABSTRACT. In this paper we find out the homology of a type of pentahedron with five vertices, and have shown that it is analogous with the homology of a chain complex, $0 \rightarrow \mathbf{R}^5 \rightarrow \mathbf{R}^8 \rightarrow \mathbf{R}^5 \rightarrow 0$ and also find out the singular homology and the Euler Characteristic of this type of pentahedron which is equal to $\sum_{n=0}^{\infty} \dim_R(H_n(P))$.

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1. INTRODUCTION

Homology classes were first defined rigorously by Henri Poincaré in his seminal paper "Analysis situs" in 1895. The homology group was further developed by Emmy Noether [1] and, independently, by Leopold Vietoris and Walther Mayer [2], in the period 1925-28.

Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology). It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is

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possible. In algebraic topology, singular homology refers to the study of a certain set of algebraic invariants of a topological space X , the so-called homology groups $H_n(X)$. Intuitively spoken, singular homology counts, for each dimension n , the n -dimensional holes of a space.

Let f and g be matrices whose product is zero. If $g.v = 0$ for some column vector v , say, of length n , we can not always write $v = f.u$. This failure is measured by the defect

$$d = n - \text{rank}(f) - \text{rank}(g).$$

In modern language, f and g represent linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

with $gf = 0$, and d is the dimension of the homology module

$$H = \ker(g)/\text{img}(f).$$

Given an R - module homomorphism $f : A \rightarrow B$, one is immediately led to study the kernel of f and image of f . Given another map $g : B \rightarrow C$, we can form the sequence

$$(1) \quad A \xrightarrow{f} B \xrightarrow{g} C.$$

We say that such a sequence is exact (at B) if $\ker(g) = \text{img}(f)$. This implies in particular that the composite $gf : A \rightarrow C$ is zero and finally brings our attention to sequence (1) such that $gf = 0$.

In this paper, first we find out that homology of a type of pentahedron is analogous to the homology of a chain complex, $0 \rightarrow \mathbf{R}^5 \rightarrow \mathbf{R}^8 \rightarrow \mathbf{R}^5 \rightarrow 0$ and we also find out the matrices of this complex. Next we show computationally, $H_2(P) \cong H_0(P) \cong R$ and $H_1(P) = 0$ and the Euler Characteristic of this type of pentahedron which is equal to $\sum_{n=0}^{\infty} \dim_R(H_n(P))$.

2. HOMOLOGY OF A TYPE OF PENTAHEDRON

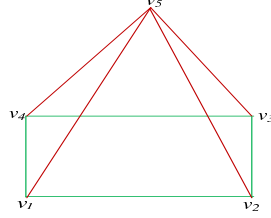


Fig. 1

We can obtain a chain complex from a geometric object. We refer to the Weibel's book [3] for some details of the construction. We illustrate it with a pentahedron P (Fig. 1). Level the vertex set of P as $V = \{v_1, v_2, v_3, v_4, v_5\}$. Then the six edges $e_{12}, e_{23}, e_{34}, e_{41}, e_{15}, e_{25}, e_{35}, e_{45}$, where $e_{ij} = e_{ji}$ for $i, j = 1, 2, 3, 4, 5$ can be ordered as

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_5\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}\}$$

and the five faces f_1, f_2, f_3, f_4, f_5 can be ordered as

$$F = \{\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_4, v_1, v_5\}\}.$$

Let R be a ring and let $C_i(P)$ be the free R -module on the set V, E, F , respectively. Define maps $\partial_0, \partial_1, \partial_2 : F \rightarrow E$ by removing the first, second, and third vertices, respectively except first face for each map. For the first face we define each map in such a way so that we can construct the homology. So, $\partial_0, \partial_1, \partial_2$ are given by

$\partial_0 : f_1 \rightarrow e_{41}$	$\partial_1 : f_1 \rightarrow e_{45}$	$\partial_2 : f_1 \rightarrow e_{15}$
$f_2 \rightarrow e_{25}$	$f_2 \rightarrow e_{15}$	$f_2 \rightarrow e_{12}$
$f_3 \rightarrow e_{35}$	$f_3 \rightarrow e_{25}$	$f_3 \rightarrow e_{23}$
$f_4 \rightarrow e_{45}$	$f_4 \rightarrow e_{35}$	$f_4 \rightarrow e_{34}$
$f_5 \rightarrow e_{15}$	$f_5 \rightarrow e_{45}$	$f_5 \rightarrow e_{41}$

The set maps ∂_i yield $k+1$ module maps $C_k \rightarrow C_{k-1}$, which we also call ∂_i ; their alternating sum $d_i = \sum (-1)^i \partial_i$ is the map $C_k \rightarrow C_{k-1}$, where $(0 \leq i \leq k \leq n)$ in the chain complex C . We can then define the map

$d_2 = \partial_0 - \partial_1 + \partial_2 : C_2 \rightarrow C_1$, which is given by

$$f_1 \rightarrow e_{41} - e_{45} + e_{15}$$

$$f_2 \rightarrow e_{25} - e_{15} + e_{12}$$

$$f_3 \rightarrow e_{35} - e_{25} + e_{23}$$

$$f_4 \rightarrow e_{45} - e_{35} + e_{34}$$

$$f_5 \rightarrow e_{15} - e_{45} + e_{41}$$

We can define maps $\partial_0, \partial_1 : E \rightarrow V$ by removing the first, second vertices, respectively. Therefore we have

$$\partial_0 : e_{12} \rightarrow v_2 \quad \partial_1 : e_{12} \rightarrow v_1$$

$$e_{23} \rightarrow v_3 \quad e_{23} \rightarrow v_2$$

$$e_{34} \rightarrow v_4 \quad e_{34} \rightarrow v_3$$

$$e_{41} \rightarrow v_1 \quad e_{41} \rightarrow v_4$$

$$e_{15} \rightarrow v_5 \quad e_{15} \rightarrow v_1$$

$$e_{25} \rightarrow v_5 \quad e_{25} \rightarrow v_2$$

$$e_{35} \rightarrow v_5 \quad e_{35} \rightarrow v_3$$

$$e_{45} \rightarrow v_5 \quad e_{45} \rightarrow v_4$$

We can define map $d_1 = \partial_0 - \partial_1$ from C_1 to C_0 , and it is given by

$$e_{12} \rightarrow v_2 - v_1$$

$$e_{23} \rightarrow v_3 - v_2$$

$$e_{34} \rightarrow v_4 - v_3$$

$$e_{41} \rightarrow v_1 - v_4$$

$$e_{15} \rightarrow v_5 - v_1$$

$$e_{25} \rightarrow v_5 - v_2$$

$$e_{35} \rightarrow v_5 - v_3$$

$$e_{45} \rightarrow v_5 - v_4$$

By viewing $C_2 = R^5$, $C_1 = R^8$, and $C_0 = R^5$, the maps d_1 and d_2 are given by the following matrices

$$d_1 = \begin{pmatrix} -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$d_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Because $d_1 d_2$ is easily computed to be zero matrix, the sequence

$$\dots \rightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \rightarrow 0$$

is a complex. We compute the homology $H_n(P)$ of this complex with the help of Maple. By finding the column space of d_1 , we find $\text{im}(d_1)$. This space has a basis consisting of the vectors

$$\{(-1, 0, 0, 0, 1), (0, -1, 1, 0, 0), (0, 0, -1, 0, 1), (0, 0, 0, -1, 1)\}.$$

We note that by adding $(0, 0, 0, 0, 1)$ that we get a basis for R^5 . Therefore $C_0/\text{im}(d_1) \cong R$. Thus, $H_0(P) = R$. Next, $\ker(d_1)$ has a basis

$$\{(1, 1, 1, 1, 0, 0, 0, 0), (-1, 0, 0, 0, 1, 0, 0, 0), (0, 1, 1, 0, 0, -1, 0, 1), (0, 1, 0, 0, 0, -1, 1, 0)\}$$

Also, $\text{im}(d_2)$ has a basis

$$\{(1, 1, 1, 1, 0, 0, 0, 0), (0, 1, 0, 0, 0, -1, 1, 0), (0, 1, 1, 0, 0, -1, 0, 1), (0, 1, 1, 1, 1, -1, 0, 0)\}$$

The first three elements are the basis vectors of $\ker(d_1)$. Also, $(0, 1, 1, 1, 1, -1, 0, 0)$ vector is $u_1 + u_2$, if u_i are the basis vectors of $\ker(d_1)$. We thus see that $\text{im}(d_2) = \ker(d_1)$. Therefore, $H_1(P) = 0$. Finally, $\ker(d_2)$ has a basis of one element $\{(-1, 0, 0, 0, 1)\}$. Therefore, $H_2(P) = \ker(d_2) = R$. To summarize, the singular homology $H_n(P)$ of the pentahedron is

$$H_0(P) = H_2(P) = R$$

$$H_1(P) = 0$$

$$H_n(P) = 0 \text{ if } n \geq 3.$$

The Euler Characteristic is a fundamental invariant for the classification of surfaces, so it is particularly useful that it can be calculated with homological algebra. The Euler characteristic of such surface P is $v - e + f$, where v is the number of vertices, e is the number of edges and f is the number of faces. Now, the Euler characteristic of pentahedron is 2, which is equal to $\sum_{n=0}^{\infty} \dim_R(H_n(P))$. This is the same as the Euler Characteristic of a sphere, which is not surprising since a pentahedron is homeomorphic to a sphere, so it is homotopic to a sphere.

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