

EXPLICIT ESTIMATES ON SOME NONLINEAR INTEGRAL INEQUALITIES IN TWO INDEPENDENT VARIABLES ON TIME SCALES AND APPLICATIONS

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ABSTRACT. The main objective in this paper is to investigate some nonlinear integral inequalities in two independent variables on time scales. The obtained inequalities can be used as handy tools to study the properties of certain partial dynamic equations on time scales.

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1. INTRODUCTION

Motivated by the paper [5], many authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales.

The main purpose of this paper is to obtain time scale version of some more fundamental nonlinear integral inequalities in two independent variables used in the theory of differential and integral equation [10]. The obtained inequalities can be used as handy tools to study the properties of certain partial dynamic equations on time scales.

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1.1. Preliminaries on time scales . In this section, we begin by giving some necessary materials for our study.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) := \inf \{s \in \mathbb{T} : s > t\} \in \mathbb{T}$ for all $t \in \mathbb{T}$, C_{rd} denotes the set of rd-continuous functions and the set $\mathbb{T}^k$ which is derived from the time scale $\mathbb{T}$ as follows : if $\mathbb{T}$ has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Throughout this paper, we always assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales and we write $x^{\Delta_t}(t, s)$ the partial delta derivatives of $x(t, s)$ with respect to t . And $\tilde{\mathbb{T}}_1 = [t_0, t_1] \cap \mathbb{T}_1^k$, $\tilde{\mathbb{T}}_2 = [s_0, s_1] \cap \mathbb{T}_2^k$ and $\tilde{\Omega} = \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$.

For more details about time scales, we advise the reader to refer to [1 – 2]

Lemma 1 (Theorem 1.117 in [1]). *Let $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^k \rightarrow \mathbb{R}$ is continuous at (t, t) , where $t \in \mathbb{T}^k$ with $t > a$. Also assume that $f^{\Delta}(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t , independent of $\tau \in [a, \sigma(t)]$, such that*

$$|f(\sigma(t), \tau) - f(s, \tau) - f^{\Delta}(t, \tau)(\sigma(t) - s)| < \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U,$$

where f^{Δ} denotes the derivative of f with respect to the first variable. Then

$$\begin{aligned} (i) \quad g(t) : &= \int_a^t f(t, \tau) \Delta \tau \quad \text{implies} \quad g^{\Delta}(t) = \int_a^t f^{\Delta}(t, \tau) \Delta \tau + f(\sigma(t), t); \\ (ii) \quad h(t) : &= \int_t^b f(t, \tau) \Delta \tau \quad \text{implies} \quad h^{\Delta}(t) = \int_t^b f^{\Delta}(t, \tau) \Delta \tau - f(\sigma(t), t); \end{aligned}$$

Lemma 2. *Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$(1.1) \quad a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}},$$

for any $K > 0$.

Proof. see [8]. □

Definition 1. *The function $\omega(u)$ is said to be subadditive and submultiplicative, if*

$$(1.2) \quad \omega(u + v) \leq \omega(u) + \omega(v) \quad \text{and} \quad \omega(uv) \leq \omega(u)\omega(v) \quad \text{for } u, v \geq 0.$$

Now we state the main results of this work.

2. MAIN RESULT

Theorem 1. *Let $u(t, s), a(t, s), b(t, s)$ and $f(t, s)$ are nonnegative functions defined for $(t, s) \in \tilde{\Omega}$ that are right-dense continuous for $(t, s) \in \tilde{\Omega}$, $\omega(x)$ be a nonnegative, nondecreasing continuous function for $x \in \mathbb{R}^+$, with $\omega(x) > 0$ for all $x > 0$. Assume that $a(t, s)$ and $b(t, s)$ are nondecreasing in each variable $(t, s) \in \tilde{\Omega}$. If*

$$(2.1) \quad u(t, s) \leq a(t, s) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(u(\tau, \eta)) \Delta\eta \Delta\tau \text{ for } (t, s) \in \tilde{\Omega},$$

then

$$(2.2) \quad u(t, s) \leq G^{-1} \left[G(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau \right],$$

for all $t_0 < t \leq t_1$ and $s_0 < s \leq s_1$. G is defined as follows

$$(2.3) \quad G^{\Delta\tau}(u(\tau, \eta)) = \frac{u^{\Delta\tau}(\tau, \eta)}{\omega(u(\tau, \eta))}, \quad (\tau, \eta) \in \tilde{\Omega}.$$

G^{-1} is the inverse function of G and t_1, s_1 are chosen so that

$$\left\{ G(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau \right\} \in \text{Dom}(G^{-1}).$$

Proof. Fixing any numbers $\bar{t}_1 \in \tilde{\mathbb{T}}_1$, $\bar{s}_1 \in \tilde{\mathbb{T}}_2$ with $t_0 < \bar{t}_1 \leq t_1$ and $s_0 < \bar{s}_1 \leq s_1$, from (2.1) we have

$$(2.4) \quad u(t, s) \leq a(\bar{t}_1, \bar{s}_1) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(u(\tau, \eta)) \Delta\eta \Delta\tau,$$

for $t_0 < t \leq \bar{t}_1$ and $s_0 < s \leq \bar{s}_1$.

Define a function $z(t, s)$ by

$$(2.5) \quad z(t, s) = a(\bar{t}_1, \bar{s}_1) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(u(\tau, \eta)) \Delta\eta \Delta\tau,$$

then we have

$$(2.6) \quad z(t_0, s) = z(t, s_0) = a(\bar{t}_1, \bar{s}_1),$$

and

$$(2.7) \quad u(t, s) \leq z(t, s).$$

From (2.5), a delta derivative with respect to t yields

$$(2.8) \quad z^{\Delta_t}(t, s) = b(\bar{t}_1, \bar{s}_1) \int_{s_0}^s f(t, \eta) \omega(u(t, \eta)) \Delta \eta,$$

it follows from (2.8) and (2.7) that

$$(2.9) \quad z^{\Delta_t}(t, s) \leq b(\bar{t}_1, \bar{s}_1) \int_{s_0}^s f(t, \eta) \omega(z(t, \eta)) \Delta \eta,$$

the fact that $z(t, s)$ is nondecreasing in $s \in [s_0, \bar{s}_1] \cap \mathbb{T}_2^k$, the inequality (2.9) implies the estimate

$$(2.10) \quad z^{\Delta_t}(t, s) \leq b(\bar{t}_1, \bar{s}_1) \omega(z(t, s)) \int_{s_0}^s f(t, \eta) \Delta \eta.$$

Dividing both sides of (2.10) by $\omega(z(t, s))$, we obtain

$$(2.11) \quad \frac{z^{\Delta_t}(t, s)}{\omega(z(t, s))} \leq b(\bar{t}_1, \bar{s}_1) \int_{s_0}^s f(t, \eta) \Delta \eta.$$

Setting $t = \tau$ in (2.11) and integrating both sides of (2.11) with respect to τ , by using (2.3) and (2.6), we get

$$(2.12) \quad G(z(t, s)) \leq G(a(\bar{t}_1, \bar{s}_1)) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta \eta \Delta \tau.$$

Letting $t = \bar{t}_1$ and $s = \bar{s}_1$ in (2.12), we have

$$(2.13) \quad G(z(\bar{t}_1, \bar{s}_1)) \leq G(a(\bar{t}_1, \bar{s}_1)) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^{\bar{t}_1} \int_{s_0}^{\bar{s}_1} f(\tau, \eta) \Delta \eta \Delta \tau.$$

Since $t_0 < \bar{t}_1 \leq t_1$ and $s_0 < \bar{s}_1 \leq s_1$ are arbitrary chosen and from (2.13), we obtain

$$(2.14) \quad G(z(t, s)) \leq G(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta \eta \Delta \tau,$$

for all $t_0 < t \leq t_1$ and $s_0 < s \leq s_1$,

from (2.14), we have

$$(2.15) \quad z(t, s) \leq G^{-1} \left(G(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau \right).$$

Combining (2.7) and (2.15), we obtain the desired inequality (2.2). $\square$

Remark 1. Letting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $t_0 = s_0 = 0$, then Theorem 1 reduced to Lemma 3.1 in [7].

Theorem 2. Assume that $u(t, s)$, $a(t, s)$ and $b(t, s)$ are nonnegative functions defined for $(t, s) \in \tilde{\Omega}$ that are right-dense continuous for $(t, s) \in \tilde{\Omega}$. If $f : \tilde{\Omega} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is right-dense continuous on $\tilde{\Omega}$ and continuous on $\mathbb{R}^+$ such that

$$(2.16) \quad 0 \leq f(t, s, x) - f(t, s, y) \leq h(t, s, y)\omega(x - y),$$

for $(t, s) \in \tilde{\Omega}$, $x \geq y \geq 0$ where $h : \tilde{\Omega} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is right-dense continuous on $\tilde{\Omega}$ and continuous on $\mathbb{R}^+$ and $\omega \in C(\mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing and satisfying (1.2). If

$$(2.17) \quad u^p(t, s) \leq a(t, s) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta, u^r(\tau, \eta)) \Delta\eta \Delta\tau \text{ for } (t, s) \in \tilde{\Omega},$$

then

$$(2.18) \quad u(t, s) \leq \left\{ \left[a(t, s) + b(t, s) G^{-1} \left(G(A(t, s)) + \int_{t_0}^t \int_{s_0}^s B(\tau, \eta) \Delta\eta \Delta\tau \right) \right] \right\}^{\frac{1}{p}},$$

where

$$(2.19) \quad A(t, s) = \int_{t_0}^t \int_{s_0}^s f(\tau, \eta, \frac{r}{p} \left(K^{\frac{r-p}{p}} a(\tau, \eta) + \frac{p-r}{r} K^{\frac{r}{p}} \right)) \Delta\eta \Delta\tau,$$

and

$$(2.20) \quad B(t, s) = h(t, s, \frac{r}{p} \left(K^{\frac{r-p}{p}} a(t, s) + \frac{p-r}{r} K^{\frac{r}{p}} \right)) \omega(\frac{r}{p} K^{\frac{r-p}{p}} b(t, s)),$$

for $p \neq 0$ and $0 \leq r \leq p$. G is defined as in (2.3), G^{-1} is the inverse function of G , for all $t_0 < t \leq t_1$ and $s_0 < s \leq s_1$, where t_1, s_1 are chosen so that

$$\left\{ G(A(t, s)) + \int_{t_0}^t \int_{s_0}^s B(\tau, \eta) \Delta\eta \Delta\tau \right\} \in \text{Dom}(G^{-1}).$$

Proof. Define a function $z(t, s)$ by

$$(2.21) \quad z(t, s) = \int_{t_0}^t \int_{s_0}^s f(\tau, \eta, u^r(\tau, \eta)) \Delta\eta \Delta\tau \text{ for } (t, s) \in \Omega,$$

then, we observe that

$$(2.22) \quad u^p(t, s) \leq a(t, s) + b(t, s)z(t, s),$$

we have also

$$u^r(t, s) = (u^p(t, s))^{\frac{r}{p}} \leq (a(t, s) + b(t, s)z(t, s))^{\frac{r}{p}},$$

using Lemma 2, we obtain

$$(2.23) \quad u^r(t, s) \leq \frac{r}{p} K^{\frac{r-p}{p}} (a(t, s) + b(t, s)z(t, s)) + \frac{p-r}{p} K^{\frac{r}{p}}, K > 0,$$

by the assumptions on f and ω , we have

$$(2.24) \quad \begin{aligned} f(\tau, \eta, u^r(\tau, \eta)) &\leq f(\tau, \eta, \frac{r}{p} K^{\frac{r-p}{p}} (a(\tau, \eta) + b(\tau, \eta)z(\tau, \eta)) + \frac{p-r}{r} K^{\frac{r}{p}}) \\ &\leq h(\tau, \eta, \frac{r}{p} \left(K^{\frac{r-p}{p}} a(\tau, \eta) + \frac{p-r}{r} K^{\frac{r}{p}} \right)) \times \\ &\quad \omega\left(\frac{r}{p} K^{\frac{r-p}{p}} b(\tau, \eta)\right) \times \omega(z(\tau, \eta)) + \\ &\quad + f(\tau, \eta, \frac{r}{p} \left(K^{\frac{r-p}{p}} a(\tau, \eta) + \frac{p-r}{r} K^{\frac{r}{p}} \right)), \end{aligned}$$

integrating (2.24) with respect to τ, η , we get

$$(2.25) \quad \begin{aligned} \int_{t_0}^t \int_{s_0}^s f(\tau, \eta, u^r(\tau, \eta)) \Delta\eta \Delta\tau &\leq \int_{t_0}^t \int_{s_0}^s f(\tau, \eta, \frac{r}{p} \left(K^{\frac{r-p}{p}} a(\tau, \eta) + \frac{p-r}{r} K^{\frac{r}{p}} \right)) \Delta\eta \Delta\tau \\ &\quad + \int_{t_0}^t \int_{s_0}^s \left[h(\tau, \eta, \frac{r}{p} \left(K^{\frac{r-p}{p}} a(\tau, \eta) + \frac{p-r}{r} K^{\frac{r}{p}} \right)) \right. \\ &\quad \left. \times \omega\left(\frac{r}{p} K^{\frac{r-p}{p}} b(\tau, \eta)\right) \omega(z(\tau, \eta)) \right] \Delta\eta \Delta\tau. \end{aligned}$$

So (2.21) can be written as

$$(2.26) \quad z(t, s) \leq A(t, s) + \int_{t_0}^t \int_{s_0}^s B(\tau, \eta) \omega(z(\tau, \eta)) \Delta\eta \Delta\tau,$$

where $A(t, s)$ and $B(\tau, \eta)$ are defined as in (2.19) and (2.20) respectively.

The above inequality is similar to (2.1), by using Theorem 1 from (2.26), we easily see that

$$(2.27) \quad z(t, s) \leq G^{-1} \left(G(A(t, s)) + \int_{t_0}^t \int_{s_0}^s B(\tau, \eta) \Delta\eta \Delta\tau \right),$$

from (2.27) and (2.22), we have

$$(2.28) \quad u^p(t, s) \leq a(t, s) + b(t, s) G^{-1} \left(G(A(t, s)) + \int_{t_0}^t \int_{s_0}^s B(\tau, \eta) \Delta\eta \Delta\tau \right).$$

The inequality (2.28) implies (2.18). $\square$

Remark 2. Letting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $t_0 = s_0 = 0$, we can see easily that Theorem 2 reduced to (b₂) of Theorem 2 in [6].

Theorem 3. Let u, a, b, f and ω be defined as in Theorem 1. Let $\varphi(x)$ be a non-negative, nondecreasing continuous function for $x \in \mathbb{R}^+$, with $\varphi(x) > 0$ for all $x > 0$.

If

$$(2.29) \quad \varphi(u(t, s)) \leq a(t, s) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(u(\tau, \eta)) \Delta\eta \Delta\tau \text{ for } (t, s) \in \tilde{\Omega}.$$

Then

$$(2.30) \quad u(t, s) \leq \varphi^{-1} \left[\Phi^{-1} \left[\Phi(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau \right] \right],$$

where

$$(2.31) \quad \Phi^{\Delta\tau}(u(\tau, s)) = \frac{u^{\Delta\tau}(\tau, s)}{\omega(\varphi^{-1}(u(\tau, s)))}, (\tau, s) \in \tilde{\Omega};$$

and Φ^{-1} is the inverse function of Φ and t_1, s_1 are chosen so that

$$\left\{ \Phi(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau \right\} \in \text{Dom}(\Phi^{-1}).$$

Proof. Define a function $z(t, s)$ by

$$(2.32) \quad z(t, s) = a(t, s) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(u(\tau, \eta)) \Delta\eta \Delta\tau,$$

then

$$(2.33) \quad \varphi(u(t, s)) \leq z(t, s),$$

so that

$$(2.34) \quad u(t, s) \leq \varphi^{-1}(z(t, s)),$$

in (2.32) fixing any numbers $\bar{t}_1 \in \tilde{\mathbb{T}}_1$, $\bar{s}_1 \in \tilde{\mathbb{T}}_2$ with $t_0 < \bar{t}_1 \leq t_1$ and $s_0 < \bar{s}_1 \leq s_1$, we obtain

$$(2.35) \quad z(t, s) = a(\bar{t}_1, \bar{s}_1) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(u(\tau, \eta)) \Delta\eta \Delta\tau,$$

with

$$(2.36) \quad z(t_0, s) = z(t, s_0) = a(\bar{t}_1, \bar{s}_1).$$

Combining (2.34) and (2.35), we get

$$(2.37) \quad z(t, s) \leq a(\bar{t}_1, \bar{s}_1) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \omega(\varphi^{-1}(z(\tau, \eta))) \Delta\eta \Delta\tau,$$

from (2.37), a delta derivative with respect to t yields

$$(2.38) \quad z^{\Delta_t}(t, s) \leq b(\bar{t}_1, \bar{s}_1) \int_{s_0}^s f(t, \eta) \omega(\varphi^{-1}(z(t, \eta))) \Delta\eta.$$

Using the fact that ω, φ and z are nondecreasing, (2.38) become

$$(2.39) \quad z^{\Delta_t}(t, s) \leq b(\bar{t}_1, \bar{s}_1) \omega(\varphi^{-1}(z(t, s))) \int_{s_0}^s f(t, \eta) \Delta\eta.$$

Dividing both sides of (2.39) by $\omega(\varphi^{-1}(z(t, s)))$, we obtain

$$(2.40) \quad \frac{z^{\Delta_t}(t, s)}{\omega(\varphi^{-1}(z(t, s)))} \leq b(\bar{t}_1, \bar{s}_1) \int_{s_0}^s f(t, \eta) \Delta\eta.$$

Setting $t = \tau$ in (2.40) and integrating both sides of (2.40) with respect to τ , by using (2.31) and (2.36), we get

$$(2.41) \quad \Phi(z(t, s)) \leq \Phi(a(\bar{t}_1, \bar{s}_1)) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau.$$

Letting $t = \bar{t}_1$ and $s = \bar{s}_1$ in (2.41), we have

$$(2.42) \quad \Phi(z(\bar{t}_1, \bar{s}_1)) \leq \Phi(a(\bar{t}_1, \bar{s}_1)) + b(\bar{t}_1, \bar{s}_1) \int_{t_0}^{\bar{t}_1} \int_{s_0}^{\bar{s}_1} f(\tau, \eta) \Delta\eta \Delta\tau.$$

Since $t_0 < \bar{t}_1 \leq t_1$ and $s_0 < \bar{s}_1 \leq s_1$ are arbitrary chosen then (2.42) implies

$$(2.43) \quad \Phi(z(t, s)) \leq \Phi(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau,$$

thus

$$(2.44) \quad z(t, s) \leq \Phi^{-1} \left(\Phi(a(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s f(\tau, \eta) \Delta\eta \Delta\tau \right).$$

Combining (2.34) and (2.44), we obtain (2.30). $\square$

Theorem 4. Assume that $u(t, s)$, $a(t, s)$ and $b(t, s)$ are nonnegative functions defined for $(t, s) \in \tilde{\Omega}$ that are right-dense continuous for $(t, s) \in \tilde{\Omega}$, $k(t, s, \tau, \eta)$, $k^{\Delta\eta}(t, s, \tau, \eta)$, $k^{\Delta\tau}(t, s, \tau, \eta)$, $k^{\Delta\eta\Delta\tau}(t, s, \tau, \eta)$ are nonnegative and right-dense continuous functions for $(t, s, \tau, \eta) \in \tilde{\Omega} \times \tilde{\Omega}$, $\omega(x)$ be a nonnegative, nondecreasing continuous function for $x \in \mathbb{R}^+$, with $\omega(x) > 0$ for all $x > 0$ and satisfying (1.2). If

$$(2.45) \quad u^p(t, s) \leq a(t, s) + b(t, s) \int_{t_0}^t \int_{s_0}^s k(t, s, \tau, \eta) \omega(u^r(\tau, \eta)) \Delta\eta \Delta\tau \quad \text{for } (t, s, \tau, \eta) \in \tilde{\Omega} \times \tilde{\Omega},$$

for $0 < r \leq p$.

Then

$$(2.46) \quad u(t, s) \leq \left\{ a(t, s) + b(t, s) \left[G^{-1} \left(G(C(t, s)) + \int_{t_0}^t \int_{s_0}^s D(\tau, \eta) \Delta\eta \Delta\tau \right) \right] \right\}^{\frac{1}{p}},$$

for all $t_0 < t \leq t_1$ and $s_0 < s \leq s_1$, where G is defined as in (2.3),

$$C(t, s) = \int_{t_0}^t \int_{s_0}^s A(\tau, \eta) \Delta\eta \Delta\tau,$$

$$\begin{aligned}
A(t, s) = & \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) \int_{t_0}^t \int_{s_0}^s k^{\Delta_\eta \Delta_\tau}(t, s, \tau, \eta) \omega(a(\tau, \eta)) \Delta \eta \Delta \tau + \\
& + \omega\left(\frac{p-r}{p}K^{\frac{r}{p}}\right) \int_{t_0}^t \int_{s_0}^s k^{\Delta_\eta \Delta_\tau}(t, s, \tau, \eta) \Delta \eta \Delta \tau \\
& + \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) \int_{t_0}^t k^{\Delta_\tau}(t, s, \tau, \sigma_2(s)) \omega(a(\tau, s)) \Delta \tau \\
& + \omega\left(\frac{p-r}{p}K^{\frac{r}{p}}\right) \int_{t_0}^t k^{\Delta_\tau}(t, s, \tau, \sigma_2(s)) \Delta \tau \\
& + \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) \int_{t_0}^t k^{\Delta_\eta}(t, s, \sigma_1(t), \eta) \omega(a(t, \eta)) \Delta \eta \\
& + \omega\left(\frac{p-r}{p}K^{\frac{r}{p}}\right) \int_{t_0}^t k^{\Delta_\eta}(t, s, \sigma_1(t), \eta) \Delta \eta \\
& + \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) k(t, s, \sigma_1(t), \sigma_2(s)) \omega(a(t, s)) \\
(2.47) \quad & + \omega\left(\frac{p-r}{p}K^{\frac{r}{p}}\right) k(t, s, \sigma_1(t), \sigma_2(s)),
\end{aligned}$$

and

$$\begin{aligned}
D(t, s) = & \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) \int_{t_0}^t \int_{s_0}^s k^{\Delta_\eta \Delta_\tau}(t, s, \tau, \eta) \omega(b(\tau, \eta)) \Delta \eta \Delta \tau \\
& + \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) \int_{t_0}^t k^{\Delta_\tau}(t, s, \tau, \sigma_2(s)) \omega(b(\tau, s)) \Delta \tau \\
& + \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) \int_{t_0}^t k^{\Delta_\eta}(t, s, \sigma_1(t), \eta) \omega(b(t, \eta)) \Delta \eta \\
(2.48) \quad & + \omega\left(\frac{r}{p}K^{\frac{r-p}{p}}\right) k(t, s, \sigma_1(t), \sigma_2(s)) \omega(b(t, s)),
\end{aligned}$$

G^{-1} is the inverse function of G and t_1, s_1 are chosen so that

$$\left\{ G(A(t, s)) + b(t, s) \int_{t_0}^t \int_{s_0}^s B(\tau, \eta) \Delta \eta \Delta \tau \right\} \in \text{Dom}(G^{-1}).$$

Proof. Define a function $z(t, s)$ by

$$(2.49) \quad z(t, s) = \int_{t_0}^t \int_{s_0}^s k(t, s, \tau, \eta) \omega(u^r(\tau, \eta)) \Delta \eta \Delta \tau,$$

then we have

$$(2.50) \quad z(t_0, s) = z(t, s_0) = 0$$

$$(2.51) \quad u^p(t, s) \leq a(t, s) + b(t, s)z(t, s),$$

$$(2.52) \quad u^r(t, s) \leq (a(t, s) + b(t, s)z(t, s))^{\frac{r}{p}},$$

by Lemma 2, (2.52) become

$$(2.53) \quad u^r(t, s) \leq \frac{r}{p} K^{\frac{r-p}{p}} (a(t, s) + b(t, s)z(t, s)) + \frac{p-r}{p} K^{\frac{r}{p}}, \quad K > 0.$$

It's follows from (2.49), that

$$(2.54) \quad z^{\Delta_t}(t, s) = \int_{t_0}^t \int_{s_0}^s k^{\Delta_\tau}(t, s, \tau, \eta) \omega(u^r(\tau, \eta)) \Delta \eta \Delta \tau + \int_{s_0}^s k(t, s, \sigma_1(t), \eta) \omega(u^r(t, \eta)) \Delta \eta,$$

with

$$(2.55) \quad z^{\Delta_t}(t, s_0) = 0,$$

and

$$(2.56) \quad \begin{aligned} z^{\Delta_t \Delta_s}(t, s) = & \int_{t_0}^t \int_{s_0}^s k^{\Delta_\eta \Delta_\tau}(t, s, \tau, \eta) \omega(u^r(\tau, \eta)) \Delta \eta \Delta \tau + \\ & + \int_{t_0}^t k^{\Delta_\tau}(t, s, \tau, \sigma_2(s)) \omega(u^r(\tau, s)) \Delta \tau \\ & + \int_{s_0}^s k^{\Delta_\eta}(t, s, \sigma_1(t), \eta) \omega(u^r(t, \eta)) \Delta \eta \\ & + k(t, s, \sigma_1(t), \sigma_2(s)) \omega(u^r(t, s)). \end{aligned}$$

From (2.53) and the assumptions on ω , we have

$$(2.57) \quad \omega(u^r(t, s)) \leq \omega\left(\frac{r}{p} K^{\frac{r-p}{p}}\right) (\omega(a(t, s)) + \omega(b(t, s)) \omega(z(t, s))) + \omega\left(\frac{p-r}{p} K^{\frac{r}{p}}\right).$$

Combining (2.57), (2.56) and taking into account that $z(t, s)$ is monotonic non-decreasing in both variable t, s and $\omega(u)$ is nondecreasing for $u \in \mathbb{R}^+$, we obtain

$$(2.58) \quad z^{\Delta_t \Delta_s}(t, s) \leq A(t, s) + D(t, s)\omega(z(t, s)),$$

where $A(t, s)$ and $D(t, s)$ are defined as in (2.47) and (2.48) respectively, integrating (2.58) with respect to t and s , using (2.55) and (2.50), we get

$$(2.59) \quad z(t, s) \leq \int_{t_0}^t \int_{s_0}^s A(\tau, \eta) \Delta \eta \Delta \tau + \int_{t_0}^t \int_{s_0}^s D(\tau, \eta) \omega(z(\tau, \eta)) \Delta \eta \Delta \tau,$$

putting in (2.59)

$$C(t, s) = \int_{t_0}^t \int_{s_0}^s A(\tau, \eta) \Delta \eta \Delta \tau,$$

then (2.59) can be rewritten as follows

$$(2.60) \quad z(t, s) \leq C(t, s) + \int_{t_0}^t \int_{s_0}^s D(\tau, \eta) \omega(z(\tau, \eta)) \Delta \eta \Delta \tau.$$

The inequality (2.60) is similar to (2.1), letting $a(t, s) = C(t, s)$, $b(t, s) = 1$ and $f(t, s) = D(t, s)$ in Theorem 1, we obtain

$$(2.61) \quad z(t, s) \leq G^{-1} \left[G(C(t, s)) + \int_{t_0}^t \int_{s_0}^s D(\tau, \eta) \Delta \eta \Delta \tau \right].$$

Combining (2.61) and (2.51), we get

$$(2.62) \quad u^p(t, s) \leq a(t, s) + b(t, s)G^{-1} \left[G(C(t, s)) + \int_{t_0}^t \int_{s_0}^s D(\tau, \eta) \Delta \eta \Delta \tau \right],$$

thus (2.62) implies (2.46). □

Remark 3. Letting $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $t_0 = s_0 = 0$, $r = 1$, we easily see that Theorem 4 reduced to Theorem 3.2 in [7].

Remark 4. If we take $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, $t_0 = s_0 = 0$, $p = r = 1$ and $b(x, y) = 1$, we obtain (b_2) of Theorem 2 in [6].

3. APPLICATION

In this section we give an application of Theorem 1. We Consider the following partial dynamic equation on time scales

$$(3.1) \quad (u(t, s))^{\Delta_t \Delta_s} = F(t, s, u(t, s)) \quad \text{for } (t, s) \in \Omega$$

with the initial boundary conditions

$$(3.2) \quad u(t, s_0) = \alpha(t), u(t_0, s) = \beta(s), u(t_0, s_0) = c$$

where $F : \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2 \times \mathbb{R} \rightarrow \mathbb{R}$ is right-dense continuous on $\tilde{\Omega}$ and continuous on $\mathbb{R}$, $\alpha : \tilde{\mathbb{T}}_1 \rightarrow \mathbb{R}$ and $\beta : \tilde{\mathbb{T}}_2 \rightarrow \mathbb{R}$ are right-dense continuous, $c \in \mathbb{R}$ is a constant.

Assume that

$$(3.3) \quad \begin{aligned} |F(t, s, u)| &\leq f(t, s)w(|u|), \\ |\alpha(t) + \beta(s) - c| &\leq a(t, s), \end{aligned}$$

where $a(t, s)$ is non negative function and nondecreasing in each variable $(t, s) \in \tilde{\Omega}$. $f(t, s)$ is nonnegative right-dense continuous function defined for $(t, s) \in \tilde{\Omega} = \tilde{\mathbb{T}}_1 \times \tilde{\mathbb{T}}_2$. And w be a nonnegative, nondecreasing continuous function for $x \in \mathbb{R}^+$, with $\omega(x) > 0$ for all $x > 0$, satisfying (1.2).

The solution of (3.1)-(3.2) satisfies :

$$(3.4) \quad u(t, s) = \alpha(t) + \beta(s) - c + \int_{t_0}^t \int_{s_0}^s F(\tau, \eta, u(\tau, \eta)) \Delta \eta \Delta \tau,$$

then by plugging (3.3) in (3.4), we have

$$(3.5) \quad |u(t, s)| \leq a(t, s) + \int_{t_0}^t \int_{s_0}^s f(\tau, \eta)w(|u(\tau, \eta)|) \Delta \eta \Delta \tau.$$

Applying Theorem 1, we obtain the bound on the solution of (3.1)-(3.2).

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