

SHADOWING PROPERTY ON FINITELY GENERATED GROUP ACTIONS

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ABSTRACT. In this paper, we study shadowing property for actions of finitely generated groups and show that conjugacy preserves shadowing property and in the actions with shadowing property, topologically transitive is equivalent to chain transitive and the set of non-wandering points is equal to the set of chain recurrent points. For actions of finitely generated free groups, we prove that on a compact metric space of non-zero dimension, every action with chain transitive and shadowing property is sensitive.

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1. INTRODUCTION

One of the main goal in dynamics is the understanding of the orbits structure. The notion of shadowing property plays an important role in the investigation of orbits structure. A dynamical system has a shadowing property if any sufficiently approximate orbit is close to some orbit. In [4], the author introduced the shadowing property for actions of finitely generated group, $\varphi : G \times X \longrightarrow X$, and showed that shadowing property has a more complicated nature. For example, in the case

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of finitely generated nilpotent group, if for some $g \in G$, $\varphi(g, \cdot) : X \rightarrow X$ has shadowing property and it is expansive, then $\varphi : G \times X \rightarrow X$ has shadowing property but in the case of finitely generated free group with at least two generators, if $\varphi : G \times X \rightarrow X$ has shadowing property then for every $g \in G$, $\varphi(g, \cdot) : X \rightarrow X$ has shadowing property, and if for some $g \in G$, $\varphi(g, \cdot) : X \rightarrow X$ is expansive then $\varphi : G \times X \rightarrow X$ has not shadowing property. This implies that if $\varphi : G \times X \rightarrow X$ is Anosov action (i.e for some $g \in G$, $\varphi(g, \cdot) : X \rightarrow X$ is Anosov, see [2]) then $\varphi : G \times X \rightarrow X$ has not shadowing property. In this paper we recall definitions of [1] and show that every non-wandering point is a chain recurrent point and in the actions with shadowing property, the set of non-wandering points equal to the set of chain recurrent points and topologically transitive equivalent to chain transitive. We also prove that on compact metric spaces of non-zero dimension, every action of finitely generated free group, with chain transitive and shadowing property is sensitive.

2. DEFINITIONS

A dynamical system in the present paper is a triple (G, X, φ) where G is a finitely generated group, (X, d) is a compact metric space and $\varphi : G \times X \rightarrow X$ is an action such that for every $g \in G$, $\varphi_g : X \rightarrow X$ is a homeomorphism. In the following, we recall definition of pseudo orbit of [4]:

Definition 2.1. Let S be a finite generating set of G . For $\delta > 0$, $f : G \rightarrow X$ is called δ -pseudo orbit whenever,

$$d(f(sg), \varphi(s, f(g))) < \delta, \forall s \in S, \forall g \in G.$$

It is clear that every orbit of x , $G(x) = \{\varphi(g, x) | g \in G\}$ is pseudo orbit. A dynamical system (G, X, φ) has shadowing property whenever for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ -pseudo orbit f , there exists $p \in X$ such that $d(f(g), \varphi_g(p)) < \epsilon$.

We say that $f : X \rightarrow X$ has expansivity if there is $c > 0$ such that if $x \neq y$ then $d(f^n(x), f^n(y)) \geq c$, for some $n \in \mathbb{Z}$. Similar to the proof of Theorem 2 in [4], we give another example for actions of finitely generated groups with shadowing property.

Proposition 2.2. *Let $G = \langle a, b \mid ba = a^2b \rangle$ be a solvable group and $\varphi : G \times \mathbb{R} \rightarrow \mathbb{R}$ generated by the maps $f_a(x) = x, f_b(x) = mx$. Then for $m > 1$, φ has shadowing property.*

Proof. It is easy to see that for $m > 1$, f_b has shadowing property and it is expansive. Choose an arbitrarily $\epsilon > 0$. Let c be the constant of expansivity and $2\eta = \min\{c, \epsilon\}$. For $\eta > 0$, there is $0 < \delta < \eta$ in the definition of shadowing property f_b . Let $F : G \rightarrow \mathbb{R}$ be δ -pseudo orbit for φ . For every element $p \in G$, the sequence $\{y_k\}_{k \in \mathbb{Z}}$ defined by $y_k = F(b^k p)$, is a δ -pseudo orbit for f_b . Thus, there is $x_p \in \mathbb{R}$ such that

$$|F(b^k p) - f_b^k(x_p)| = |y_k - f_b^k(x_p)| < \eta.$$

Claim. $x_{bp} = f_b(x_p), x_{ap} = x_p$.

Proof of the claim. Since $|F(b^k bp) - f_b^k(x_{bp})| < \eta$ and $|F(b^{k+1}p) - f_b^k(f_b(x_p))| < \eta$, we have

$$|f_b^k(x_{bp}) - f_b^k(f_b(x_p))| < 2\eta \leq c, \quad \forall k \in \mathbb{Z}.$$

Thus $x_{bp} = f_b(x_p)$. Since $\varphi_a(x) = x$, therefore $|F(ag) - F(g)| < \delta$, for every $g \in G$. From $ba = a^2b$, we have $b^k a = a^{2^k} b^k$, thus

$$|F(a^{2^k-i} b^k p) - F(a^{2^k-i-1} b^k p)| < \delta, \quad \text{for } i = 0, \dots, 2k$$

By Triangle inequality:

$$|F(a^{2^k-i} b^k p) - F(b^k p)| < \delta(2k)$$

From $F(a^{2^k} b^k p) = F(b^k ap)$, $|F(b^k ap) - f_b^k(x_{ap})| < \eta$ and $|F(b^k p) - f_b^k(x_p)| < \eta$, we have:

$$m^k |x_{ap} - x_p| < 2\eta + (2k)\delta, \quad \forall k \in \mathbb{Z}.$$

Since $m > 1$, $x_{ap} = x_p$. Therefore, we have proved that $x_{bp} = f_b(x_p)$ and $x_{ap} = f_a(x_p)$. This means that $x_p = \varphi(p, x_e)$. Thus $|F(p) - \varphi(p, x_e)| < \epsilon$.

This shows that φ has shadowing property □

Above proposition gives an example that φ has shadowing property but for $a \in G$, φ_a has not shadowing property. In the case of finitely free group actions we have:

Theorem 2.3. . *Let G be a finitely generated free group with at least two generators. Let φ be an action of G on X . If φ has shadowing property, then for any $g \in G$, $\varphi(g, \cdot) : X \rightarrow X$ has shadowing property.*

Proof. See [4]. □

Definition 2.4. Dynamical systems (G, X, φ) and (G, Y, ψ) are topologically conjugate whenever there is a homeomorphism $h : X \rightarrow Y$ such that $h(\varphi(g, x)) = \psi(g, h(x))$.

In discrete dynamical systems, conjugacy preserves shadowing property. In the following proposition we show that in the group actions conjugacy preserves shadowing property too:

Proposition 2.5. *Let (G, X, φ) and (G, Y, ψ) be topologically conjugate. Then φ has shadowing property if and only if ψ has shadowing property.*

Proof. Suppose φ has shadowing property. We show that ψ has shadowing property too. Since $h : X \rightarrow Y$ is homeomorphism, for $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(h(x), h(y)) < \epsilon$. For $\delta > 0$ there is $\eta > 0$ by the definition of shadowing property of φ , and because $h^{-1} : Y \rightarrow X$ is homeomorphism, for $\eta > 0$ there is $\xi > 0$ such that if $d(x, y) < \xi$ then $d(h^{-1}(x), h^{-1}(y)) < \eta$. Let $f : G \rightarrow Y$ be ξ -pseudo orbit for (G, Y, ψ) . We define $P : G \rightarrow X$ by $P(g) = h^{-1}(f(g))$. We claim that P is η -pseudo orbit for (G, X, φ) . To see it note that

$$\begin{aligned} d(P(sg), \varphi(s, P(g))) &= d(h^{-1}(f(sg)), h^{-1}(\psi(s, h(P(g)))) \\ &= d(h^{-1}(f(sg)), h^{-1}(\psi(s, f(g)))) \end{aligned}$$

since $d(f(sg), \psi(s, f(g))) < \xi$, we have $d(h^{-1}(f(sg)), h^{-1}(\psi(s, f(g)))) < \eta$ i.e. $d(P(sg), \varphi(s, P(g))) < \eta$. Since φ has shadowing property for η -pseudo orbit $P : G \rightarrow X$, there is $z_0 \in X$ such that $d(P(g), \varphi(g, z_0)) < \delta$. Let $h(z_0) = z$ then $d(f(g), \psi(g, z)) = d(h(P(g)), h\varphi(g, z_0)) < \epsilon$ i.e. $d(f(g), \psi(g, z)) < \epsilon$. Thus ψ has shadowing property. Similarly if ψ has shadowing property, then φ has shadowing property too. □

A dynamical system (X, G, φ) is topologically transitive whenever for any non-empty open sets $U, V \subseteq X$, there is a $g \in G$ such that $\varphi_g(U) \cap V \neq \emptyset$.

Lemma 2.6. *In a dynamical system (G, X, φ) the following are equivalent:*

- (1) φ is topologically transitive,
- (2) $\{x \in X | \overline{Gx} = X\}$ is a dense subset of X ,
- (3) $\overline{Gx} = X$, for some $x \in X$.

Proof. We show that $1 \Rightarrow 2$.

Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable basis for the compact metric space (X, d) . For $x \in X$, we have:

$$\begin{aligned}
 \overline{Gx} \neq X &\Leftrightarrow Gx \cap U_n = \emptyset \text{ for some } n \\
 &\Leftrightarrow \forall g \in G, gx \in X \setminus U_n \text{ for some } n \\
 &\Leftrightarrow x \in \bigcap_{g \in G} g^{-1}X \setminus U_n \text{ for some } n \\
 &\Leftrightarrow x \in \bigcup_{n=1}^{\infty} \bigcap_{g \in G} g^{-1}X \setminus U_n
 \end{aligned}$$

Since φ is topologically transitive, it is easy to see that GU_n is dense in X , thus $\bigcap_{g \in G} X \setminus g^{-1}U_n$ is nowhere dense. So $\bigcup_{n=1}^{\infty} \bigcap_{g \in G} g^{-1}U_n$ is a first category. Since X is a compact metric space, $\{x \in X | \overline{Gx} = X\}$ is dense in X .

The proofs of $2 \Rightarrow 3$ and $3 \Rightarrow 1$ are clear. $\square$

Definition 2.7. A dynamical system (G, X, φ) is chain transitive whenever for every $x, y \in X$ and $\delta > 0$, there exists δ -pseudo orbit $f : G \rightarrow X$ such that $f(g_1) = x$ and $f(g_2) = y$ for some $g_1, g_2 \in G$.

It is clear that if (G, X, φ) is chain transitive then for $\delta > 0$ and every $x, y \in X$ there is δ -pseudo orbit $f : G \rightarrow X$ such that $f(e) = x, f(g_1) = y$ for some $g_1 \in G$.

Proposition 2.8. In a dynamical system (G, X, φ) , topologically transitive implies chain transitivity.

Proof. Fix $\epsilon > 0$ and $x, y \in X$, for $\epsilon > 0$ there exists $0 < \delta < \frac{\epsilon}{2}$ such that for every $s \in S$ if $d(a, b) < \frac{\delta}{2}$ then $d(\varphi(s, a), \varphi(s, b)) < \frac{\epsilon}{2}$. By topological transitivity, there is $p \in X$ such that $\varphi(g_1, p) \in B(\frac{\delta}{2}, x)$ and $\varphi(g_2, p) \in B(\frac{\delta}{2}, y)$ for some $g_1, g_2 \in G$. Put $z_1 := \varphi(g_1, p)$ and $z_2 := \varphi(g_2, p)$. For simplicity, assume that $z_1 \neq z_2$. Define:

- $J_1 = \{g \in G | \varphi(g, p) = z_1\}$,
- $J_2 = \{g \in G | \varphi(g, p) = z_2\}$,

- $J_3 = \{g \in G \mid \varphi(g, p) \notin \{z_1, z_2\}\}$

Define $f : G \longrightarrow X$ in the following way:

- $f(g) := x, \forall g \in J_1,$
- $f(g) := y, \forall g \in J_2,$
- $f(g) := \varphi(g, p), \forall g \in J_3$

Claim. $f : G \longrightarrow X$ is a ϵ -pseudo-orbit

Proof of claim.

Case 1. $g \in J_1$ and $sg \in J_2$. It means that $\varphi(s, z_1) = z_2$. In this case $f(g) = x$, $f(sg) = y$. Note that $d(f(sg), \varphi(s, f(g))) = d(y, \varphi(s, x)) \leq d(y, z_2) + d(z_2, \varphi(s, z_1)) + d(\varphi(s, z_1), \varphi(s, x)) < \epsilon$. Thus $d(f(sg), \varphi(s, f(g))) < \epsilon$

Case 2. $g \in J_1$ and $sg \in J_3$. In this case $f(g) = x$ and $f(sg) = \varphi(sg, p) = \varphi(s, \varphi(g, p)) = \varphi(s, z_1)$.

$d(f(sg), \varphi(s, f(g))) = d(\varphi(s, x), \varphi(s, z_1)) < \epsilon$.

Case 3. $g \in J_2$ and $sg \in J_3$. This case is similar to the Case 2.

Case 4. $g \in J_3, sg \in J_1$, in this case $f(g) = \varphi(g, p), f(sg) = x$. Note that by definition of J_1 , $\varphi(s, \varphi(g, p)) = z_1$. Indeed, $d(f(sg), \varphi(s, f(g))) = d(x, z_1) \leq \frac{\delta}{2} < \epsilon$.

Case 5. $g \in J_3, sg \in J_2$. This case is completely similar to the Case 4.

Case 6. $g \in J_1, sg \in J_1$. In this case $f(g) = f(sg) = x$. By definition of J_1 , $\varphi(s, z_1) = z_1$. We have $d(f(sg), \varphi(s, f(g))) = d(x, \varphi(s, x)) \leq d(x, z_1) + d(z_1, \varphi(s, z_1)) + d(\varphi(s, z_1), \varphi(s, x)) < \epsilon$

Case 7. $g \in J_2, sg \in J_2$. This case is completely similar to the Case 6.

Thus $f : G \longrightarrow X$ is ϵ -pseudo trajectory joining x and y . $\square$

Lemma 2.9. *If φ has shadowing property then chain transitivity implies topologically transitive.*

Proof. Let U, V be none empty open subsets of X . For $x \in X$ and $y \in Y$, suppose that $B_\epsilon(x) \subseteq U$ and $B_\epsilon(y) \subseteq V$. For $\epsilon > 0$ there is $\delta > 0$ in the definition of shadowing property of φ . Suppose that $f : G \rightarrow X$ is δ -pseudo orbit, $f(g_0) = x$ and $f(g_1) = y$, by shadowing property of φ , there is $z \in X$ such that $d(f(g), \varphi(g, z)) < \epsilon$. For $g = g_0, g_1$ we have $d(y, \varphi(g_0, z)) < \epsilon$ and $d(x, \varphi(g_1, z)) < \epsilon$, this means that $\varphi(g_1 g_0^{-1}, V) \cap U \neq \emptyset$ i.e. φ is topologically transitive. $\square$

A point $x \in X$ is called a non-wandering point whenever for every non-empty open set U in X , there is $g \in G$ such that $\varphi(g, U) \cap U \neq \emptyset$ and the set of non-wandering points denotes by $\Omega(\varphi)$. We also denote by $\mathcal{R}(f)$, the set of all $x \in X$ such that for every $\delta > 0$, there is δ -pseudo orbit from x to x (x is called to be recurrent point).

Lemma 2.10. *In (G, X, φ) , $\Omega(\varphi) \subseteq \mathcal{R}(\varphi)$.*

Proof. For $\epsilon > 0$ there is $0 < \delta < \frac{\epsilon}{4}$ such that if $d(x, y) < \delta$ then $d(\varphi(s, x), \varphi(s, y)) < \frac{\epsilon}{2}$ for all $s \in S$. Let $x \in \Omega(\varphi)$ thus for $\delta > 0$, there are $y \in X$ and $h \in G$ such that $d(x, y) < \delta$ and $d(x, \varphi(h, y)) < \delta$. Define:

- $J_1 = \{g \in G | \varphi(g, y) = \varphi(h, y)\},$
- $J_2 = \{g \in G | \varphi(g, y) = y\},$
- $J_3 = \{g \in G | \varphi(g, y) \notin \{y, \varphi(h, y)\}\}.$

Define $f : G \longrightarrow X$ by the following:

- $f(g) := x, \forall g \in J_1 \cup J_2,$
- $f(g) := \varphi(g, y), \forall g \in J_3.$

Claim. $f : G \longrightarrow X$ is ϵ -pseudo orbit.

Case1. $sg, g \in J_1$

We have $\varphi(s, \varphi(h, y)) = \varphi(h, y)$. Thus, $d(f(sg), \varphi(s, f(g))) = d(x, \varphi(s, x)) \leq d(x, \varphi(h, y)) + d(\varphi(h, y), \varphi(s, x)) < \epsilon$.

Case2. $g \in J_1, sg \in J_2$. In this case, $\varphi(s, \varphi(h, y)) = y$. Thus, $d(f(sg), \varphi(s, f(g))) = d(x, \varphi(s, x)) \leq d(x, \varphi(h, y)) + d(\varphi(h, y), \varphi(s, \varphi(h, y))) + d(\varphi(s, \varphi(h, y)), \varphi(s, x)) < \epsilon$

Case3. $g \in J_1, sg \in J_3$. In this case, $\varphi(sg, y) = \varphi(s, \varphi(h, y))$. Thus

$$d(f(sg), \varphi(s, f(g))) = d(\varphi(s, \varphi(h, y)), \varphi(s, x)) < \epsilon.$$

Case4. $g \in J_2, sg \in J_3$. In this case, $\varphi(g, y) = y$ and $\varphi(sg, y) = \varphi(s, y)$. Thus we have $d(f(sg), \varphi(s, f(g))) = d(\varphi(sg, y), \varphi(s, x)) = d(\varphi(s, y), \varphi(s, x)) < \epsilon$.

Case5. $g, sg \in J_2$. In this case, $\varphi(g, y) = \varphi(s, y) = y$. Thus $d(f(sg), \varphi(s, f(g))) = d(x, \varphi(s, x)) \leq d(x, y) + d(\varphi(s, y), \varphi(s, x)) < \epsilon$.

Case6. $g \in J_2, sg \in J_1$. In this case, $\varphi(g, y) = y, \varphi(s, y) = \varphi(h, y)$. Thus, $d(f(sg), \varphi(s, f(g))) = d(x, \varphi(s, x)) \leq d(x, \varphi(h, y)) + d(\varphi(s, y), \varphi(s, x)) < \epsilon$.

Case7. $g \in J_3, sg \in J_1$. In this case, $\varphi(sg, y) = \varphi(h, y)$. So $d(f(sg), \varphi(s, f(g))) = d(x, \varphi(sg, y)) = d(x, \varphi(h, y)) < \epsilon$.

Case8. $g \in J_3, sg \in J_2$. In this case $d(f(sg), \varphi(s, f(g))) = d(x, y) < \epsilon$.

Case9. $g, sg \in J_3$. In this Case $d(f(sg), \varphi(s, f(g))) = 0$.

Thus $f : G \rightarrow X$ is ϵ -pseudo orbit and $x \in \mathcal{R}(\varphi)$.

□

Proposition 2.11. *If φ has shadowing property then $\Omega(\varphi) = \mathcal{R}(\varphi)$.*

Proof. By Lemma 2.10, $\Omega(\varphi) \subseteq \mathcal{R}(\varphi)$. We show that $\mathcal{R}(\varphi) \subseteq \Omega(\varphi)$. Suppose $x \in \mathcal{R}(\varphi)$ and $\epsilon > 0$, let δ be in the definition of shadowing property φ . For $\delta > 0$ and $x \in \mathcal{R}(\varphi)$, there is a δ -pseudo orbit $f : G \rightarrow X$ such that $f(e) = f(h) = x$, thus by shadowing property there exists $z \in X$ such that $d(f(g), \varphi(g, z)) < \epsilon$. This means that $d(x, z) < \epsilon$ and $d(x, \varphi(h, z)) < \epsilon$. Thus $\varphi(h, B(\epsilon, x)) \cap B(\epsilon, x) \neq \emptyset$. Therefore $x \in \Omega(\varphi)$. □

If every point of X has dense orbit then (G, X, φ) is called minimal dynamical system.

Definition 2.12. A point $x \in X$ is called an equicontinuous point if for $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in X$, $d(x, y) < \delta$ implies $d(\varphi_g(x), \varphi_g(y)) < \epsilon$ for all $g \in G$. We denote by $Eq(X)$ the set of equicontinuous points in the dynamical system (G, X, φ) .

A system is said to be equicontinuous if every point of X is equicontinuous point i.e. $Eq(X) = X$. A point which is not an equicontinuous point will be called a sensitive point, thus x is sensitive point if there are $\epsilon > 0$, $\{x_n\} \subseteq X$ and $\{g_n\} \subseteq G$ such that $x_n \rightarrow x$ and $d(\varphi(g_n, x_n), \varphi(g_n, x)) \geq \epsilon$, in this case x is called ϵ -sensitive point.

Definition 2.13. A dynamical system (G, X, φ) is said to have sensitive dependence on initial conditions (or is sensitivity) if there is some $\epsilon > 0$ such that for all $x \in X$ and for every neighborhood U of x there exists $y \in U$ and $g \in G$ such that $d(\varphi_g(x), \varphi_g(y)) > \epsilon$.

It is clear that the assumption that a system has sensitive dependence on initial condition is stronger than the assertion that every point is sensitive. Also, it is clear that if two dynamical systems (G, X, φ) and (G, Y, ψ) are topologically conjugate with the homeomorphism $h : X \rightarrow Y$, then x in (G, X, φ) is a transitive point,

equicontinuous point, sensitive point if and only if $h(x)$ in (G, Y, ψ) is a transitive point, equicontinuous point, sensitive point respectively.

Lemma 2.14. *Let A_ϵ denote the set of ϵ -sensitive points, then $\overline{A_\epsilon} \subseteq A_{\frac{\epsilon}{2}}$.*

Proof. Suppose $x \in \overline{A_\epsilon}$, thus there is $\{x_n\} \subseteq A_\epsilon$ such that $x_n \rightarrow x$. Since $x_n \in A_\epsilon$, there are $\{y_k^n\} \subseteq X$ and $\{g_k^n\} \subseteq G$ such that $\lim_{k \rightarrow \infty} y_k^n = x_n$ and $d(\varphi(g_k^n, x_n), \varphi(g_k^n, y_k^n)) \geq \epsilon$. Thus:
 $d(\varphi(g_k^n, x_n), \varphi(g_k^n, x)) \geq \frac{\epsilon}{2}$ or $d(\varphi(g_k^n, x), \varphi(g_k^n, y_k^n)) \geq \frac{\epsilon}{2}$
 (otherwise $d(\varphi(g_k^n, y_k^n), \varphi(g_k^n, x_n)) < \epsilon$, which is a contradiction). Thus $x \in A_{\frac{\epsilon}{2}}$, i.e. $\overline{A_\epsilon} \subseteq A_{\frac{\epsilon}{2}}$. $\square$

In (G, X, φ) , it is easy to see that if $x \in X$ is ϵ -sensitive point, then for every $g \in G$, $\varphi(g, x)$ is ϵ -sensitive point. Thus if x is ϵ -sensitive point, then $\overline{Gx}$ is $\frac{\epsilon}{2}$ -sensitive point. This means that:

Corollary 2.15. *If (G, X, φ) is minimal and contains a sensitive point then there exists $\epsilon > 0$ such that every point is ϵ -sensitive point.*

Let $Trans(X) = \{x \in X | \overline{Gx} = X\}$.

Proposition 2.16. *If $Eq(X) \neq \emptyset$ then $Trans(X) \subseteq Eq(X)$.*

Proof. Let $x \in Trans(X)$, i.e. $\overline{Gx} = X$. For $y \in Eq(X)$ and $\epsilon > 0$ there is $\delta > 0$ such that if $d(z, y) < \delta$ then $d(\varphi(g, z), \varphi(g, y)) < \epsilon$, for every $g \in G$. Since $\overline{Gx} = X$, there is $h \in G$ such that $\varphi(h, x) \in B_{\frac{\delta}{2}}(y)$. Since $\varphi_h : X \rightarrow X$ is a homeomorphism, for $\frac{\delta}{2} > 0$ there is $0 < \delta' < \frac{\delta}{2}$ such that $d(w, x) < \delta'$ implies that $d(\varphi_h(w), \varphi_h(x)) < \frac{\delta}{2}$. Therefore, $\varphi_h(w) \in B_\delta(y)$. Since $y \in Eq(X)$, for every $g \in G$, $d(\varphi(gh, w), \varphi(g, y)) < \frac{\epsilon}{2}$. On the other hand for $g \in G$, $d(\varphi(gh, x), \varphi(g, y)) < \frac{\epsilon}{2}$. This means that $d_{\sup}(w, x) < \epsilon$, i.e. $x \in Eq(X)$. $\square$

The condition $Eq(X) \neq \emptyset$ in the above proposition is essential, because there exist some systems that are topologically transitive and sensitive dependence i.e. $Eq(X) = \emptyset$ and $Trans(X) \neq \emptyset$. If $\overline{Eq(X)} = X$ then φ is called almost equicontinuous. By Lemma 2.6 and Proposition 2.16, every topologically transitive is either sensitive or almost equicontinuous and by Corollary 2.15 every minimal dynamical system is either sensitive or equicontinuous.

Theorem 2.17. *If $\varphi : G \times X \longrightarrow X$, is chain transitive with shadowing property such that $Eq(X) \neq \emptyset$, then φ is minimal and equicontinuous.*

Proof. Suppose φ is not minimal, there is $x \in X$ such that $\overline{Gx} \neq X$. It is easy to see that for every $y \in X \setminus \overline{Gx}$, $\overline{Gx} \cap \overline{Gy} = \emptyset$. Suppose $d(\overline{Gx}, \overline{Gy}) > 3\eta > 0$. By Lemma 2.9, φ is topologically transitive. Thus from $Eq(X) \neq \emptyset$ we have $\overline{Eq(X)} = X$. Choose $h \in Eq(X)$ such that $d(h, x) < \frac{\eta}{2}$. There is $0 < \epsilon < \frac{\eta}{2}$ such that If $d(y, h) < \epsilon$ then $d(\varphi(g, y), (\varphi(g, h))) < \frac{\eta}{2}$. For $\epsilon > 0$ there is $\delta > 0$ in the definition of shadowing property φ . Let $f : G \rightarrow X$ be δ -chain from h to y such that $f(e) = h$ and $f(g_0) = y$.

Since φ has shadowing property, there is $z \in X$ such that $d(f(g), \varphi(g, z)) < \epsilon$. Since $f(e) = h$ and $d(h, z) < \epsilon$, we have $d(x, z) < \eta$, This means that $d(Gz, Gy) > 2\eta$. But from $f(g_0) = y$ we have $d(y, \varphi(g_0, z)) < \epsilon$, which is a contradiction, i.e. $Trans(X) = X$ implies that $Eq(X) = X$. So we have proved that φ is minimal and equicontinuous. $\square$

It is clear that if (G, X, φ) is equicontinuous then for every $h \in G$, $\varphi_h : X \rightarrow X$ is equicontinuous and by Theorem 2.3, if G is a finitely generated free group with at least two generators and (G, X, φ) has shadowing property then for every $h \in G$, $\varphi_h : X \rightarrow X$ has shadowing property. By Theorem 4 in [3] any equicontinuous surjective dynamical system (X, f) has the shadowing property if and only if X is totally disconnected, thus we have:

Proposition 2.18. *Let (X, d) be a compact metric space, G be a finitely generated free group with at least two generators and (G, X, φ) is equicontinuous with shadowing property. Then X is totally disconnected.*

In [4] Authors present the following conjecture :

Conjecture 2.19. Any uniformly continuous action of a finitely generated free group with at least two generators on a manifold does not have the shadowing property.

By Proposition 2.18, we give a positive answer to the above conjecture in the case of equicontinuous action:

Corollary 2.20. *Let G be a finitely generated free group with at least two generators and $\dim(X) \geq 1$ then equicontinuous action, (G, X, φ) , has not shadowing property.*

A compact metric space (X, d) is totally disconnected if and only if it has zero dimension. By Theorem 2.17 and Proposition 2.18 we have:

Corollary 2.21. *Let (X, d) be compact manifold with $\dim(X) \geq 1$ or connected compact metric space and (G, X, φ) is chain transitive with shadowing property then (G, X, φ) has sensitive dependence.*

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REFERENCES

- [1] Joseph Auslander, Minimal flows and their extensions, North-Holland mathematics studies, 153(1930).
- [2] Steven Hurder, Affine Anosov actions. The Michigan Mathematical Journal, Vol. 40, Issue 3 , 419-618 (1993)
- [3] Subrahmonian Moothathu, Implications of pseudo-orbit tracing property for continuous maps on compacta, Topology and its Applications 158, 2232-2239(2011)
- [4] Alexey V. Osipov, Sergey B. Tikhomirov, Shadowing for actions of some finitely generated groups, preprint (arxiv.org/abs/1305.7394).