ON SMARANDACHE CURVES LYING IN LIGHTCONE IN MINKOWSKI 3-SPACE

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Abstract. In this paper we define the spacelike and the null lightcone Smarandache curves of a spacelike lightcone curve α according to the lightcone Frenet frame of a α in Minkowski 3-space. We obtain the lightcone curvature and the expressions for the lightcone Frenet frame's vectors of a spacelike Smarandache curve of α. We prove that if the null straight line is the lightcone Smarandache curve of α, then α has non-zero constant lightcone curvature. Finally, we give some examples of lightcone Smarandache curves.

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1. Introduction

It is known that a Smarandache geometry is a geometry which has at least one Smarandachely denied axiom ([2]). An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the Theory of relativity and the Parallel Universes. Smarandache curves are the objects of Smarandache geometry. By definition, if the position vector of a curve β is composed by the Frenet frame's vectors of another curve α, then the curve β is called a Smarandache curve ([9]). Special Smarandache curves in the Euclidean and Minkowski spaces are studied by some authors ([1, 7, 8, 3, 4]). In this paper we define the spacelike and the null lightcone Smarandache curves of a spacelike lightcone curve α according to the lightcone Frenet frame of a α in Minkowski 3-space. We obtain the lightcone curvature and the expressions for the lightcone Frenet frame's vectors of a spacelike Smarandache curve of α. We prove that if the null straight line is the lightcone Smarandache curve of α, then α has non-zero constant lightcone curvature. Finally, we give some examples of lightcone Smarandache curves in Minkowski 3-space.

2. Basic Concepts

The Minkowski 3-space $\mathbb{R}^3_1$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the standard flat metric given by

$$\langle \cdot\, , \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2, \tag{1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}^3_1$. Since g is an indefinite metric, recall that a non-zero vector $\vec{x} \in \mathbb{R}^3_1$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle \vec{x},\ \vec{x} \rangle > 0$, timelike if $\langle \vec{x},\ \vec{x} \rangle < 0$ and null (lightlike) if $\langle \vec{x},\ \vec{x} \rangle = 0$. In particular, the norm (length) of a vector $\vec{x} \in \mathbb{R}^3_1$ is given by $\|\vec{x}\| = \sqrt{|\langle \vec{x},\ \vec{x} \rangle|}$ and two vectors $\vec{x}$ and $\vec{y}$ are said to be orthogonal, if $\langle \vec{x},\ \vec{y} \rangle = 0$. Next, recall that an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}^3_1$, can locally be *spacelike, timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike, timelike* or *null (lightlike)* for all $s \in I$([6]). A spacelike or a timelike curve α is parameterized by arclength parameter s if $\langle \alpha'(s),\ \alpha'(s) \rangle = 1$ or $\langle \alpha'(s),\ \alpha'(s) \rangle = -1$ respectively. For any two vectors $\vec{x} = (x_1,\ x_{2,}\ x_3)$ and $\vec{y} = (y_1,\ y_2,\ y_3)$ in the space $\mathbb{R}^3_1$, the *pseudo vector product* of $\vec{x}$ and $\vec{y}$ is defined by

$$\begin{aligned}\vec{x}\times\vec{y} &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \\ &= (-x_2y_3+x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1). \end{aligned} \tag{2}$$

Lemma 2.1. *Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be the vectors in $\mathbb{R}^3_1$. Then:*

$$\begin{aligned} (i) \qquad \langle \vec{x}\times\vec{y},\ \vec{z}\rangle &= \det(\vec{x},\ \vec{y},\ \vec{z}), \\ (ii) \qquad \vec{x}\times(\vec{y}\times\vec{z}) &= -\langle \vec{x},\ \vec{z}\rangle\,\vec{y} \ +\ \langle \vec{x},\ \vec{y}\rangle\,\vec{z}, \\ (iii) \quad \langle \vec{x}\times\vec{y},\ \vec{x}\times\vec{y}\rangle &= -\langle \vec{x},\ \vec{x}\rangle\langle \vec{y},\ \vec{y}\rangle \ +\ \langle \vec{x},\ \vec{y}\rangle^2, \end{aligned}$$

where $\times$ is the pseudo vector product in $\mathbb{R}^3_1$.

Lemma 2.2. *In the Minkowski 3-space $\mathbb{R}^3_1$, following properties are satisfied ([6]):*
(i) two timelike vectors are never orthogonal;
(ii) two null vectors are orthogonal if and only if they are linearly dependent;
(iii) timelike vector is never orthogonal to a null vector.

The *lightcone* in the Minkowski 3-space $\mathbb{R}^3_1$ is a quadric defined by

$$C = \left\{ \vec{x}\in\mathbb{R}^3_1 \middle|\, \langle \vec{x},\vec{x}\rangle = 0 \right\}.$$

Let $\alpha : I\subset\mathbb{R}\longrightarrow C$ be a curve lying fully in the lightcone C in $\mathbb{R}^3_1$. Then α is a spacelike curve or a null straight line.

Assume that α is a spacelike lightcone curve. Then there exists the lightcone Frenet frame $\{\alpha, T, \xi\}$ along the curve α, such that the following conditions hold:

$$\langle\alpha,\alpha\rangle = \langle\xi,\xi\rangle = \langle\alpha,T\rangle = \langle T,\xi\rangle = 0, \quad \langle\alpha,\xi\rangle = \langle T,T\rangle = 1. \tag{3}$$

It follows that

$$\alpha\times T = \alpha, \quad T\times\xi = \xi, \quad \xi\times\alpha = T. \tag{4}$$

The Frenet formulae of α, according to the lightcone Frenet frame read [5]

$$\begin{bmatrix} \alpha' \\ T' \\ \xi' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ k & 0 & -1 \\ 0 & -k & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}. \tag{5}$$

where $k(s) = \langle T'(s), \xi(s)\rangle \neq 0$ is the lightcone curvature of α and s is the arclength parameter of α. If α is a null straight line lying in lightcone, then it hasn't got lightcone Frenet frame, so we exclude this case in this paper.

3. Lightcone Smarandache Curves in Minkowski 3-space

In this section, we consider the spacelike lightcone curve α and define its lightcone Smarandache curve according to the lightcone Frenet frame $\{\alpha, T, \xi\}$ of α in Minkowski 3-space.

Definition 3.1. *Let $\alpha : I \subset R \mapsto C$ be a unit speed spacelike curve lying fully in lightcone C with the lightcone Frenet frame $\{\alpha, T, \xi\}$. The lightcone $\alpha T\xi$-Smarandache curve β of α is defined by*

$$\beta(s) = a\alpha(s) + bT(s) + c\xi(s), \tag{6}$$

where s is arclength parameter of α, $a, b, c \in R_0$ and $b^2 + 2ac = 0$.

Note that the lightcone $\alpha T\xi$-Smarandache curve β of α can be a spacelike or a null curve.

In the sequel, we obtain the lightcone curvature and the expressions for the ligthcone Frenet frame's vectors of a spacelike lightcone $\alpha T\xi$-Smarandache curve β of α. We also prove that if β is a null lightcone $\alpha T\xi$-Smarandache curve of α, then α has non-zero constant lightcone curvature.

Finally, we give some examples of lightcone Smarandache curves in Minkowski 3-space.

Theorem 3.1. *Let $\alpha : I \subset R \mapsto C$ be a unit speed spacelike curve lying fully in lightcone C with the lightcone Frenet frame $\{\alpha,\ T,\ \xi\}$ and the lightcone curvature $k(s) \neq 0$. If $\beta : I \subset R \mapsto C$ is a spacelike lightcone $\alpha T\xi$-Smarandache curve of α, then its frame $\{\beta,\ T_\beta,\ \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} a & b & c \\ \frac{bk(s)}{|a+ck(s)|} & \frac{a-ck(s)}{|a+ck(s)|} & -\frac{b}{|a+ck(s)|} \\ \frac{ck^2(s)}{(a+ck(s))^2} & -\frac{bk(s)}{(a+ck(s))^2} & \frac{a}{(a+ck(s))^2} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix},$$

and the corresponding lightcone curvature k_β reads

$$k_\beta(s) = \frac{1}{\varepsilon_0(a + ck(s))^3}[a\varepsilon_1(s) - bk(s)\varepsilon_2(s) + ck^2(s)\varepsilon_3(s)]$$

where

$$\begin{aligned}
\varepsilon_0 &= sgn\{a+ck(s)\} = \pm 1,\\
\varepsilon_1(s) &= \frac{abk'^2 - c^2k^2(s))}{\varepsilon_0(a+ck(s))^2},\\
\varepsilon_2(s) &= \frac{2bk(s)(a+ck(s)) - 2ack'(s)}{\varepsilon_0(a+ck(s))^2},\\
\varepsilon_3(s) &= \frac{bck'^2 + c^2k^2(s)}{\varepsilon_0(a+ck(s))^2},
\end{aligned}$$

$a, b, c \in R_0$, $b^2 + 2ac = 0$ *and* $a + ck(s) \neq 0$ *for all* s.

Proof. Assume that β is a spacelike lightcone $\alpha T\xi$-Smarandache curve of α. Then β has parameter equation

$$\beta(s) = a\alpha(s) + bT(s) + c\xi(s), \tag{7}$$

where $a, b, c \in R_0$, $b^2 + 2ac = 0$ and s is the arclength parameter of α. Denote by s^* the arclength parameter of β. Differentiating the equation (7) with respect to s and using (5) we obtain

$$\beta'(s) = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = bk(s)\alpha(s) + (a - ck(s))\,T(s) - b\xi(s). \tag{8}$$

Since $T_\beta = d\beta/ds^\star$ and

$$\frac{ds^*}{ds} = ||\beta'(s)|| = \sqrt{(a-ck(s))^2 - 2b^2k(s)} = |a + ck(s)|, \tag{9}$$

relation (8) becomes

$$T_\beta(s) = \frac{1}{|a+ck(s)|}\left(bk(s)\alpha(s) + (a-ck(s))\,T(s) - b\xi(s)\right) \tag{10}$$

where $|a + ck(s)| \neq 0$ for all s.

Next we will determine the vector $\xi_\beta(s)$ of the lightcone Frenet frame $\{\beta, T_\beta, \xi_\beta\}$, such that it satisfies the following conditions:

$$\langle \xi_\beta, \xi_\beta\rangle = 0, \quad \langle \xi_\beta, T_\beta\rangle = 0, \quad \langle \beta, \xi_\beta\rangle = 1. \tag{11}$$

With respect to the lightcone Frenet frame $\{\alpha, T, \xi\}$, the vector ξ_β can be decomposed as

$$\xi_\beta(s) = m(s)\alpha(s) + n(s)T(s) + p(s)\xi(s), \tag{12}$$

where $m(s), n(s)$ and $p(s)$ are some scalar functions in arclength parameter s of α. Relations (3), (7), (10), (11) and (12) imply the system of equations

$$\begin{aligned} n^2(s) + 2m(s)p(s) &= 0, \\ ap(s) + bn(s) + cm(s) &= 1, \\ bp(s)k(s) + (a - ck(s))n(s) - bm(s) &= 0. \end{aligned}$$

A straightforward calculation shows that the solution of the previous system of equations is given by

$$m(s) = \frac{ck^2(s)}{(a + ck(s))^2} \quad n(s) = -\frac{bk(s)}{(a + ck(s))^2}, \quad p(s) = \frac{a}{(a + ck(s))^2}.$$

Substituting this in (12), we obtain

$$\xi_\beta(s) = \frac{ck^2(s)}{(a + ck(s))^2}\alpha(s) - \frac{bk(s)}{(a + ck(s))^2}T(s) + \frac{a}{(a + ck(s))^2}\xi(s). \tag{13}$$

Differentiating the equation (10) with respect to s and using (9), it follows that

$$\frac{dT_\beta}{ds^*}\frac{ds^*}{ds} = T'_\beta|a + ck(s)| = \varepsilon_1(s)\alpha(s) + \varepsilon_2(s)T(s) + \varepsilon_3(s)\xi(s), \tag{14}$$

where

$$\begin{aligned} \varepsilon_1(s) &= \frac{abk'^2 - c^2k^2(s))}{\varepsilon_0(a + ck(s))^2}, \\ \varepsilon_2(s) &= \frac{2bk(s)(a + ck(s)) - 2ack'(s)}{\varepsilon_0(a + ck(s))^2}, \\ \varepsilon_3(s) &= \frac{bck'^2 + c^2k^2(s)}{\varepsilon_0(a + ck(s))^2}, \end{aligned}$$

Finally, from (13) and (14) we obtain that the lightcone curvature k_β of β is given by

$$k_\beta(s) = \langle T'_\beta(s), \xi_\beta(s)\rangle = \frac{1}{\varepsilon_0(a + ck(s))^3}(a\varepsilon_1(s) - bk(s)\varepsilon_2(s) + ck^2(s)\varepsilon_3(s)), \tag{15}$$

where $\varepsilon_0 = sgn\{a + ck(s)\} = \pm 1$. □

Theorem 3.2. *Let $\alpha : I \subset R \mapsto C$ be a unit speed spacelike curve lying fully in lightcone C with the lightcone Frenet frame $\{\alpha, T, \xi\}$ and the lightcone curvature $k(s) \neq 0$. If γ is a null lightcone $\alpha T\xi$-Smarandache curve of α, then α has non-zero constant lightcone curvature given by*

$$k(s) = \frac{b^2}{2c^2},$$

where $a, b, c \in R_0$ and $b^2 + 2ac = 0$.

Proof. Assume that γ is a null lightcone $\alpha T\xi$-Smarandache curve of α. Then γ is given by

$$\gamma(s) = a\alpha(s) + bT(s) + c\xi(s),$$

where s is arclength parameter of α, $a, b, c \in R_0$ and $b^2 + 2ac = 0$. Differentiating the previous equation with respect to s and using (5), we obtain

$$T_\gamma(s) = \gamma'(s) = bk(s)\alpha(s) + (a - ck(s))T(s) - b\xi(s).$$

The condition $\langle T_\gamma(s), T_\gamma(s)\rangle = 0$ implies

$$(a - ck(s))^2 - 2b^2 k(s) = 0. \tag{16}$$

By assumption there holds $b^2 + 2ac = 0$ and thus $b^4 - 4a^2c^2 = 0$. By using the last equation and (16), we find

$$k(s) = \frac{b^2 \pm \sqrt{b^4 - 4a^2c^2}}{2c^2} = \frac{b^2}{2c^2},$$

which proves the Theorem. $\square$

4. Examples

Example 4.1. *Let α be a unit speed spacelike curve lying in lightcone in Minkowski 3-space with parameter equation (see Figure 1)*

$$\alpha(s) = (2\cosh(\frac{s}{2}), 2\sinh(\frac{s}{2}), 2). \tag{17}$$

The lightcone Frenet frame $\{\alpha, T, \xi\}$ along the curve α is given by

$$\begin{aligned} \alpha(s) &= (2\cosh(\frac{s}{2}), 2\sinh(\frac{s}{2}), 2), \\ T(s) &= \alpha'(s) = (\sinh(\frac{s}{2}), \cosh(\frac{s}{2}), 0), \\ \xi(s) &= (-\frac{\cosh(\frac{s}{2})}{4}, -\frac{\sinh(\frac{s}{2})}{4}, \frac{1}{4}). \end{aligned}$$

In particular, the lightcone curvature $k(s)$ of the curve α is given by

$$k(s) = \langle T'(s), \xi(s)\rangle = \frac{1}{8}. \tag{18}$$

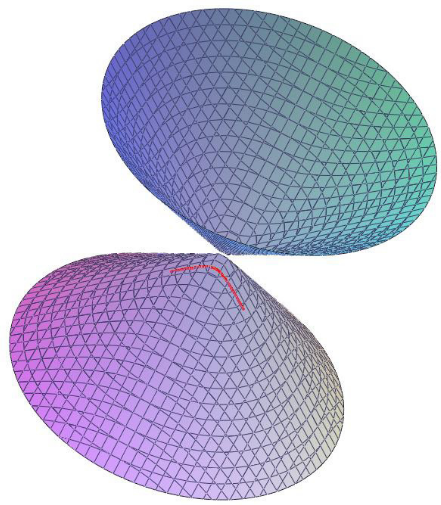

FIGURE 1. The curve α

Taking $a = 1$, $b = \sqrt{2}$ and using the relation $b^2 + 2ac = 0$ we find $c = -1$. It follows that the lightcone $\alpha T\xi$-Smarandache curve β of α is given by (see Figure 2)

$$\begin{aligned} \beta(s) &= \alpha(s) + \sqrt{2}T(s) - \xi(s) \\ &= \left(\frac{e^{\frac{s}{2}}(9+4\sqrt{2}) + e^{-\frac{s}{2}}(9-4\sqrt{2})}{8}, \frac{e^{\frac{s}{2}}(9+4\sqrt{2}) - e^{-\frac{s}{2}}(9-4\sqrt{2})}{8}, \frac{7}{4}\right) \end{aligned}$$

It can be easily checked that $\langle \beta'(s), \beta'(s)\rangle > 0$, which means that β is a spacelike curve. According to Theorem 3.1, its lightcone Frenet frame $\{\beta, T_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} & -1 \\ \frac{\sqrt{2}}{7} & \frac{9}{7} & -\frac{8\sqrt{2}}{7} \\ -\frac{1}{49} & -\frac{8\sqrt{2}}{49} & \frac{64}{49} \end{bmatrix} \begin{bmatrix} \alpha \\ T \\ \xi \end{bmatrix}.$$

By using relation (15), we obtain that the lightcone curvature k_β of β reads

$$k_\beta(s) = \frac{8}{49}.$$

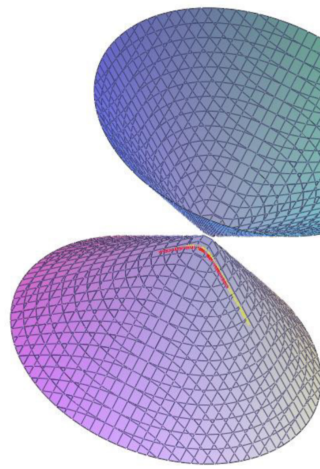

FIGURE 2. The lightcone $\alpha T\xi$-Smarandache curve β and the curve α

Example 4.2. *Let us consider a unit speed spacelike curve α lying on lightcone in the Minkowski 3-space with parameter equation (17). Then its lightcone curvature is given by (18). Denote by β a null lightcone $\alpha T\xi$-Smarandache curve of α. According to Theorem 3.2, the curve α has non-zero constant lightcone curvature given by*

$$k(s) = \frac{b^2}{2c^2}, \quad b, c \in R_0.$$

The last relation together with (18) implies $b = \pm\frac{c}{2}$. By Definition 3.1 there holds $b^2 + 2ac = 0$ and therefore $a = -\frac{c}{8}$. Choosing $c = 1$ and $b = \frac{c}{2}$, we find $a = -\frac{1}{8}$ and $b = \frac{1}{2}$. Consequently, the null lightcone $\alpha T\xi$-Smarandache curve β of α is given by (see Figure 3)

$$\beta(s) = -\frac{1}{8}\alpha(s) + \frac{1}{2}T(s) + \xi(s) = \frac{1}{2}e^{-s}(-1, 1, 0),$$

which means that β is a null straight line.

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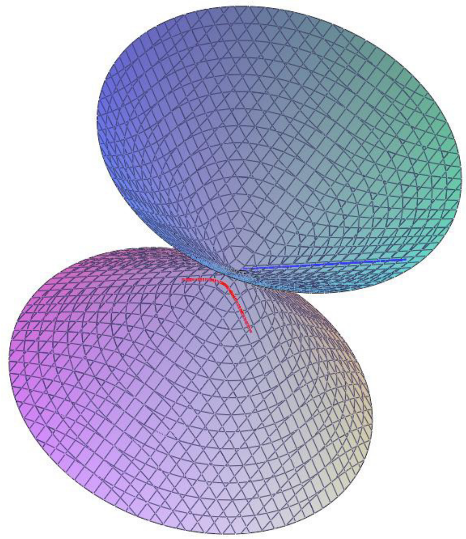

FIGURE 3. The null lightcone $\alpha T\xi$ -Smarandache curve β and the curve α

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