

ON PRACTICAL STABILITY OF TIME SCALE PERTURBED SYSTEMS

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ABSTRACT. This work addresses the problem of uniform practical exponential stability for nonlinear perturbed systems on time scales. Based on Lyapunov techniques and LMI's approach, new sufficient conditions are derived, to find the positive definite function which the derivative is negative definite and provides the uniform practical asymptotic aspect for the system. We present numerical examples to illustrate the proposed result.

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1. INTRODUCTION

Stability theory in sense of Lyapunov of general nonlinear dynamical systems and applications to control theory, has been widely studied during the last decades resulting in important progression both in analysis and control. For an excellent introduction to stability for differential equations and difference equations, we refer the reader to monographs [10, 12] and [2], respectively. In this sequel, Hilger introduces in 1988 the theory of dynamic equations on time scales in order to unify

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the analysis for the two classes of equations. The general idea for the stability theory is to prove result for a dynamical equation where the domain of the drift function is a so-called time scale $\mathbb{T}$. This latest set is an arbitrary closed subset of $\mathbb{R}$. Bohner and Peterson [6] give an overview of time scale dynamical systems. The work of Aulbach and Hilger [1] presents the theory with a particular focus on linear systems. Time scale methods allow the examination and manipulation of dynamical systems according to particular domains of the system, e.g. continuous, discrete or mixed. Let n be a positive integer, consider the regressive non-autonomous perturbed time-scale system described by the following equation

$$(1) \quad x^\Delta(t) = A(t)x + F(t, x)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{T}$. The value function $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is a regressive matrix and $F : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ a rd-continuous valued vector in its first argument. The function F is considered as a perturbation of the nominal system

$$(2) \quad x^\Delta(t) = A(t)x.$$

Many research has been devoted to the analysis of system whom the origin is an equilibrium point, i.e $F(t, 0) = 0$, for all $t \in \mathbb{T}$, using Lyapunov techniques. For instance, recently Dacunha [8] uses Lyapunov-type functions to investigate some suitable conditions that ensure all trajectories to be uniformly asymptotically stable. Earlier, K. Mukdasai and P. Niamsup [11] use some similar ideas to obtain, in addition to this, other stability types: h -stability and ψ -stability.

In this work we are interested to develop stability for some special cases where the origin is not required to be an equilibrium point for the system (1) below. We study the uniform asymptotical stability problem not for the origin but for a neighborhood of the origin approximated by a small ball of radius R . The stability is given by matrix inequalities involving a symmetric definite positive matrix that provides quadratic stability in practical sense. We perform numerical examples to demonstrate the efficiency of the proposed method.

The rest of this paper is organized as follows. In section 2, we introduce some general definitions and results for time-scale dynamical theory. Section 3 is devoted to general notions of Lyapunov stability and results from the literature. In section 4, we present the main results. Finally, in section 5 we give some illustrative examples.

2. TIME SCALE CALCULUS

We start by introducing definitions and notations related to time scales theory.

Definition 2.1. ([5]) Let $t \in \mathbb{T}$. We define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T}; s > t\} \text{ for all } t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T}; s < t\} \text{ for all } t \in \mathbb{T}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum M , then we define $\mathbb{T}^k = \mathbb{T} - M$, otherwise $\mathbb{T}^k = \mathbb{T}$. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t, \quad \forall t \in \mathbb{T}.$$

Definition 2.2. ([5]) A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called Δ -differentiable at a point $t \in \mathbb{T}^k$ if there exist $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W -neighborhood of $t \in \mathbb{T}^k$ satisfying

$$|[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in W$$

In this case we shall write $f^\Delta(t) = \gamma$

Theorem 2.1. Assume that $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are differential at $t \in \mathbb{T}$. Then

(i) For any constant a and b , the sum $af + bg : \mathbb{T} \rightarrow \mathbb{R}$ is differential at t with

$$(af + bg)^\Delta(t) = af^\Delta(t) + bg^\Delta(t).$$

(ii) If $f^\Delta(t)$ exists, then f is continuous at t .

(iii) If $f^\Delta(t)$ exists, then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

(iv) The product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differential at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

Definition 2.3. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and left-sided limits exists (finite) at left-dense points in $\mathbb{T}$ and denotes by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.4. If $F^\Delta(t) = f(t)$, $t \in \mathbb{T}$, then it is said that F is a (delta) anti-derivative of f and the Cauchy (delta) integral by

$$\int_t^s f(t) \Delta t = F(s) - F(\tau), \forall s, \tau \in \mathbb{T}.$$

For a more general definition of the delta integral, see [5] and [6].

Definition 2.5. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regressive* provided that

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^k.$$

the set of all regressive and rd-continuous function is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. The set $\mathcal{R}^+$ of all positively regressive function is

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^k\}.$$

We use the cylinder transformation to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$.

Definition 2.6. If $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then we define the generalized exponential function $e_p(t, s)$ by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right) \text{ for all } s, t \in \mathbb{T}$$

where $\xi_h(z)$ is the cylinder transformation given by

$$\xi_h(z) = \begin{cases} \frac{1}{h} \log(1 + zh) & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

Here $\log$ is the principal logarithm function.

Remark 2.1. Consider the regressive dynamics initial value-problem

$$(3) \quad x^\Delta(t) = p(t)x, \quad x(t_0) = x_0, \quad t_0 \in \mathbb{T}.$$

The exponential function $x(t) = e_p(t, t_0)x_0$ is the unique solution of (3).

Remark 2.2. For convenience of notation we let throughout, for a $t_0 \in \mathbb{T}$ the set $\mathbb{T}_{t_0}^+ := \{s \in \mathbb{T}; s \geq t_0\}$.

The inner product of the vector space $\mathbb{R}^n$ is a function $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n \text{ for } x = (x_1, x_2, \dots, x_n)^T, y = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n.$$

For each vector $x \in \mathbb{R}^n$, the norm (also called the length) of x is defined as the number

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x^T x}.$$

It is clear that the next result cited by Bohner and Peterson [6], in each case, reversing the inequalities in the assumptions yields corresponding lower (instead of upper) estimates for the solution.

Theorem 2.2. (Comparison Theorem [6]). Let $t_0 \in \mathbb{T}$, $x, f \in \mathcal{C}_{rd}$ and $p \in \mathcal{R}^+$, then

$$x^\Delta(t) \leq p(t)x(t) + f(t)$$

implies

$$x(t) \leq x(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau, \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

Now, we find a comparison relation in [4] between the general exponential function on a time scale and the classical exponential function.

Lemma 2.1. For nonnegative p with $-p \in \mathcal{R}^+$ we have the inequality

$$(4) \quad 1 - \int_a^t p(u)\Delta u \leq e_{-p}(t, a) \leq \exp\left\{-\int_a^t p(u)\Delta u\right\} \text{ for all } t \in \mathbb{T}_a^+.$$

Lemma 2.2. ([9]). If $x, y \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$, then

$$\text{for all } \delta > 0, 2x^T A B y \leq \delta x^T A A^T x + \frac{1}{\delta} y^T B^T B y.$$

3. LYAPUNOV STABILITY THEORY

We consider the dynamic system defined on a time scale $\mathbb{T}$ as below :

$$(5) \quad x^\Delta(t) = f(t, x), \quad t \in \mathbb{T}_{t_0}^+, \quad x(t_0, t_0, x_0) = x_0, \quad t_0 \in \mathbb{T}, \quad x_0 \in \mathbb{R}^n$$

where $x \in \mathbb{R}^n$, $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is rd-continuous in its first argument with $\|f(t, 0)\| \leq f_0$. Also we assume that conditions for the existence of a unique solution of system (5) on $\mathbb{T}_{t_0}^+$ are satisfied for all initial state $(t_0, x_0) \in \mathbb{T} \times \mathbb{R}^n$. Let denote this solution by $x(t) = x(t, t_0, x_0)$.

Definitions of stabilities of time-scale systems are obtained by a slight modification of the standard definitions for ordinary differential equations.

In that follow, we give some definitions

Definition 3.1. *Equation (5) is said to be uniformly stable if there exists a constant $\gamma > 0$ such that for every initial state $x_0 \in \mathbb{R}^n$ the corresponding solution satisfies*

$$(6) \quad \|x(t)\| < \gamma \|x_0\|, \quad \forall t \in \mathbb{T}_{t_0}^+.$$

For the next definition, we define a stability property that is not concerning only the boundedness of a solution to (5), but also the asymptotical aspect of trajectories as well.

Definition 3.2. *Equation (5) is said to be uniformly exponentially stable if there exists $\lambda > 0$ with $-\lambda \in \mathcal{R}^+$ and $\gamma \geq 1$ independent on any initial point t_0 such that*

$$(7) \quad \|x(t)\| \leq \gamma e_{-\lambda}(t, t_0) \|x_0\| \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

It is obvious by inspection of the previous definitions that trajectories approach zero exponentially as $t \rightarrow \infty$.

Definition 3.3. *Equation (5) is said to be globally uniformly practically stable if there exists a class $\mathcal{K}$ function ϱ , and a constant $r > 0$ such that, given any initial condition x_0 , the trajectory satisfies*

$$\|x(t)\| \leq \varrho(\|x_0\|) + r, \quad \forall t \in \mathbb{T}_{t_0}^+.$$

Definition 3.4. *Equation (5) is said to be globally uniformly practically asymptotically stable if there exists a class $\mathcal{K}$ function ϱ , constants $0 \leq \lambda < \nu$, $\omega > 0$ and $r > 0$ such that*

$$(8) \quad \|x(t)\| \leq \varrho(\|x_0\|)(e_{\lambda \ominus \nu}(t, t_0))^\omega + r \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

This stability aspect relates the case where the size of initial states may be at random increase and the corresponding trajectories converge, in a stable way by a decaying time scale exponential function, to a small neighborhood for the origin.

Now, we present results of Dacunha [8] stating sufficient criteria for uniform stability and uniform exponential stability of perturbed systems.

Theorem 3.1. *If there exists a continuously differentiable positive definite function $V(t, x(t)) \in \mathcal{C}_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$, and $a, b \in \mathbb{R}^+$ such that*

- (i): $V^\Delta(t, x) \leq 0$,
- (ii): $a\|x(t)\|^2 \leq V(t, x) \leq b\|x(t)\|^2$,

then equation (5) is uniformly stable.

Theorem 3.2. *If there exists a continuously differentiable positive definite function $V(t, x(t)) \in \mathcal{C}_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$, and $a, b, c \in \mathbb{R}^+$ with $\frac{-c}{b} \in \mathcal{R}^+$ such that*

- (i): $V^\Delta(t, x) \leq -c\|x(t)\|^2$,
- (ii): $a\|x(t)\|^2 \leq V(t, x) \leq b\|x(t)\|^2$,

then equation (5) is uniformly exponentially stable.

4. MAIN RESULTS

In this section we present the main results of the paper.

4.1. Lyapunov function for GUPAS. The ensuing result delivers sufficient conditions in terms of Lyapunov function to ensure global uniform practical asymptotic stability for dynamical equations (5). Here, we suppose that $\mathbb{T}$ is a time scale with bounded graininess, that is to say $\mu(t) \leq \hat{\mu} < \infty$ for all $t \in \mathbb{T}$.

Theorem 4.1. *Suppose there exists a continuously differentiable function Lyapunov $V : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ that satisfies*

$$(9) \quad \alpha_1 \|x\|^2 \leq V(t, x) \leq \alpha_2 \|x\|^2 + A(1 + \mu(t)\delta)e_{\ominus\varphi}(t, t_0)$$

$$(10) \quad V^\Delta(t, x) \leq \frac{-\alpha_3}{1 + \mu(t)\delta} \|x\|^2$$

where $A, \alpha_1, \alpha_2, \alpha_3, \delta, \varphi$ are positive constants with $\varphi + \frac{\alpha_3}{\alpha_2} < \delta$.

Then the system (5) is globally uniformly practically asymptotically stable.

Proof. Let $t_0 \in \mathbb{T}$, $t \in \mathbb{T}_{t_0}^+$ and $x \in \mathbb{R}^n$. Designate $W(t, x) = V(t, x)e_\delta(t, t_0)$. Then, with time scale differentiating

$$W^\Delta(t, x) = V^\Delta(t, x)e_\delta(\sigma(t), t_0) + \delta V(t, x)e_\delta(t, t_0)$$

Taking into account inequality (10), we obtain

$$(11) \quad W^\Delta(t, x) \leq -\alpha_3 \|x\|^2 e_\delta(t, t_0) + \delta V(t, x)e_\delta(t, t_0)$$

By the left-side of relation (10) and (11), we deduce that

$$W^\Delta(t, x) \leq (\delta - \frac{\alpha_3}{\alpha_2})V(t, x) + \frac{A\alpha_3}{\alpha_2}(1 + \mu(t)\delta)e_{\ominus\varphi}(t, t_0)e_\delta(t, t_0)$$

It's equivalent to

$$(12) \quad W^\Delta(t, x) \leq \delta' W(t, x) + \frac{A\alpha_3}{\alpha_2}(1 + \mu(t)\delta)e_{(\ominus\varphi) \oplus \delta}(t, t_0)$$

, where $\delta' = \delta - \frac{\alpha_3}{\alpha_2}$. Note that $\delta' > 0$ implies $\delta' \in \mathcal{R}^+$.

To further simplify (12) we resort to comparison theorem (2.2), we get

$$W(t, x) \leq W(t_0, x_0)e_{\delta'}(t, t_0) + \frac{A\alpha_3}{\alpha_2} \int_{t_0}^t e_{\delta'}(\sigma(t), s)(1 + \mu(s)\delta)e_{\ominus\varphi \oplus \delta}(s, t_0)\Delta s$$

Since $V(t, x) = W(t, x)e_{\ominus\delta}(t, t_0)$ the last inequality gives

$$(13) \quad V(t, x) \leq [V(t_0, x_0)e_{\delta'}(t, t_0) + \frac{A\alpha_3}{\alpha_2} \int_{t_0}^t e_{\delta'}(t, \sigma(s))(1 + \mu(s)\delta)e_{(\ominus\varphi) \oplus \delta}(s, t_0)\Delta s]e_{\ominus\delta}(t, t_0)$$

It's easy to see, with $\varphi < \delta'$, that for all $s \in \mathbb{T}_{t_0}^+, s \leq t$

$$(1 + \mu(s)\delta)e_{\delta'}(t, \sigma(s))e_{(\ominus\varphi) \oplus \delta}(s, t_0)e_{\ominus\delta}(t, t_0) \leq (1 + \hat{\mu}\delta) \frac{e_{(\ominus\varphi)}(s, t_0)}{(1 + \mu(s)\varphi)}$$

Using the fact that $e_{\ominus\varphi}(t, t_0) \rightarrow 0, t \rightarrow \infty$ with $\varphi \in \mathcal{R}^+$, we have

$$\int_{t_0}^{+\infty} \frac{e_{(\ominus\varphi)}(s, t_0)}{(1 + \mu(s)\varphi)} \Delta s = -e_{\ominus\varphi}(s, t_0)|_{t_0}^{+\infty} = \frac{1}{\varphi}$$

From (10), we obtain

$$W(t_0, x_0) = V(t_0, x_0) \leq \alpha_2 \|x_0\|^2 + A(1 + \hat{\mu}\delta)$$

By substituting the above inequality into (13), we arrive at

$$(14) \quad V(t, x) \leq \alpha_2 \|x_0\|^2 e_{\delta' \ominus \delta}(t, t_0) + A(1 + \hat{\mu}\delta) + \frac{A\alpha_3(1 + \hat{\mu}\delta)}{\alpha_2 \varphi}$$

Combining with the left inequality in relation (9) the last estimation leads to the desired asymptotic aspect

$$\|x(t)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|x_0\| \sqrt{e_{\delta' \ominus \delta}(t, t_0)} + \sqrt{\frac{A(1 + \hat{\mu}\delta)}{\alpha_1}} (1 + \frac{\alpha_3}{\alpha_2 \varphi})$$

The proof is completed. ■

4.2. Quadratic Stability. The classical approach for studying the stability of nonlinear systems consists in finding a definite positive function which decreases along the trajectories of the considered system. Related to quadratic stability of perturbed time scale systems (2), we can illustrate works of K. Mukdasai and P. Niamsup [11] in which they have investigated sufficient conditions to ensure uniform asymptotic stability using scalar functions of the form $V_1(t, x(t)) = x^T(t)B(t)x(t)$, with $B(t) \in \mathcal{C}_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$. We focus on the practical stability of system (2), that is to say, we ask whether all trajectories of system (1) converge to a neighborhood of the origin as $t \rightarrow \infty$. Our main contribution consists in guaranteeing some adequate hypothesis for both global uniform stabilities: practical and asymptotical practical. The conditions are formulated in terms of two linear matrix inequalities moderated by integrability of relaxing term to upper bound perturbation. Precisely, we discuss boundedness properties and asymptotic behavior of trajectories for systems with perturbing mapping satisfying the growth rate condition

$$(15) \quad \|F(t, x)\| \leq P(t, \|x\|) + \beta(t), \quad t \in \mathbb{T}, \quad x \in \mathbb{R}^n$$

where P verifies the relation

$$(16) \quad 0 \leq P(t, u) - P(t, v) \leq Q(t, v)(u - v), \quad t \in \mathbb{T}, \quad u \geq v \geq 0$$

with $\beta(\cdot), Q(\cdot, u) \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+) \forall u \in \mathbb{R}^n, P(t, 0) = 0 \forall t \in \mathbb{T}$.

It is trivial to see that the growth rate condition of perturbed term F has the form,

$$(17) \quad \|F(t, x)\| \leq \alpha(t)\|x(t)\| + \beta(t)$$

where $\alpha(t) := Q(t, 0)$ for all $t \in \mathbb{T}$. The asymptotical aspect of any solution $x(t)$ of system (1) results from the evolution of the quadratic energy function mentioned below.

The delta derivative of the quadratic function $V_1(t, x(t))$ is given by

$$\begin{aligned}
[V_1(t, x(t))]^\Delta &= [x^T(t)B(t)]^\Delta x(t) + x^T(\sigma(t))B(\sigma(t))x^\Delta(t) \\
&= x^T(t)B^\Delta(t)x(t) + x^T(t)A^T(t)B(t)x(t) + x^T(t)B(t)A(t)x(t) \\
&\quad + \mu(t)x^T(t)B^\Delta(t)A(t)x(t) + \mu(t)x^T(t)A^T(t)B^\Delta(t)x(t) \\
&\quad + \mu(t)x^T(t)A^T(t)B(t)A(t)x(t) + \mu(t)^2x^T(t)A^T(t)B^\Delta(t)A(t)x(t) \\
&\quad + \mu(t)F^T(t, x)B(t)F(t, x) + \mu(t)^2F^T(t, x)B^\Delta(t)F(t, x) \\
&\quad + \mu(t)F^T(t, x)B(t)A(t)x(t) + \mu(t)x^T(t)A^T(t)B(t)F(t, x) \\
&\quad + \mu(t)F^T(t, x)B^\Delta(t)x(t) + \mu(t)x^T(t)B^\Delta(t)F(t, x) \\
&\quad + F^T(t, x)B(t)x(t) + x^T(t)B(t)F(t, x) \\
&\quad + \mu(t)^2F^T(t, x)B^\Delta(t)A(t)x(t) + \mu(t)^2x^T(t)A^T(t)B^\Delta(t)F(t, x).
\end{aligned}$$

Assumption 4.1. *There exists a positive definite symmetric matrix function $B(t) \in \mathcal{C}_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$ and positive constants a_1, a_2, b_1, b_2 such that $a_1 I_n \leq B(t) \leq a_2 I_n$ and $a_1 I_n \leq B^\Delta(t) \leq a_2 I_n$.*

Assumption 4.2. *Suppose that $W_1(t) \leq -cI_n$. where*

$$\begin{aligned}
W_1(t) &= b_2 I_n + A^T(t)B(t) + B(t)A(t) + \mu(t)B^\Delta(t)A(t) + \mu(t)A^T(t)B^\Delta(t) \\
&\quad + \mu(t)A^T(t)B(t)A(t) + \mu(t)^2A^T(t)B^\Delta(t)A(t) + 2\alpha(t)^2\bar{\mu}(t)I_n \\
&\quad + \delta_2 B^\Delta(t)B^\Delta(t) + \delta_1 A^T(t)B(t)B(t)A(t) + \delta_3 A^T(t)B^\Delta(t)B^\Delta(t)A(t),
\end{aligned}$$

$c \in \mathbb{R}_+$ with $\frac{-c}{a_2} \in \mathcal{R}^+$,

and $\bar{\mu}(t) := a_2\mu(t) + b_2\mu(t)^2 + \frac{1}{\delta_3}\mu(t)^2 + \frac{1}{\delta_2}\mu(t)^2 + \frac{1}{\delta_3}\mu(t)^4 + \frac{1}{\delta_4}$.

Assumption 4.3. *Assume that $\int_{t_0}^t \bar{\mu}(s)[\beta(s)]^2 \Delta s \leq \bar{\beta}$.*

Theorem 4.2. *Under Assumptions 4.1-4.3 with $c = 0$, the time-scale perturbed system (1) is globally uniformly practically stable.*

Proof. Let us consider $x(\cdot) = x(\cdot, t_0, x_0)$ an arbitrary trajectory of system (1).

Define $\Phi : \mathbf{M}_{n,1}(\mathbb{R}) \times \mathbf{M}_n(\mathbb{R}) \times \mathbf{M}_{n,1}(\mathbb{R}) \rightarrow \mathbb{R}^+$,

$$(x, M, y) \mapsto [x^T M y + y^T M^T x]$$

We can estimate some of the quantities of time-scale derivative $[V_1(t, x(t))]^\Delta$ in this way, by using lemma (2.2)

$$\Phi(x(t), A^T(t)B(t), \mu(t)F(t, x))$$

$$\leq \frac{1}{\delta_1} \mu(t)^2 \|F(t, x)\|^2 + \delta_1 x^T(t) A^T(t) B(t) B(t) A(t) x(t)$$

$$\Phi(x(t), B^\Delta(t), \mu(t) F(t, x)) \leq \frac{1}{\delta_2} \mu(t)^2 \|F(t, x)\|^2 + \delta_2 x^T(t) B^\Delta(t) B^\Delta(t) x(t)$$

$$\begin{aligned} & \Phi(x(t), A^T(t) B^\Delta(t), \mu(t)^2 F(t, x)) \\ & \leq \frac{1}{\delta_3} \mu(t)^4 \|F(t, x)\|^2 + \delta_3 x^T(t) A^T(t) B^\Delta(t) B^\Delta(t) A(t) x(t) \end{aligned}$$

$$\Phi(x(t), B(t), F(t, x)) \leq \frac{1}{\delta_4} \|F(t, x)\|^2 + \delta_4 x^T(t) B(t) B(t) x(t)$$

Taking the time derivative $[V_1(t, x(t))]^\Delta$ and using above inequalities and the fact that the perturbed term has the upper-bound (17), one obtains

$$\begin{aligned} [V_1(t, x(t))]^\Delta & \leq x^T(t) \{b_2 I_n + A^T(t) B(t) + B(t) A(t) + \mu(t) B^\Delta(t) A(t) \\ & \quad + \mu(t) A^T(t) B^\Delta(t) + \mu(t) A^T(t) B(t) A(t) + \mu(t)^2 A^T(t) B^\Delta(t) A(t) \\ & \quad + 2\alpha(t)^2 \bar{\mu}(t) + B^\Delta(t) B^\Delta(t) + A^T(t) B(t) B(t) A(t) \\ & \quad + A^T(t) B^\Delta(t) B^\Delta(t) A(t)\} x(t) + 2\bar{\mu}(t) [\beta(t)]^2 \\ & = x^T(t) W(t) x(t) + 2\bar{\mu}(t) [\beta(t)]^2. \end{aligned}$$

From hypothesis (4.2) with $c = 0$, we have

$$(18) \quad [V_1(t, x(t))]^\Delta \leq 2\bar{\mu}(t) [\beta(t)]^2, \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

Integrating both sides of the inequality above (18) from t_0 to t , we further have :

$$(19) \quad V_1(t, x(t)) \leq V_1(t_0, x_0) + 2 \int_{t_0}^t \bar{\mu}(s) [\beta(s)]^2 \Delta s, \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

By using assumption (4.1) and (19) we deduce that,

$$\|x(t)\| \leq \sqrt{\frac{V(t, x(t))}{a_1}} \leq \sqrt{\frac{a_2}{a_1}} \|x(t_0)\| + \sqrt{\frac{2\beta}{a_1}}, \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

Consequently, it can be concluded that the perturbed system is globally uniformly practically stable. We get the coveted result. $\blacksquare$

Theorem 4.3. *Under Assumptions 4.1-4.3, the time-scale perturbed system is (1) is globally uniformly practically asymptotically stable.*

Proof. Consider the Lyapunov function $V_1(t, x(t)) = x^T(t)B(t)x(t)$ along an arbitrary trajectory $x(t, t_0, x_0)$ of system (1).

As following the same approach of the proof of previous theorem, we arrive at,

$$[V_1(t, x(t))]^\Delta \leq -c\|x(t)\|^2 + 2\bar{\mu}(t)[\beta(t)]^2, \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

It is easy to see that

$$a_1\|x(t)\|^2 \leq V(t, x) \leq a_2\|x(t)\|^2, \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

Hence,

$$(20) \quad [V_1(t, x(t))]^\Delta \leq -\frac{c}{a_2}V(t, x(t)) + 2\bar{\mu}(t)[\beta(t)]^2, \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

In the view of comparison theorem 2.2 and lemma 2.1, we obtain the following estimation from (20),

$$V_1(t, x(t)) \leq V(t_0, x_0)e_{-\frac{c}{a_2}}(t, t_0) + 2 \int_{t_0}^t \bar{\mu}(s)[\beta(s)]^2 \Delta s, \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

Consequently,

$$(21) \quad \|x(t)\| \leq \sqrt{\frac{a_2}{a_1}}\|x(t_0)\| \sqrt{e_{-\frac{c}{a_2}}(t, t_0)} + \sqrt{\frac{2\bar{\beta}}{a_1}}, \quad \text{for all } t \in \mathbb{T}_{t_0}^+.$$

Immediately, the perturbed system is globally uniformly practically stable with rate $-\frac{c}{a_2}$ and radius of convergence $r = \sqrt{\frac{2\bar{\beta}}{a_1}}$.

This finishes the proof of the theorem. ■

Now, we consider the following Lyapunov function for the system (1)

$$(22) \quad V_3(t, x(t)) = \langle B(t)x(t), x(t) \rangle + \xi(t)\|x(t)\|^2,$$

where ξ is delta-differentiable on $\mathbb{T}$ such that $\max\{\xi(t), \xi^\Delta(t)\} \leq \bar{\xi}, \forall t \in \mathbb{T}$.

The delta derivative of the scalar function

$$V_2(t, x(t)) = \xi(t)\|x(t)\|^2 = \xi(t)x^T(t)x(t)$$

with respect to t is given by

$$\begin{aligned}
 [V_2(t, x(t))^{\Delta T} &= \xi^{\Delta}(t)\|x(t)\|^2 + (\xi(t) + \mu(t)\xi^{\Delta}(t))[\|x(t)\|^2]^{\Delta} \\
 &= \xi^{\Delta}(t)\|x(t)\|^2 + (\xi(t) + \mu(t)\xi^{\Delta}(t))\{[x^T(t)x(t)]^{\Delta} + x^T(\sigma(t))x^{\Delta}(t)\} \\
 &= \xi^{\Delta}(t)\|x(t)\|^2 + (\xi(t) + \mu(t)\xi^{\Delta}(t))\{x^T(t)[A^T(t) + A(t) + \\
 &\quad \mu(t)A^T(t)A(t)]x(t) + F^T(t, x)x(t) + x^T(t)F(t, x) \\
 &\quad + \mu(t)x^T(t)A^T(t)F(t, x) + \mu(t)F^T(t, x)A(t)x(t) \\
 &\quad + \mu(t)\|F(t, x)\|^2\}.
 \end{aligned}$$

Assumption 4.4. Assume that $W_2(t) \leq -cI_n$, where

$$W_2(t) = \bar{\xi}I_n + (1 + \mu(t))\bar{\xi}\{A^T(t) + A(t) + (1 + \mu(t))A^T(t)A(t) + \{1 + 2\alpha(t)^2\bar{\mu}(t)\}I_n\} + W_1(t),$$

$c \in \mathbb{R}_+$ with $\frac{-c}{a_2} \in \mathcal{R}^+$ and $\underline{\mu}(t) = 1 + \mu(t) + \mu(t)^2$.

Assumption 4.5. Adopt that $\int_{t_0}^t (\bar{\mu}(s) + \underline{\mu}(s))[\beta(s)]^2 \Delta s \leq \hat{\beta}$.

Theorem 4.4. Under Assumptions 4.1, 4.4 and 4.5 with $c = 0$, the time-scale perturbed system is (1) is globally uniformly practically asymptotically stable.

Proof. Let us consider $x(t) := x(t, t_0, x_0)$ an arbitrary solution of system (1).

Now we are in a position to show that $V(t, x(t))$ as defined in (22) is a Lyapunov function. To verify this, we substitute the above inequalities with growth rate condition of the perturbed term and we obtain

$$\begin{aligned}
 (23) \quad F^T(t, x)x(t) + x^T(t)F(t, x) &\leq \|x(t)\|^2 + \|F(t, x)\|^2 \\
 \Phi(x(t), A^T(t), \mu(t)F(t, x)) &\leq \mu(t)^2\|F(t, x)\|^2 + x^T(t)A^T(t)A(t)x(t).
 \end{aligned}$$

The delta-derivative of the Lyapunov function V_3 along trajectories of system (1) is given by

$$[V_3(t, x(t))^{\Delta} = [V_1(t, x(t))^{\Delta} + [V_2(t, x(t))^{\Delta}.$$

Taking into consideration inequalities (??), (23) and (4.4), we get

$$[V_3(t, x(t))^{\Delta} \leq -c\|x(t)\|^2 + 2[\bar{\mu}(t) + \underline{\mu}(t)][\beta(t)]^2.$$

By the same way as in the proof of theorem 4.2, the required feature of stability holds. The proof is completed. ■

Theorem 4.5. *Under Assumptions 4.1, 4.4 and 4.5, the time-scale perturbed system is (1) is globally uniformly practically asymptotically stable.*

Proof. The proof follows exactly as in theorem 4.3, with the observation that the Lyapunov function V_3 exponentially decreases over time in evolution to a ball centered at the origin.

We omit details. ■

Of course, the problem at hand is obtaining a proper scalar function. Now we can introduce a new form of scalar function of energy named quasi-quadratic Lyapunov functions.

Definition 4.1. *Let $B(t)$ be a symmetric matrix such that $B(t) \in \mathcal{C}_{rd}^1(\mathbb{T}, \mathbb{R}^{n \times n})$, a quasi-quadratic Lyapunov function is given by*

$$(24) \quad V_4(t, x(t)) = x^T(t)B(t)x(t) + a(\|x\|), \quad a \in \mathcal{K} \cap \mathcal{C}^1(\mathbb{R}^+).$$

The delta derivative of V along the trajectory of system (1) is given by

$$[V_4(t, x(t))]^\Delta = [x^T B(t) x(t)]^\Delta + [a(\|x(t)\|)]^\Delta.$$

Using chain Rule version 1 [6], there exists c_1, c_2 in the real interval $[t, \sigma(t)]$ such that

$$[a(\|x(t)\|)]^\Delta = \frac{a'(\|x(c_1)\|)}{2\|x(c_2)\|} [x^T(t)x(t)]^\Delta.$$

We define then the operator $\omega(t) := \frac{a'(\|x(c_1)\|)}{2\|x(c_2)\|}$ for $t \in \mathbb{T}$.

Assumption 4.6. *Suppose that $W_3(t) \leq -cI_n$ where*

$$W_3(t) = \omega(t)[A^T(t) + A(t) + (1 + \mu(t))A^T(t)A(t) + \{1 + 2\alpha(t)^2 \underline{\mu}(t)\}I_n] + W_1(t),$$

$c \in \mathbb{R}_+$ with $\frac{-c}{a_2} \in \mathcal{R}^+$.

Assumption 4.7. *Take that $\int_{t_0}^t (\underline{\mu}(s))[\beta(s)]^2 \Delta s \leq \underline{\beta}$*

Theorem 4.6. *Under Assumptions 4.1, 4.6 and 4.7 with $c = 0$, the time-scale perturbed system is (1) is globally uniformly practically stable.*

Proof. Let consider $x(t) := x(t, t_0, x_0)$ be an arbitrary solution of system (1). Now it is possible to show that $V_4(t, x(t))$ as defined in (24) is a Lyapunov function. Using the same approach of proof of precedent theorem we arrive at

$$V_4^\Delta(t, x(t)) \leq 2\underline{\mu}(t)[\beta(t)]^2, \text{ for all } t \in \mathbb{T}_{t_0}^+.$$

Integrating it from t_0 to t , we get $V(t, x(t)) \leq V(t_0, x_0) + 2 \int_{t_0}^t \underline{\mu}(s)[\beta(s)]^2 \Delta s$. Combine the inequality above with assumption (4.7),

$$V(t, x(t)) \leq a_2 \|x_0\|^2 + a(\|x_0\|) + \frac{2\beta}{a_1}, a \in \mathcal{K}.$$

Consequently,

$$\|x(t)\|^2 \leq \frac{a_2}{a_1} \|x_0\|^2 + \frac{1}{a_1} a(\|x_0\|) + \frac{2\beta}{a_1}, \text{ for all } t \in \mathbb{T}_{t_0}^+$$

which implies that trivial solution of system (1) is globally uniformly practically stable. ■

Theorem 4.7. *Under Assumptions 4.1, 4.6 and 4.7, the time-scale perturbed system is (1) is globally uniformly practically asymptotically stable.*

Proof. The proof of the theorem follows by an argument similar to that in the proof of the above theorems. We omit details. ■

5. EXAMPLES

To show the utility of the result 4.2, we include this example.

Example 5.1. *Taking a time-scale varying system of the form (1)*

$$\text{with } A(t) = \begin{pmatrix} -c(t) & -1 \\ 1 & -c(t) \end{pmatrix},$$

$$F(t, x) = \begin{pmatrix} \frac{\sqrt{x_1^2(t) + x_2^2(t)}}{8\sqrt{2}(1+\mu(t))} + \frac{\sqrt{t+\sigma(t)+2}}{(t+1)(\sigma(t)+1)\sqrt{t^2+1}} \\ -\frac{\sqrt{x_1^2(t) + x_2^2(t)}}{8\sqrt{2}(1+\mu(t))} + \frac{\sqrt{t+\sigma(t)+2}}{(t+1)(\sigma(t)+1)\sqrt{t^2+1}} \end{pmatrix}$$

, $c(t) = -e_{\ominus 8}(t, 0) + 1$, and $F(t, x(t))$ is rd-continuous in the first argument with $\|f(t, 0)\| < M = \sqrt{2}$ for all $t \in \mathbb{T}$. Assume that $\sup\{\mu(t); t \in \mathbb{T}\} = 1/4$.

We note that assumption 4.1 is fulfilled with $a_1 = 1/8$, $a_2 = 1/4$, $b_1 = -1$, $b_2 = 0$, $\delta_1 = 1/2$, $\delta_2 = 1/16$, $\delta_3 = 1/16$, $\delta_4 = 1$.

To see this we consider a candidate valued matrix function

$$B(t) = \begin{pmatrix} \frac{1}{8}(e_{\ominus 8}(t, 0) + 1) & 0 \\ 0 & \frac{1}{8}(e_{\ominus 8}(t, 0) + 1) \end{pmatrix},$$

that solving assumptions 4.1-4.2.

Furthermore the relaxing term $\beta(t)^2 = \sqrt{2} \frac{t+\sigma(t)+2}{(t+1)^2(\sigma(t)+1)^2(t^2+1)}$ of perturbed upper bound is delta-integrable, using the Drichlet-Abel Test [7].

Finally this example satisfies all requirements of the main result and the desired aspect of solutions is acquired.

To illustrate the usefulness of theorem 4.3, we state the following example.

Example 5.2. We consider a time-scale varying system of the form (1)

$$\text{with } A(t) = \begin{pmatrix} -c(t) & -1 \\ 1 & -c(t) \end{pmatrix},$$

$$F(t, x) = \begin{pmatrix} \frac{-\sin(t)}{8(1+\mu(t))}x_2(t) + \arctan(1+x_2(t))\frac{e_\varphi(t,0)}{e_\psi(t,0)} \\ \frac{\cos(t)}{8(1+\mu(t))}x_1(t) + \arctan(1+x_1(t))\frac{e_\varphi(t,0)}{e_\psi(t,0)} \end{pmatrix}$$

, $c(t) = e_{\ominus 8}(t, 0) + 1$, and $F(t, x(t))$ is rd-continuous in the first argument with $\|f(t, 0)\| < M = \frac{\pi}{\sqrt{2}}$ for all $t \in \mathbb{T}$. Assume that $\sup\{\mu(t); t \in \mathbb{T}\} = 1/4$.

We remark that hypothesis 4.1 is verified with $a_1 = 1/8$, $a_2 = 1/4$, $b_1 = -1$,

$b_2 = 0$, $\delta_1 = 1/2$, $\delta_2 = 1/16$, $\delta_3 = 1/16$, $\delta_4 = 1$.

To ensure this we consider a candidate valued matrix function

$$B(t) = \begin{pmatrix} \frac{1}{8}(e_{\ominus 8}(t, 0) + 1) & 0 \\ 0 & \frac{1}{8}(e_{\ominus 8}(t, 0) + 1) \end{pmatrix},$$

that us satisfy hypothesis 4.1-4.2.

Moreover, the relaxing term $\beta(t)^2 = \frac{\pi}{2} \left(\frac{e_\varphi(t,0)}{e_\psi(t,0)} \right)^2$ is delta-integrable for $0 \leq \varphi < \psi$.

Therefore this system is globally uniformly practically asymptotically stable.

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