

FINSLERIAN HYPERSURFACES OF A FINSLER SPACES WITH (γ, β) -METRIC

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ABSTRACT. In the present paper we have study the Finslerian hypersurfaces of a Finsler spaces with (γ, β) -metric. The relations between the Finslerian hypersurface given by the Finsler space with cubic metric and the other Finslerian hypersurfaces which is given by Finsler space with (γ, β) -metric. We have also proved that a (γ, β) -metric make three types of hypersurfaces invariant under certain condition. These three types of a hypersurfaces are hyperplanes of first, second and third kind.

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1. INTRODUCTION

A cubic metric as a simplest non-Riemannian Finsler space has been studied in detail by M. Matsumoto [5] in 1979 after a lapse of 60 years since P. Finsler

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introduced the concept of Finsler space in 1918. A cubic metric is defined as

$$(1) \quad L(x, y) = \{a_{ijk}(x)y^i y^j y^k\}^{\frac{1}{3}},$$

where $a_{ijk}(x)$ are components of a symmetric tensor field of (0,3)-type depending on the position x alone, and a Finsler space with a cubic metric is called the *cubic Finsler space*.

Let (M^n, L) be an n -dimensional Finsler space on a differentiable manifold M^n , equipped with the fundamental function $L(x, y)$. In 1984, Shibata [10] introduced the transformation of Finsler metric:

$$(2) \quad L'(x, y) = f(L, \beta),$$

where $\beta = b_i(x)y^i$, $b_i(x)$ are components of a covariant vector in (M^n, L) and f is positively homogeneous function of degree one in L and β . This change of metric is called a β -change.

In 2011 Pandey and Chaubey [6] introduced the concept of (γ, β) -metric, where $\gamma = \{a_{ijk}(x)y^i y^j y^k\}^{\frac{1}{3}}$ is a cubic metric and $\beta = b_i(x)y^i$ is a one-form and obtained its basic torsion tensors, curvature tensors and T-tensor. Pandey and Chaubey [7, 8] also studied its main scalar in two-dimensional Finsler space and variational problem in Lagrange spaces with this metric.

On the other hand in 1985 M. Matsumoto [4] investigated the theory of Finslerian hypersurface. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.

In 2005, Prasad and Tripathi [9] studied the Finslerian hypersurfaces and Kropina change of the Finsler metric in detail and obtained so many result. In the present paper, using the field of linear frame [2, 3, 1] we shall consider Finslerian hypersurfaces given by a Finsler space with (γ, β) -metric. Our purpose is to give some relations between the original Finslerian i.e. cubic Finsler space and the other which is Finslerian hypersurface given by a (γ, β) -metric. We also obtained three types of hypersurfaces invariant under certain condition in a Finsler space with (γ, β) -metric.

2. FINSLERIAN HYPERSURFACES

Let M^n be an n -dimensional smooth manifold and $F^n = (M^n, \gamma)$ be an n -dimensional Finsler space equipped with the fundamental function $\gamma(x, y)$ on M^n . The metric tensor $g_{ij}(x, y)$ and Cartan's C-tensor $C_{ijk}(x, y)$ are given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 \gamma^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

respectively and we introduced the Cartan's connection $CT = (F_{jk}^i, N_j^i, C_{jk}^i)$ in F^n .

A hypersurface $M^{(n-1)}$ of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on $M^{(n-1)}$ and Greek indices vary from 1 to $(n-1)$. Here we shall assume that the matrix consisting of the projection factors $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ is of rank $(n-1)$. The following notations are also employed:

$$B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B_{0\beta}^i = v^\alpha B_{\alpha\beta}^i.$$

If the supporting element y^i at a point (u^α) of $M^{(n-1)}$ is assumed to be tangential to $M^{(n-1)}$, we may then write $y^i = B_\alpha^i(u) v^\alpha$ i.e. v^α is thought of as the supporting element of $M^{(n-1)}$ at the point (u^α) . Since the function $\bar{\gamma}(u, v) = \gamma\{x(u), y(u, v)\}$ gives rise to a Finsler metric of $M^{(n-1)}$, we get a $(n-1)$ -dimensional Finsler space $F^{(n-1)} = \{M^{(n-1)}, \bar{\gamma}(u, v)\}$.

At each point (u^α) of $F^{(n-1)}$, the unit normal vector $N^i(u, v)$ is defined by

$$(3) \quad g_{ij} B_\alpha^i N^j = 0, \quad g_{ij} N^i N^j = 1.$$

If (B_i^α, N_i) is the inverse matrix of (B_α^i, N^i) , we have

$$B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i N_i = 0, \quad N^i N_i = 1 \quad \text{and} \quad B_\alpha^i B_j^\alpha = N^i N_j = \delta_j^i.$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$(4) \quad B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j, \quad N_i = g_{ij} N^j.$$

For the induced Cartan's connection $ICT = (F_{\beta\gamma}^\alpha, N_\alpha^\beta, C_{\beta\gamma}^\alpha)$ on $F^{(n-1)}$, the second fundamental h -tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are respectively given by [3]

$$(5) \quad H_{\alpha\beta} = N_i (B_{\alpha\beta}^i + F_{jk}^i B_\alpha^j B_\beta^k) + M_\alpha H_\beta, \quad H_\alpha = N_i (B_{0\beta}^i + N_j^i B_\beta^j),$$

where

$$M_\alpha = C_{ijk} B_\alpha^i N^j N^k.$$

Contracting $H_{\alpha\beta}$ by v^α , we immediately get $H_{0\beta} = H_{\alpha\beta} v^\alpha = H_\beta$. Furthermore the second fundamental v -tensor $M_{\alpha\beta}$ is given by [9]

$$(6) \quad M_{\alpha\beta} = C_{ijk} B_\alpha^i B_\beta^j N^k.$$

3. FINSLER SPACE WITH (γ, β) -METRIC

A Finsler metric $L^*(x, y)$ is called a (γ, β) -metric [5] when L^* is positively homogeneous function of first degree in two variables γ and β , where $\gamma = \{a_{ijk}(x)y^i y^j y^k\}^{\frac{1}{3}}$ is cubic metric and $\beta = b_i(x)y^i$ is a one-form metric.

Throughout the present paper the following notations are adopted:

$$a_{ijk}(x)y^j y^k = \gamma^2 a_i, \quad a_{ijk}y^k = \gamma a_{ij}, \quad g^{ij}b_j = B^i, \quad g^{ij}l_j = l^i,$$

where (g^{ij}) is the inverse matrix of (g_{ij}) . Further, subscripts γ, β denote partial differentiation with respect to γ, β respectively.

The normalized supporting element l_i^* for a (γ, β) -metric $L^*(\gamma, \beta)$, is written as

$$\dot{\partial}_i L^* = l_i^* = L_\gamma^* a_i + L_\beta^* b_i,$$

$$(7) \quad l_i^* = L_\gamma^* l_i + L_\beta^* b_i,$$

where $a_i = l_i$ [4] is the normalized supporting element for a Finsler space with cubic metric.

Again differentiating equation (7) with respect to y^j , the angular metric tensor $h_{ij}^* = g_{ij}^* - l_i^* l_j^* = L^* \dot{\partial}_i \dot{\partial}_j L^*$ can be written as

$$(8) \quad h_{ij}^* = p h_{ij} + q_0 b_i b_j + q_1 (l_i b_j + l_j b_i) + q_2 l_i l_j,$$

where h_{ij} is the angular metric tensor for a Finsler space with cubic metric, and

$$p = \frac{L^* L_\gamma^*}{\gamma}, \quad q_0 = L^* L_{\beta\beta}^*, \quad q_1 = L^* L_{\gamma\beta}^*, \quad q_2 = L^* L_{\gamma\gamma}^* \quad (8)'$$

Owing to the homogeneity or $h_{ij}^* y^{*j} = 0$, we have two identities

$$(9) \quad \begin{cases} q_0 \beta + q_1 \gamma = 0, \\ q_1 \beta + q_2 \gamma = 0. \end{cases}$$

Remark: In (8) the subscripts of coefficients p, q_0, q_1, q_2 are used to indicate respective degrees of homogeneity.

From (7) and (8), the fundamental tensor g_{ij}^* of a (γ, β) -metric is given by

$$(10) \quad g_{ij}^* = h_{ij}^* + l_i^* l_j^* = p g_{ij} + p_0 b_i b_j + p_1 (l_i b_j + l_j b_i) + p_2 l_i l_j,$$

where g_{ij} is the fundamental tensor for a Finsler space with cubic metric, and

$$p_0 = q_0 + L_\beta^{*2}, \quad p_1 = q_1 + L_\gamma^* L_\beta^*, \quad p_2 = q_2 + L_\gamma^{*2} - \frac{2L_\gamma^* L_\beta^*}{\gamma}. \quad (10)'$$

From (9) we get identities

$$(11) \quad \begin{cases} p_0 \beta + p_1 \gamma = L^* L_\beta^* \\ p_1 \beta + p_2 \gamma = 0 \end{cases}$$

The determinant g of g_{ij}^* are given by [5]

$$(12) \quad g^* = p^{n-1} g(p + c^2)(1 + d^2) = p^{n-1} g \tau_{(-2)},$$

where g is the determinant of g_{ij} , $\tau_{(-2)} = p(p + p_0 b^2 + p_1 \bar{l}) + (p_1 p \bar{l} - p_1^2 \bar{l}) + p_2 p + p_2 c^2$, $B^i b_i = b^2 = g^{im} b_m b_i$, $l_i B^i = g^{im} l_i b_m = l^i b_i = \bar{l}$, $\pi^2 b^2 = c^2 = c_i c^i$, and $l_i l^i = 1$.

The inverse matrix (g^{*ij}) of (g_{ij}^*) is given by

$$(13) \quad g^{*ij} = \frac{1}{p} g^{ij} - s_1 B^i B^j - s_0 (B^i l^j + B^j l^i) - s_{-1} l^i l^j,$$

where

$$(13)' \quad \begin{aligned} s_1 &= \frac{\pi_0 p^2 + \pi^2 (\tau_{-2} + \pi^2 p_2 \bar{l}^2 - 2p_1 \bar{l})}{\tau_{-2} p (p + c^2)}, & s_0 &= \frac{p_1 p - \pi^2 p_2 \bar{l}}{\tau_{(-2)} p} \\ s_{-1} &= \frac{p p_2 + c^2 p_2}{\tau_{-2} p}. \end{aligned}$$

Now differentiating (10) w.r.t. y^k , we get

$$(14) \quad 2p C_{ijk}^* = 2p^2 C_{ijk} + r_1 b_i b_j b_k + \Pi_{(ijk)} (P_i h_{jk} + r_2 l_i b_j b_k + r_3 l_i l_j b_k) + r_4 l_i l_j l_k,$$

where $\Pi_{(ijk)}$ represents the sum of cyclic permutation of i, j, k ;

$$(15) \quad P_i = p_2 l_i + p_1 b_i,$$

and

$$(14)' \quad \begin{aligned} r_1 &= p p_{0\beta} - 3p_1 q_0, & r_2 &= p p_{1\beta} - q_0 p_2 - 2p_1 q_1, \\ r_3 &= p p_{2\beta} - 2p_2 q_1 - p_1 q_2, & r_4 &= p \frac{p_{2\gamma}}{\gamma} - 3p_2 q_2. \end{aligned}$$

The components C_{jk}^{*i} of the (h)hv-torsion tensor CT are given by

$$(16) \quad 2pC_{jk}^{*i} = 2p^2C_{jk}^i + (\delta_j^i P_k - l^i l_j P_k) + (\delta_k^i P_j - l^i l_k P_j) + \frac{r_2}{p_2^3} P_j P_k + h_{jk} P^i,$$

where $P_i g^{ij} = P^j$, $g^{ij} l_j = l^i$, $C_{rjk} g^{ri} = C_{jk}^i$.

4. HYPERSURFACES GIVEN BY (γ, β) -METRIC

Consider a Finslerian hypersurface $F^{(n-1)} = \{M^{(n-1)}, \bar{L}(u, v)\}$ of the F^n and another Finslerian hypersurface $F^{*(n-1)} = \{M^{(n-1)}, \bar{L}^*(u, v)\}$ of the F^{*n} given by the Finsler space with (γ, β) -metric. Let N^i be the unit vector at each point of $F^{(n-1)}$ and (B_α^i, N_i) be the inverse matrix of (B_α^i, N^i) . The function B_α^i may be considered as components of $(n-1)$ linearly independent tangent vectors of $F^{(n-1)}$ and they are invariant under Randers conformal change. Thus we shall show that a unit normal vector $N^{*i}(u, v)$ of $F^{*(n-1)}$ is uniquely determined by

$$(17) \quad g_{ij}^* B_\alpha^i N^{*j} = 0, \quad g_{ij}^* N^{*i} N^{*j} = 1.$$

Contracting (10) by $N^i N^j$ and paying attention to (3) and the fact that $l_i N^i = 0$, we have

$$(18) \quad g_{ij}^* N^i N^j = p + p_0 (b_i N^i)^2.$$

Therefore we obtain

$$g_{ij}^* \left\{ \pm \frac{N^i}{\sqrt{p + p_0 (b_i N^i)^2}} \right\} \left\{ \pm \frac{N^j}{\sqrt{p + p_0 (b_i N^i)^2}} \right\} = 1.$$

Hence we can put

$$(19) \quad N^{*i} = \frac{N^i}{\sqrt{p + p_0 (b_i N^i)^2}},$$

where we have chosen the positive sign in order to fix an orientation.

Using equation (10), (19) and from first condition of (17) we have

$$(20) \quad (p_0 b_i B_\alpha^i + p_1 l_i B_\alpha^i) \frac{b_j N^j}{\sqrt{p + p_0 (b_i N^i)^2}} = 0.$$

If $(p_0 b_i B_\alpha^i + p_1 l_i B_\alpha^i) = 0$ then contracting it by v^α and using $y^i = B_\alpha^i v^\alpha$ we get $p_0 \beta + p_1 \gamma = 0$ which is contradiction to the assumption that $L^* > 0$. Hence

$b_i N^i = 0$. Therefore equation (19) can be written as

$$(21) \quad N^{*i} = \frac{1}{\sqrt{p}} N^i.$$

Summarizing the above we have:

Proposition 4.1. *If $\{(B_\alpha^i, N^i), \alpha = 1, 2, \dots, (n-1)\}$ be the field of linear frame of the Finsler space with cubic metric F^n , there exist a field of linear frame $\{(B_\alpha^i, N^{*i} = \frac{1}{\sqrt{p}} N^i), \alpha = 1, 2, \dots, (n-1)\}$ of the Finsler space with (γ, β) -metric F^{*n} such that (17) is satisfied along $F^{*(n-1)}$ and then b_i is tangential to both the hypersurfaces $F^{(n-1)}$ and $F^{*(n-1)}$.*

The quantities $B_i^{*\alpha}$ are uniquely defined along $F^{*(n-1)}$ by

$$B_i^{*\alpha} = g^{*\alpha\beta} g_{ij}^* B_\beta^j,$$

where $g^{*\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}^*$. Let $(B_i^{*\alpha}, N_i^*)$ be the inverse matrix of (B_α^i, N^{*i}) , then we have

$$B_\alpha^i B_i^{*\beta} = \delta_\alpha^\beta, \quad B_\alpha^i N_i^* = 0, \quad N^{*i} N_i^* = 1.$$

Furthermore $B_\alpha^i B_j^{*\alpha} + N^{*i} N_j^* = \delta_j^i$. We also get $N_i^* = g_{ij}^* N^{*j}$ which in view of (7), (10) and (21) gives

$$(22) \quad N_i^* = \sqrt{p} N_i.$$

We denote the Cartan's connection of F^n and F^{*n} by $(F_{jk}^i, N_j^i, C_{jk}^i)$ and $(F_{jk}^{*i}, N_j^{*i}, C_{jk}^{*i})$ respectively and put $D_{jk}^i = F_{jk}^{*i} - F_{jk}^i$ which will be called difference tensor. We choose the vector field b_i in F^n such that

$$(23) \quad D_{jk}^i = A_{jk} b^i - B_{jk} l^i,$$

where A_{jk} and B_{jk} are components of symmetric covariant tensors of second order. Since $b_i N^i = 0$ and $N_i l^i = 0$, from (23) we get $N_i D_{jk}^i = 0$ and $N_i D_{0k}^i = 0$. Therefore from (5) and (22) we get

$$(24) \quad H_\alpha^* = \sqrt{p} H_\alpha.$$

If each path of a hypersurface $F^{(n-1)}$ with respect to the induced connection is also a path of the enveloping space F^n then $F^{(n-1)}$ is called a hyperplane of the first kind. A hyperplane of the first kind is characterized by $H_\alpha = 0$ [4]. Hence from (24) we have:

Theorem 4.1. *If $b_i(x)$ be a vector field in a Finsler space F^n satisfying (23) then a hypersurface $F^{(n-1)}$ is a hyperplane of the first kind if and only if the hypersurface $F^{*(n-1)}$ is a hyperplane of the first kind.*

Next contracting (14) by $B_\alpha^i N^{*j} N^{*k}$ and paying attention to (21),

$$M_\alpha = C_{ijk} B_\alpha^i N^j N^k,$$

$a_i N^i = 0$, $b_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B_\alpha^i N^j = 0$, we get

$$(25) \quad M_\alpha^* = M_\alpha - \frac{1}{2p^2} P_i B_\alpha^i.$$

From (5), (22), (23), (24) and (25) we have

$$(26) \quad H_{\alpha\beta}^* = \sqrt{p} \{ H_{\alpha\beta} - \frac{1}{2p^2} P_i B_\alpha^i H_\beta \}.$$

If each h-path of a hypersurface $F^{(n-1)}$ with respect to the induced connection is also h-path of the enveloping space F^n , then $F^{(n-1)}$ is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by $H_{\alpha\beta} = 0$ [4]. Since $H_{\alpha\beta} = 0$ implies that $H_\alpha = 0$ from (24) and (26) we have the following:

Theorem 4.2. *If $b_i(x)$ be a vector field in a Finsler space F^n satisfying (23) then a hypersurface $F^{(n-1)}$ is a hyperplane of the second kind if and only if the hypersurface $F^{*(n-1)}$ is a hyperplane of the second kind.*

Finally contracting (14) by $B_\alpha^i B_\beta^j N^{*k}$ and paying attention to (6), (19), $a_i N^i = 0$ and $h_{ij} B_\alpha^i N^j = 0$, we have

$$(27) \quad M_{\alpha\beta}^* = \sqrt{p} M_{\alpha\beta}.$$

If the unit normal vector of $F^{(n-1)}$ is parallel along each curve of $F^{(n-1)}$ then $F^{(n-1)}$ is called a hyperplane of the third kind. A hyperplane of the third kind is characterized by $H_{\alpha\beta} = 0$, $M_{\alpha\beta} = 0$ [4]. From (26) and (27) we have:

Theorem 4.3. *If $b_i(x)$ be a vector field in a Finsler space F^n satisfying (23) then a hypersurface $F^{(n-1)}$ is a hyperplane of the third kind if and only if the hypersurface $F^{*(n-1)}$ is a hyperplane of the third kind.*

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