

AVERAGE SHADOWING PROPERTY WITH NON UNIFORMLY HYPERBOLICITY ON PERIODIC POINTS

ALIREZA ZAMANI BAHABADI

DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD,
MASHHAD, IRAN
E-MAIL: ZAMANY@UM.AC.IR

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ABSTRACT. In this short paper we show that generically if f is tame and non uniformly hyperbolic on the its periodic points, then average shadowing property and hyperbolicity are equivalent.

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1. INTRODUCTION

Hyperbolic dynamics characterized by the presence of expanding and contracting directions for the derivative. This is a situation where the differential alone provides global information about the dynamics. The hyperbolicity theory has many applications within other theory in dynamical systems such as chaos theory and structural stability. It also serves to show the important notions that motivate the study of nonuniform hyperbolicity. In fact, it is the theory of nonuniformly hyperbolic dynamical systems that in applications provides the mathematical foundation of the theory of chaos. One of the important tools in dynamical systems is the shadowing

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property. In [15] has been shown that hyperbolic basic sets have the shadowing property. So many authors studied about more shadowing-type properties which relate to the hyperbolicity, for example see [8-14]. Recently some authors try to obtain hyperbolicity from the non uniformly hyperbolicity on periodic points. For example, A. Castro and et al in [19] obtained by nonuniformly hyperbolic periodic points, the uniform hyperbolicity and the shadowing property had an important rule in their theorems. A well-known kind of the shadowing property is average shadowing property (ASP) introduced by Blank [3] and studied by Sakai [9] in a greater detail. It is a natural question, how we can address relation between ASP and hyperbolicity. Some authors try to answer this question. For example, Sakai [9] proved that on a closed C^∞ surface, the C^1 interior of the set of all C^1 diffeomorphisms with the average shadowing property is characterized by the set of Anosov diffeomorphisms. Arbieto and Ribeiro [2] extended the work of Sakai for flows and showed that the interior of the set of flows with the average shadowing property on a three-dimensional closed manifold is formed by transitive Anosov flows. We studied the work of Sakai for conservative diffeomorphisms in a higher dimension and obtained dominated splitting for these systems (see [5]). In this context, for conservative flows and hyperbolicity see [17]. Here we are interest to know the relation between ASP and hyperbolicity C^1 generically. So what happen for systems having ASP generically? In [20] we showed that in a manifold with dimension three, generically if a diffeomorphism f is tame and has the average shadowing property, then f is Anosov or there is a non-hyperbolic ergodic measure with support on the whole of the manifold M . In this short paper we show that generically if f is tame and non uniformly hyperbolic on the its periodic points, then average shadowing property and hyperbolicity are equivalent. In section 2, some basic terminology and main results are presented. The proofs are given in section 3.

2. SOME BASIC TERMINOLOGY

Let (X, d) be a compact metric space and let $f : X \rightarrow X$ be a homeomorphism (a discrete dynamical system on X). A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is called an orbit of f , denote by $o(x, f)$, if for each $n \in \mathbb{Z}$, $x_{n+1} = f(x_n)$ and is called a δ -pseudo-orbit of f if

$$d(f(x_n), x_{n+1}) \leq \delta, \forall n \in \mathbb{Z}.$$

A map f is called chain transitive if for any $x, y \in X$ and for every $\epsilon > 0$, there exists an ϵ -pseudo orbit (ϵ -chain) from x to y . For given $\delta > 0$, the sequence $\{x_i\}_{i=-\infty}^{\infty}$ in X is called δ -average-pseudo-orbit of f if there exists a positive integer $N = N(\delta)$ such that for all $n \geq N$ and $k \in \mathbb{N}$, we have

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta,$$

and

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{-i+k}), x_{-i+k+1}) < \delta.$$

We say that f has the average shadowing property (ASP) if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ -average-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ -shadowed in average by the orbit of some point $y \in X$, that is

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon,$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^{-i}(y), x_{-i}) < \epsilon.$$

Throughout this paper, $N_\epsilon(x)$ denotes the open ball with center x and radius ϵ . It is well known that if f has ASP, then f^n has ASP for every positive integer n .

Let M be a smooth compact manifold of dimension d . We denote by $\text{Diff}^1(M)$ the space of all C^1 diffeomorphism endowed with C^1 topology. We say that a compact invariant set Λ is *hyperbolic* if there is a Df -invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle restricted to Λ and a constant $\lambda < 1$ such that (for some choice of a Riemannian metric on M) for every $x \in \Lambda$

$$\|Df|E_x^s\| < \lambda$$

and

$$\|Df^{-1}|E_x^u\| < \lambda.$$

$f \in \text{Diff}^1(M)$ is said to be Anosov if M is a hyperbolic set for f .

A periodic point p of period k is a hyperbolic periodic point if the derivative of f^k at p has no eigenvalue with norm equal to one. If all of the eigenvalues of $Df^k(p)$ are less (greater) than one, then p is sink (source). If p is sink, then an open neighborhood of p is absorbed by p . Such neighborhood is called basin of p . the hyperbolic periodic point p which is neither sink nor source is called saddle.

For the diffeomorphism f and the point $x \in M$, the stable and unstable sets of x are given by

$$W^s(x) = \{y \in M : d(f^k(x), f^k(y)) \rightarrow 0 \text{ as } k \rightarrow +\infty\}$$

and

$$W^u(x) = \{y \in M : d(f^k(x), f^k(y)) \rightarrow 0 \text{ as } k \rightarrow -\infty\},$$

respectively. Let $\mathcal{O}(p)$ be a hyperbolic periodic orbit of diffeomorphism f , then the homoclinic class of $\mathcal{O}(p)$ is the set

$$H(p, f) = \overline{W^s(\mathcal{O}(p)) \cap W^u(\mathcal{O}(p))}.$$

A compact invariant set Λ admits an (l, λ) -dominated splitting for f if there exists a decomposition $T_\Lambda M = F \oplus G$ and a number $\lambda < 1$ such that for every $x \in \Lambda$:

$$\|Df^l|_{F(x)}\| \|Df^{-l}|_{G(f^l(x))}\| < \lambda.$$

A property P is said to be generic for elements of a topological space M if the set of points $x \in M$ satisfying P is residual, i.e, it includes a countable intersection of open and dense subsets of M .

Every C^1 -generic diffeomorphism comes in one of two types: tame diffeomorphisms, which have a finite number of homoclinic classes and whose nonwandering sets admits partitions into a finite number of disjoint transitive sets; and wild diffeomorphisms, which have an infinite number of homoclinic classes and whose nonwandering sets admit no such partitions. For a greater detail, see [7].

As we mentioned earlier, we consider the problem of the relation between ASP and hyperbolicity generically. We prove the following theorem. Let M be a compact manifold. There is a residual subset $R \subset \text{Diff}^1(M)$ such that if $f \in R$ is tame and is NUH on $\text{Per}(f)$, then the following two conditions are equivalent:

- (1) f has average shadowing property
- (2) f is Anosov.

3. PROOF OF THEOREM 2.1

Remark that ASP has no sources and sinks [20]. If $f \in \text{diff}^1(M)$ has ASP, then f has no sink and source.

There is a residual set $R_1 \subset \text{Diff}^1(M)$ such that every $f \in R_1$ satisfies the following properties:

- (1) $\overline{\text{Per}(f)} = \Omega(f)$ [18],
- (2) any ergodic invariant measure μ of f is the limit of a sequence of ergodic invariant measures supported by periodic points [6].
- (3) A compact f -invariant set Λ is chain transitive if and only if Λ is the Hausdorff limit of a sequence of periodic orbits of f [4].

We say that f is non uniformly hyperbolic (NUH) on a set $H \subset M$ if there is $\lambda > 0$ and invariant splitting $TM = E \oplus F$ such that for each $x \in H$,

$$\liminf \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))|_{E(f^i(x))}\| < -\lambda.$$

$$\liminf \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))|_{F(f^i(x))}\| > \lambda.$$

We say that f is NUH on $\text{Per}(f)$ if in the above definition $H = \text{Per}(f)$.

Now we prove the main theorem.

Proof of Theorem 2.1 This is well known that hyperbolicity implies average shadowing property. Therefore we just need to show that generically if f is tame and is NUH on $\text{Per}(f)$ and ASP, then f is Anosov. Let R_1 be as in Lemma 3.2. Consider $f \in R_1$ is tame and has ASP. It is well known that M is chain transitive for any ASP (see [16]). So by Lemma 3.2, there are hyperbolic saddles $\{p_n\}$ such that $\lim_{n \rightarrow \infty} H(p_n, f) = M$. Note that f has neither sink nor source. Since f is tame, then it has finite homoclinic classes which implies that $M = H(f, p)$ for some hyperbolic saddle point p . On other hand this is proved in [1] that there is a residual set $R' \subset \text{Diff}^1(M)$ such that for $f \in R'$ any chain transitive k has a dominated splitting or else it is contained in closure of infinite numbers of periodic sinks or sources. Put $R_2 = R' \cap R_1$. If $f \in R_2$ is tame and has a ASP, since $M = H(p, f)$ for some hyperbolic saddle point p and f has neither sink nor source so f has dominated splitting $TM = E \oplus F$. Suppose that $f \in R_2$ is tame and has ASP and is NUH on $\text{Per}(f)$. We can see that $\overline{\text{Per}(f)} = M$. To complete the proof of Theorem, it is enough to show that Df is contracting on E and Df is expanding on F . Suppose that f is not expanding on F , so there exists $b \in M$ such that $\|Df^k|_{F(b)}\| \leq 1$ for every $k \in \mathbb{N}$. Let μ be accumulation point of sequence $\{\frac{1}{l} \sum_{i=0}^{l-1} \delta_{f^i(b)} : l \in \mathbb{N}\}$. Since $\|Df^k|_{F(b)}\| \leq 1$ for every $k \in \mathbb{N}$, so we have

$$\int \log \|Df|_{F(x)}\| d\mu \leq 0.$$

By the ergodic decomposition theorem, there is an ergodic measure ν such that

$$\int \log \|Df|_{F(x)}\| d\nu \leq 0.$$

We can take a sequence of ergodic invariant measures $\{\nu_n\}$ supported by periodic points p_n and $\nu_n \rightarrow \nu$ in weak* topology. So we have

$$\lim_{n \rightarrow \infty} \int \log \|Df|_{F(x)}\| d\nu_n = \int \log \|Df|_{F(x)}\| d\nu.$$

On the other hand we have

$$\int \log \|Df|_{F(x)}\| d\nu_n = \frac{1}{m_n} \sum_{i=0}^{m_n-1} \log \|Df(f^i(p_n))|_{F(f^i(p_n))}\|,$$

where m_n is the period of p_n . Remark that we can assume $m_n \rightarrow \infty$ as $n \rightarrow \infty$.

Since f is NUH on $\text{Per}(f)$, we have $\lim_{n \rightarrow \infty} \int \log \|Df|_{F(x)}\| d\nu_n > 0$. Therefore $\int \log \|Df|_{F(x)}\| d\nu > 0$, which is a contradiction. We can see similarly Df is contracting on E . Therefore f is Anosov.

REFERENCES

- [1] F. Abdenur, C. Bonatti, And S. Crovisier, Global dominated splittings and the C^1 newhouse phenonenon., Proc. Am. Math. Soc., 134 (2006), 2229-2237.
- [2] A. Arbieto, R. Ribeiro, Flows with the (asymptotic) average shadowing property on three-dimensional closed manifolds, Dynamical Systems 2011, 1-8.
- [3] M.L. Blank, Metric properties of ϵ -trajectory of dynamical systems with stochastic behavior, Ergodic Theory Dyn. Syst., 1988, 8: 365-378.
- [4] Crovisier S., Periodic orbit and chain transitive sets of C^1 -diffeomorphisms, Publ. Math.Inst. Hautes Sci. 104(2006), 87-141.
- [5] A. Zamani Bahabadi, Average shadowing and asymptotic average shadowing and dominated splittings for conservative systems.J Adv Res Pure Math 2012;4(3):34-39.
- [6] K. Lee and X. Wen, Shadowable Chain Transitive Sets of C^1 -Generic Diffeomorphisms. Bull. Korean Math. Soc. 49 (2012), 263-270.
- [7] F. Abdenur, Generic robustness of spectral decompositions, Ann. Scient. Ec. Norm. Sup.36(2003), 213-224.
- [8] K. Sakai, Diffeomorphisms with the shadowing property, J. Austral. Math. Soc. Ser. A 61(3) (1996), 396-399.
- [9] K. Sakai, Diffeomorphisms with the average-shadowing property on two-dimensional closed manifolds, Rocky Mountain J. Math. 30(3) (2000), 1129-1137.
- [10] K. Sakai, Diffeomorphisms with weak shadowing, Fund. Math. 168(1) (2001), 57-75.
- [11] T. Eirola, O. Nevanlinna, and S.Y. Pilyugin, Limit shadowing property, Numer. Funct. Anal. Optim. 18(12) (1997), 75-92.

- [12] S.Y. Pilyugin and O.B. Plamenevskaya, Shadowing is generic, *Topology Appl.* 97(3) (1999), 253-266.
- [13] S.Y. Pilyugin, A.A. Rodionova, and K. Sakai, Orbital and weak shadowing properties, *Discrete Contin. Dyn. Syst.* 9(2) (2003), 287-308.
- [14] S.Y. Pilyugin and S.B. Tikhomirov, Sets of vector fields with various properties of shadowing of pseudotrajectories, *Dokl. Akad. Nauk* 422(1) (2008).
- [15] B. Hasselblatt and A. Katok, Introduction to the Modern Theory of Dynamical Systems, *Encyclopedia of Mathematics and its Applications*, Vol. 54, Cambridge University Press, Cambridge, 1995.
- [16] Y. Zhang, On the average-shadowing property, *Acta Scientiarum Naturalium Universitatis Pekinensis* 37 (2001), 648-651.
- [17] A. Zamani Bahabadi, Divergence-free vector fields with average and asymptotic average shadowing property, *Chaos, Solitons & Fractals* 45 (2012).
- [18] C. Pugh, The closing lemma, *Amer. J. Math.* 89 (1967), 965-1009.
- [19] A. Castro, K. Oliveria and V. Pinheiro, Shadowing by non-uniformly hyperbolic periodic points and uniform hyperbolicity. *Nonlinearity* 20(2007) 75-85.
- [20] A. Zamani Bahabadi, On the average shadowing property and hyperbolicity, *New Zealand Journal of Mathematics*, 42 (2012), 189-194.