

# ATTRACTORS IN RELATIVE SEMI-DYNAMICAL SYSTEMS

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(Received: 21 October 2013, Accepted: 17 December 2013)

**ABSTRACT.** In this paper we present a new concept of  $\omega$ -limit set and attracting set. Relative semi-dynamical system and some of its properties are presented. Relative semi-attractor set is considered. semi-attractor is a natural generalization of the concept of attractor in the work of Huang Tusen [9].

**AMS Classification:** 34D45.

**Keywords:**  $\omega_\mu$ -Limit Set; Relative Semi-Attractor; Observer; Invariant Set

## 1. INTRODUCTION

Attractors have played a major role in natural and scientific phenomena. This is why attractors have received many attentions in the recent years from researchers who studying dynamical systems. Many works have been considered from different points of view, an interested reader may see [1, 3, 9, 10, 11], for details.

Let  $X$  be a compact metric space,  $Y \subset X$ , and  $f : X \times R \longrightarrow X$  be a flow on  $X$ .  $\omega(Y, f)$  is defined as a maximal invariant set in the closure of  $Y.[0, \infty)$  and it is

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*JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES*  
VOL. 12, NUMBER 1 (2014) 1-10.

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called  $\omega$ -limit of  $Y$ . A set  $A$  is called an attracting set if there exists a neighborhood  $N$  of  $A$  such that  $A = \omega(N, f)$ .

Many efforts have been taken in recent years to consider relative dynamical systems and their properties [5, 6, 7, 8].

In a relative dynamical system, we have an **observer**  $\mu$ , which is a function:  $\mu : X \rightarrow [0, 1]$ . It's well known that if  $\mu$  is an observer then a topology  $\tau_\mu$  of the family of the subsets of  $\mu$  can be defined accordingly [6, 8].

In this work we try to define limit sets on relative topology. We use of this concept to define relative attractors for relative dynamical systems.

The significant result of this study is that the given definition for relative attractors is in direct connection with that of [9], and more generally, generalizes it. This work ends with questions: *In what conditions components of an relative attractors are relative attractors themselves?* And conversely *When the union of several relative attractors build up a relative attractors?*

In this direction Tusen has been considered some properties of attractors and quasi-attractors [9]. In this paper we are going to present attractors based on relative topology [6, 7]. In fact  $(X, \tau_\mu)$  is called a relative topological space and members of  $\tau_\mu$  are called  $\mu$ -open sets.

## 2. NOTATIONS

In this section we give some definitions and lemmas which are necessary in the following sections.

Let  $X$  be a compact metric space with metric  $\rho$ . The metric on  $X \times X$  is denoted by  $\rho$  also. For  $Y \subseteq X$ ,  $Cl(Y)$  denotes the closure of  $Y$ .

Flows on  $X$  are denoted by  $f : X \times R \longrightarrow X$  such that  $f(x, 0) = x$  and  $f(x, s + t) = f(f(x, s), t)$ .

For  $Y \subseteq X$  and  $J \subseteq R$ ,  $Y \cdot J \equiv f(Y \times J)$ . A set  $Y$  is invariant under  $f$  if  $Y \cdot R = Y$ .

**Definition 2.1.** [4]. A dynamical system is a pair  $(\{f^s\}_{s \geq 0}, X)$ , where  $X$  is a state space and  $f^s : X \rightarrow X$  is a family of evolution operators satisfies the following properties:

- i)  $f^0 = id$ , where  $id$  is the identity map on  $X$ .
- ii)  $f^{t+s} = f^t \circ f^s$ . It means that

$$f^{t+s}(x) = f^t(f^s(x))$$

for all  $x \in X$  and  $t, s \geq 0$ .

**Definition 2.2.** [1]. For  $Y \subseteq X$ ,  $\omega(Y, f)$  is defined to be the maximal invariant set in the closure of  $Y \cdot [0, +\infty]$ . This set is called the  $\omega$ -limit set of  $Y$ .

The set  $\omega(Y, f)$  can be expressed by  $\bigcap \{Cl(Y \cdot [t, +\infty)) | t \geq 0\}$ . For  $Y = \{x\}$ ,  $\omega(x, f)$  is the usual  $\omega$ -limit set of  $x$ .

**Definition 2.3.** [1]. A set  $A$  is called an attractor for  $f$  if  $A$  admits a (closed) neighborhood  $N$  such that  $A = \omega(N, f)$ .

Let  $\{X, \tau\}$  be a topological space and  $f : X \times R \longrightarrow X$  be a flow. If we define  $f^s(x) = f(s, x)$ , then  $(\{f^s\}_{s \geq 0}, X)$  is a semi-dynamical system.

In this study we assume that the relative sets are members of  $L^X$ , where  $L$  is  $[0, 1]$  and  $X$  is a non-empty set.

**Definition 2.4.** [6]. An observer of a set  $X$  is a function  $\mu : X \longrightarrow [0, 1]$ .

**Definition 2.5.** Let  $\mu \in L^X$  and  $\tau_\mu$  be a collection of the elements of  $L^X$  which are contained in  $\mu$  ( $\lambda \in \tau_\mu$  implies  $\lambda \subseteq \mu$ ). If  $\tau_\mu$  has the following properties: [8]

- i)  $\mu, \chi_\emptyset \in \tau_\mu$ , where  $\chi_\emptyset(x) = 0$ , for all  $x \in X$ .
- ii) If  $\lambda_1, \lambda_2 \in \tau_\mu$ , then  $\lambda_1 \cap \lambda_2 \in \tau_\mu$ , where  $\lambda_1 \cap \lambda_2 : X \longrightarrow [0, 1]$  is defined by  $(\lambda_1 \cap \lambda_2)(x) = \min\{\lambda_1(x), \lambda_2(x)\}$ .
- iii) If  $\{\lambda_\gamma \mid \gamma \in \Gamma\} \subset \tau_\mu$ , then  $\bigcup_{\gamma \in \Gamma} \lambda_\gamma \in \tau_\mu$  where  $\bigcup \lambda_\gamma : X \longrightarrow [0, 1]$  is defined by  $\bigcup \lambda_\gamma(x) = \sup\{\lambda_\gamma(x) \mid \gamma \in \Gamma\}$ . Then  $\tau_\mu$  is called a  $\mu$ -relative topology for  $\mu$  and elements of  $\tau_\mu$  are called  $\mu$ -open sets.

**Example 2.6.** [8]. Let  $X$  be the set of all differentiable real valued functions on  $(1, \infty)$  with positive derivative of order one and let  $L$  be the set of all real valued functions on  $(1, \infty)$ , consider  $\mu : X \rightarrow L$  by  $\mu(f) = f' + e^x$ . For non-negative integer  $n$  defining  $\lambda^n : X \rightarrow L$  by  $(\lambda^n(f))(x) = f'(x) + \sum_{i=0}^n \frac{x^i}{i!}$ .

If  $\tau_\mu = \{\mu, \chi_\emptyset, \lambda^n : n = 0, 1, 2, \dots\}$ , then  $(\mu, \tau_\mu)$  is a relative topological space.

Note that  $\lambda^n \cap \lambda^m = \lambda^m$ ,  $\lambda^n \cup \lambda^m = \lambda^n$  whenever  $n > m$ .

Let  $\lambda \subset \mu$  and  $\lambda^\mu : X \rightarrow [0, 1]$  be a function defined by  $\lambda^\mu(x) = \mu(x) - \lambda(x)$ . Then  $\lambda$  is called relative closed set if  $\lambda^\mu$  is relative open set, i.e.  $\lambda^\mu \in \tau_\mu$  [6].

**Definition 2.7.** [8]. Suppose  $(\mu, \tau_\mu)$  and  $(\gamma, \tau_\gamma)$  are two relative topological spaces, where  $\mu \in [0, 1]^X$  and  $\gamma \in [0, 1]^Y$ . Then a mapping  $f : X \longrightarrow Y$  is called  $(\mu, \gamma)$ -continuous if:  $f^{-1}(\eta) \cap \mu \in \tau_\mu$ , for all  $\eta \in \tau_\gamma$ , where  $f^{-1}(\eta)(x) = \eta(f(x))$ .

Let  $(X, \tau_\mu)$  be a relative topological space and  $(\{f^s\}_{s \geq 0}, X)$  be a collection of functions on  $X$  with the following conditions:

- i)  $f^0(x) = x$ ; for every  $x \in X$ .
- ii)  $f^{s+t}(x) = f^s(f(x, t)) = f(x, s + t)$ ; for every  $s, t \geq 0$ .

Then  $(\{f^s\}_{s \geq 0}, X, \tau_\mu)$ , is called a relative semi-dynamical system.

**Example 2.8.** [6]. Let  $f : [0, 1] \rightarrow [0, 1]$  be a map defined by  $f(x) = \begin{cases} x & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$ ,

and let  $\mu : [-1, 1] \rightarrow [0, 1]$  be the constant function  $\mu(x) \frac{1}{2}$ , where  $x \in [-1, 1]$ . Moreover let  $\tau_\mu = \{\lambda \subset \mu : \lambda(x) = 0 \text{ if } x \leq 0\} \cup \{\chi_\phi, \mu\}$ . Then  $(f, [-1, 1], \tau_\mu)$  is a relative semi-dynamical system.

### 3. RELATIVE INVARIANT SETS

In this section, the invariant sets for  $(\{f^s\}_{s \geq 0}, X, \tau_\mu)$  and their properties are considered at first, and what comes next is the concept of closure of a relative set. In this paper,  $X$  denotes a nonempty set.

**Definition 3.1.** Let  $\eta \subseteq \mu$  be an observer in  $\tau_\mu$ . We define the observer  $f^s(\eta)$  by  $f^s(\eta)(x) := \sup(\{\eta(y) \mid f^s(y) = x\}, \{0\})$ , where  $s \in \mathbb{R}^+$ .

**Definition 3.2.** Let  $\eta \subseteq \mu$ ,  $\eta$  is invariant under  $f$  if  $f^s(\eta) \subseteq \eta$ .

In the following lemma, one of the properties of relative topology which we need in Definition 3.7 and it's next proposition is considered.

**Lemma 3.3.** Any finite union of relative closed sets is a relative closed set.

*Proof.* Let  $\lambda_1, \lambda_2$  are two relative closed sets. So  $\lambda_1^\mu, \lambda_2^\mu \in \tau_\mu$ , then  $\lambda_1^\mu \cap \lambda_2^\mu \in \tau_\mu$ . We show that  $(\lambda_1^\mu \cap \lambda_2^\mu)(x) = (\lambda_1 \cup \lambda_2)^\mu(x)$ , for all  $x \in X$ . If  $(\lambda_1^\mu \cap \lambda_2^\mu)(x) = \lambda_1^\mu(x)$  then  $\lambda_1^\mu(x) \leq \lambda_2^\mu(x)$ , so  $\lambda_2(x) \leq \lambda_1(x)$ . Therefore  $(\lambda_1 \cup \lambda_2)(x) = \lambda_1(x)$ . Then  $(\lambda_1 \cup \lambda_2)^\mu(x) = \lambda_1^\mu(x) = (\lambda_1^\mu \cap \lambda_2^\mu)(x)$ . Without loss of generality this prove that  $\lambda_1^\mu \cap \lambda_2^\mu = (\lambda_1 \cup \lambda_2)^\mu$ . So the union of two relative closed sets is a relative closed set. By induction any finite union of relative closed sets is a relative closed set.  $\square$

We recall [5] that the "intersection of all relative closed sets  $\gamma$  such that  $\lambda \subseteq \gamma$  is called the closure of  $\lambda$  and denoted by  $\bar{\lambda}$ . One must pay attention to this point that in the observational approach  $\lambda$  is not necessary a subset of  $\bar{\lambda}$ ".

**Lemma 3.4.** If  $\lambda_1, \lambda_2$  are two observers then  $\overline{\lambda_1 \cup \lambda_2} = \bar{\lambda}_1 \cup \bar{\lambda}_2$ .

*Proof.* Suppose  $\overline{\lambda_1 \cup \lambda_2} = \bigcap_{i \in \Gamma} \gamma_i$  where  $\lambda_1 \cup \lambda_2 \subseteq \gamma_i$  and  $\gamma_i$  are relative closed sets, for all  $i \in \Gamma$ . Therefore  $\lambda_1 \subseteq \gamma_i$  and  $\lambda_2 \subseteq \gamma_i$  for all  $i \in \Gamma$ , so  $\overline{\lambda_1} \subseteq \bigcap_{i \in \Gamma} \gamma_i$  and  $\overline{\lambda_2} \subseteq \bigcap_{i \in \Gamma} \gamma_i$ . Then  $\overline{\lambda_1} \cup \overline{\lambda_2} \subseteq \bigcap_{i \in \Gamma} \gamma_i$ .

On the other hand, we show that  $\overline{\lambda_1 \cup \lambda_2} \subseteq \overline{\lambda_1} \cup \overline{\lambda_2}$ . If  $\overline{\lambda_1 \cup \lambda_2}$  is not a subset of  $\overline{\lambda_1} \cup \overline{\lambda_2}$ , then there exist  $t \in [0, 1]$  and  $x \in X$  such that  $\overline{\lambda_1 \cup \lambda_2}(x) > t > (\overline{\lambda_1} \cup \overline{\lambda_2})(x)$ . So there exist relative closed sets  $\gamma_1 \supseteq \lambda_1$  and  $\gamma_2 \supseteq \lambda_2$  such that  $t > \gamma_1(x)$  and  $t > \gamma_2(x)$ . Thus  $\overline{\lambda_1 \cup \lambda_2}(x) > t > (\gamma_1 \cup \gamma_2)(x)$  which is a contradiction. By Lemma 3.3.,  $\gamma_1 \cup \gamma_2$  is a relative closed set containing  $\lambda_1 \cup \lambda_2$ .  $\square$

In order to prove Proposition 3.8., we need the following lemma.

**Lemma 3.5.** Let  $\{\eta_i\}_{i \in N} \subseteq \tau_\mu$  then  $f^s(\bigcup_{i \in N} \eta_i) = \bigcup_{i \in N} f^s(\eta_i)$ .

*Proof.* We first show that  $f^s(\eta_1 \cup \eta_2) = f^s(\eta_1) \cup f^s(\eta_2)$ . Let  $x \in X$  and  $f^s(\eta_1)(x) = t_1$ ,  $f^s(\eta_2)(x) = t_2$ . Without loss of generality suppose that  $t_1 < t_2$ . This implies that if  $f^s(y) = x$  for some  $y \in X$ , then  $\eta_1(y) \leq t_1 < t_2$  and  $\eta_2(y) \leq t_2$  and therefore  $(\eta_1 \cup \eta_2)(y) \leq t_2$ . Hence  $f^s(\eta_1 \cup \eta_2)(x) \leq t_2$ . On the other hand if  $u < t_2$ , then there exists  $y \in X$  such that  $f^s(y) = x$  and  $u < \eta_2(y) \leq t_2$ . So  $u < f^s(\eta_1 \cup \eta_2)(x) \leq t_2$  for all  $u < t_2$ . Hence

$$f^s(\eta_1 \cup \eta_2)(x) = t_2 = f^s(\eta_1) \cup f^s(\eta_2)(x).$$

Then  $f^s(\eta_1 \cup \eta_2) = f^s(\eta_1) \cup f^s(\eta_2)$ . Therefore by induction  $f^s(\bigcup_{i=1}^n \eta_i) = \bigcup_{i=1}^n f^s(\eta_i)$  for all  $n \in N$ . Now we prove the equality for infinite union.

Since  $\bigcup_{i=1}^\infty \eta_i \supseteq \bigcup_{i=1}^n \eta_i$  then  $f^s(\bigcup_{i=1}^\infty \eta_i) \geq f^s(\bigcup_{i=1}^n \eta_i)$ . So  $f^s(\bigcup_{i=1}^\infty \eta_i) \geq \bigcup_{i=1}^n f^s(\eta_i)$  for all  $n \in N$ . This implies that  $f^s(\bigcup_{i=1}^\infty \eta_i) \geq \bigcup_{i=1}^\infty f^s(\eta_i)$ .

Conversely, if  $\bigcup_{i=1}^\infty f^s(\eta_i)$  is not greater or equal than  $f^s(\bigcup_{i=1}^\infty \eta_i)$  then there exist  $x \in X$  and  $t \in [0, 1]$  such that  $f^s(\bigcup_{i=1}^\infty \eta_i)(x) > t > \bigcup_{i=1}^\infty f^s(\eta_i)(x)$ . Therefore there exists  $y \in X$  that  $f^s(y) = x$  and  $(\bigcup_{i=1}^\infty \eta_i)(y) > t > f^s(\eta_i)(x)$  for all  $i \geq 1$ . So there is a  $k \in N$  that  $\eta_k(y) > t > f^s(\eta_i)(x)$  for all  $i \geq 1$ , and this implies that  $\eta_k(y) > t > f^s(\eta_k)(x)$ , where  $f^s(y) = x$ . By definition of  $f^s(\eta_k)(x)$  this is a contradiction.  $\square$

The first question which comes into mind is that; *What sets are relative invariant?*

**Proposition 3.6.** If  $\eta_1, \eta_2$  are two invariant observers. Then their union and intersection are invariant observers.

*Proof.* Since  $\eta_1$  and  $\eta_2$  are invariant observers then  $f^s(\eta_1) \subset \eta_1$  and  $f^s(\eta_2) \subset \eta_2$ . So  $f^s(\eta_1) \cup f^s(\eta_2) \subseteq \eta_1 \cup \eta_2$ . Therefore it is sufficient to show that  $f^s(\eta_1 \cup \eta_2) \subseteq f^s(\eta_1) \cup f^s(\eta_2)$ . If  $f^s(\eta_1 \cup \eta_2)(x) > (f^s(\eta_1) \cup f^s(\eta_2))(x)$ , for some  $x \in X$ , then there exists  $t \in (0, 1)$  such that  $f^s(\eta_1 \cup \eta_2)(x) > t > (f^s(\eta_1)(x))$  and  $f^s(\eta_1 \cup \eta_2)(x) > t > (f^s(\eta_2)(x))$ . So there exists  $y \in X$  such that  $f^s(y) = x$  and  $(\eta_1 \cup \eta_2)(y) > t$ . Suppose  $\eta_1(y) = \max\{\eta_1(y), \eta_2(y)\}$ . This implies that  $\eta_1(y) > t > f^s(\eta_1)(x)$  that is a contradiction based on the definition of  $f^s(\eta_k)(x)$ . So  $f^s(\eta_1 \cup \eta_2) \subseteq f^s(\eta_1) \cup f^s(\eta_2) \subseteq \eta_1 \cup \eta_2$ .

Now we show that  $f^s(\eta_1 \cap \eta_2) \subseteq \eta_1 \cap \eta_2$ . If  $\lambda \subseteq \gamma$  then for every  $y \in X$ ,  $\lambda(y) \leq \gamma(y)$ . So  $\sup\{\lambda(y) \mid f^s(y) = x\} \leq \sup\{\gamma(y) \mid f^s(y) = x\}$ . This implies that  $f^s(\lambda) \subseteq f^s(\gamma)$ . Since  $\eta_1 \cap \eta_2 \subseteq \eta_1$  and  $\eta_1 \cap \eta_2 \subseteq \eta_2$ , then  $f^s(\eta_1 \cap \eta_2) \subseteq f^s(\eta_1)$  and  $f^s(\eta_1 \cap \eta_2) \subseteq f^s(\eta_2)$ . Then  $f^s(\eta_1 \cap \eta_2) \subseteq f^s(\eta_1) \cap f^s(\eta_2) \subseteq \eta_1 \cap \eta_2$ .  $\square$

We need to define the limit sets for relative semi-dynamical systems.  $\omega$ -limit set and their properties are defined in various ways, for example, see [9].

**Definition 3.7.** Let  $\tau_\mu$  be a relative topological space. For  $\eta \subseteq \mu$ ,  $\omega_\mu(\eta, f)$  is  $\bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\eta)}$ . For  $\eta(x_0) = \chi_{x_0} \in [0, 1]$ , we have  $\omega_\mu(\eta(x_0), f) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\eta(x_0))}$ .

In the following proposition, some properties of  $\omega_\mu$ -limit sets are presented.

**Proposition 3.8.** i) Let  $\eta_i \subseteq \mu$ ,  $\omega_\mu(\bigcup_{i=1}^n \eta_i, f) = \bigcup_{i=1}^n \omega_\mu(\eta_i, f)$ .  
ii) If  $\eta_1 \subset \eta_2$ , then  $\omega_\mu(\eta_1, f) \subset \omega_\mu(\eta_2, f)$ .

*Proof.* i) Lemma 3.5 implies that  $f^s(\bigcup_{i=1}^n \eta_i) = \bigcup_{i=1}^n f^s(\eta_i)$  for all  $n \in N$ . Also by Lemma 3.4.  $\overline{\bigcup_{i=1}^n \eta_i} = \bigcup_{i=1}^n \overline{\eta_i}$ .

So  $\overline{\bigcup_{s \geq t} f^s(\bigcup_{i=1}^n \eta_i)} = \overline{\bigcup_{s \geq t} \bigcup_{i=1}^n f^s(\eta_i)} = \overline{\bigcup_{i=1}^n \bigcup_{s \geq t} f^s(\eta_i)} = \bigcup_{i=1}^n \overline{\bigcup_{s \geq t} f^s(\eta_i)}$ . Therefore  $\omega_\mu(\bigcup_{i=1}^n \eta_i, f) = \bigcup_{i=1}^n \omega_\mu(\eta_i, f)$ .

ii) If  $\eta_1 \subset \eta_2$ , then for every  $y \in X$ ,  $\eta_1(y) \leq \eta_2(y)$ . So

$$\sup\{\eta_1(y) \mid f^s(y) = x\} \leq \sup\{\eta_2(y) \mid f^s(y) = x\}.$$

This implies that  $f^s(\eta_1) \subseteq f^s(\eta_2)$  and therefore  $f^s(\eta_1)(x) \subseteq f^s(\eta_2)(x)$ , for all  $x \in X$ . Hence  $(\bigcup_{s \geq t} f^s(\eta_1))(x) = \sup_{s \geq t} f^s(\eta_1)(x) \leq \sup_{s \geq t} f^s(\eta_2)(x) = (\bigcup_{s \geq t} f^s(\eta_2))(x)$ . It follows that  $\overline{\bigcup_{s \geq t} f^s(\eta_1)} \subseteq \overline{\bigcup_{s \geq t} f^s(\eta_2)}$ . So  $\bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\eta_1)} \subseteq \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\eta_2)}$ , because  $(\bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\eta_1)})(x) = \inf_{t \geq 0} \sup_{s \geq t} f^s(\eta_1)(x) \leq \inf_{t \geq 0} \sup_{s \geq t} f^s(\eta_2)(x)$ , for

all  $x \in X$ .

Hence  $\omega_\mu(\eta_1, f) \subset \omega_\mu(\eta_2, f)$ . □

#### 4. THE RELATIVE SEMI-ATTRACTOR

The notion of attractor set have extended from different ways [1, 3, 9]. We would like to present a relative extension of it.

**Definition 4.1.** If  $(\{f^s\}_{s \geq 0}, X, \tau_\mu)$  be a relative semi-dynamical system.  $\lambda \subseteq \mu$  is called relative semi-attractor if there exists relative open set  $\eta$ , such that  $\lambda \subseteq \eta \subseteq \mu$  and  $\omega_\mu(\eta, f) = \lambda$ .

This definition is the natural generalization of the concept of attractor which was introduced at first by Huang Tusen [9]. If  $\mu = \chi_X$ ,  $\tau_\mu = \{\chi_U \mid U \in \tau\}$  then  $A$  is an attractor, if there exists a closed set as  $N$  such that  $\chi_A = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\chi_N)}$ . We first prove the following lemma in order to prove the claim we just made.

**Lemma 4.2.** Let  $(\{f^s\}_{s \geq 0}, X)$  be a semi-dynamical systems and  $(\{f^s\}_{s \geq 0}, X, \tau_\mu)$  be a relative semi-dynamical systems, where  $\tau_\mu = \{\chi_U \mid U \subseteq X \text{ is an open set in metric space } (X, \rho)\}$ . Then the following properties hold:

- i)  $A \subseteq B$  if and only if  $\chi_A \subseteq \chi_B$  in relative topology  $(X, \tau_\chi)$ .
- ii) The set  $A$  is an open(closed)set if and only if  $\chi_A$  is a relative open(relative closed) set.
- iii) If  $Cl(B) = C$  then  $\overline{\chi_B} = \chi_C$ .
- iv) Let  $N \subseteq X$ , if  $N \cdot [t, +\infty) = C$  then  $\bigcup_{s \geq t} f^s(\chi_N) = \chi_C$ .

*Proof.* The proof of parts *i* and *ii* are clear by their definitions.

iii) Suppose  $\{B_i\}_{i \in I}$  is the family of all closed set containing  $B$ , such that  $Cl(B) = \bigcap_{i \in I} B_i$ . If  $\chi_B \subseteq \chi_D$ , where  $\chi_D$  is a relative closed set, then  $B \subseteq D$  and  $D$  is a closed set. So there exists  $i \in I$  such that  $D = B_i$ . Thus  $\{\chi_{B_i}\}_{i \in I}$  is the family of all relative closed set containing  $\chi_B$ . So  $\overline{\chi_B} = \bigcap_{i \in I} \chi_{B_i} = \chi_{\bigcap_{i \in I} B_i} = \chi_C$ .

iv) Let  $x \in N \cdot [t, +\infty)$ . Then there exist  $n \in N$  and  $s \in [t, \infty)$  such that  $x = f(n, s)$ . So  $x = f^s(n)$ , then  $\sup(\{\chi_N(y) \mid f^s(y) = x\}, \{0\}) = 1$ .

Therefore  $f^s(\chi_N)(x) = 1$  and consequently

$\bigcup_{s \geq t} f^s(\chi_N)(x) = 1$ . So  $\chi_C \subseteq \bigcup_{s \geq t} f^s(\chi_N)$ . Conversely if  $\bigcup_{s \geq t} f^s(\chi_N)(x) = 1$  then we can find  $s \geq t$  such that  $f^s(\chi_N)(x) = 1$ . So there exists  $n \in N$  such that

$f^s(n) = x$  consequently  $f(n, s) = x$ . Therefore  $x \in N \cdot [t, +\infty)$  and  $x \in C$ . This implies that  $\bigcup_{s \geq t} f^s(\chi_N) \subseteq \chi_C$ .  $\square$

**Theorem 4.3.** Let  $(\{f^s\}_{s \geq 0}, X)$  be a semi-dynamical systems and  $(\{f^s\}_{s \geq 0}, X, \tau_\mu)$  be a relative semi-dynamical systems, where  $\tau_\mu = \{\chi_U | U \subseteq X \text{ is an open set in metric space } (X, \rho)\}$ . Then  $A \subseteq X$  is an attractor for a semi-dynamical system  $(\{f^s\}_{s \geq 0}, X)$  if and only if  $\chi_A$  is a relative semi-attractor for the relative semi-dynamical system  $(\{f^s\}_{s \geq 0}, X, \tau_\mu)$ .

*Proof.* We prove the necessity, the proof of sufficiency should be done in a similar way.

Suppose  $A$  is an attractor set and  $\omega(N, f) = A$ . We show that  $\omega_\mu(\chi_N, f) = \chi_A$ . Let  $x \in A$ . Since  $\omega(N, f) = \bigcap \{Cl(N \cdot [t, +\infty)) | t \geq 0\}$ , then  $x \in Cl(N \cdot [t, +\infty))$  for all  $t \geq 0$ . If  $B$  is a closed set contain  $N \cdot [t, +\infty)$  then  $x \in B$  and by Lemma 4.2.  $\chi_B$  is a relative closed set contain  $\bigcup_{s \geq t} f^s(\chi_N)$ . So  $\overline{\bigcup_{s \geq t} f^s(\chi_N)}(x) = 1$  for all  $t \geq 0$ . This implies that  $\bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\chi_N)}(x) = 1$ , so  $\chi_A \subseteq \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\chi_N)}$ . On the other hand, if  $\bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\chi_N)}(x) = 1$  then  $\overline{\bigcup_{s \geq t} f^s(\chi_N)}(x) = 1$  for all  $t \geq 0$ . So  $\chi_B(x) = 1$  for all relative closed set containing  $\bigcup_{s \geq t} f^s(\chi_N)$  then  $x \in B$  for all closed set  $B$  containing  $N \cdot [t, +\infty)$ . Therefore  $x \in Cl(N \cdot [t, +\infty))$  for all  $t \geq 0$ . Then  $x \in \bigcap \{Cl(N \cdot [t, +\infty)) | t \geq 0\}$ , so  $x \in \omega(N, f) = A$ . This prove that  $\bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} f^s(\chi_N)} \subseteq \chi_A$ . Hence  $\omega_\mu(\chi_N, f) = \chi_A$ .  $\square$

**Example 4.4.** Let  $X = [0, \infty)$  and let  $\mu(x) = e^{-x}$ . Suppose  $\eta_n : X \rightarrow [0, 1]$  is the observer  $\eta_n(x) = e^{-nx}$  for  $n \geq 2$  and  $\lambda : X \rightarrow [0, 1]$  is the observer

$$\lambda(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then  $\tau_\mu = \{\eta_n, \lambda, \mu, \chi_\emptyset | n \geq 2\}$  is a relative topology. Let  $f^s : X \rightarrow X$  be a map defined by  $f^s(x) = 2^{-s}x$  we show that  $\omega_\mu(\eta_n, f) = \lambda$  for every  $n \geq 2$ . Let  $x \in X$  and  $n \geq 2$  then  $f^s(\eta_n)(x) = \sup\{\eta_n(y) | f^s(y) = x\} = \eta_n(2^s x)$ , therefore  $f^s(\eta_n)(x) = e^{-n(2^s x)}$ . This implies that  $\bigcup_{s \geq t} f^n(\eta_n(x)) = e^{-n(2^t x)}$ , for all  $t \geq 0$ . So  $\omega_\mu(\eta_n, f) = \lambda$ .

The existence of a relative semi-attractor is an important problem in the study of properties of a relative semi-dynamical system. By definition of relative semi-attractor and Proposition 2.6., every relative semi-attractor is invariant. Obviously

$\omega_\mu(\mu, f) = \mu$ . Therefore  $\chi_\emptyset, \mu$  are both relative semi-attractors for  $f$ , with observers  $\chi_\emptyset, \chi_X$ , respectively.  $\chi_\emptyset, \mu$  are called trivial relative semi-attractors.

**Theorem 4.5.** Let  $\lambda_1 \cup \lambda_2 = \lambda$  and  $\lambda_1, \lambda_2$  be two relative semi-attractors for  $f$ , then  $\lambda$  is a relative semi-attractor for  $f$ .

*Proof.* Since  $\omega_\mu(\eta_1, f) = \lambda_1$ ,  $\omega_\mu(\eta_2, f) = \lambda_2$  then according to Proposition 3.8.,  $\lambda_1 \cup \lambda_2 = \omega_\mu(\eta_1, f) \cup \omega_\mu(\eta_2, f) = \omega_\mu(\eta_1 \cup \eta_2, f)$ . Since  $\lambda_1 \subseteq \eta_1 \subseteq \mu$  and  $\lambda_2 \subseteq \eta_2 \subseteq \mu$  then  $\lambda_1 \cup \lambda_2 \subseteq \eta_1 \cup \eta_2 \subseteq \mu$ . Because  $\max\{\lambda_1(x), \lambda_2(x)\} \leq \max\{\eta_1(x), \eta_2(x)\}$ , for all  $x \in X$ .  $\square$

## 5. CONCLUSION

The most significant result of this study is a new definition for invariant sets and  $\omega$ -limit sets, as well as considering their properties. Two questions remain here unresolved, and therefore may be considered as a future study.

The first one is to see; *Whether or not a relative attractor is invariant set according to the Definition 3.2?* We guess that someone can answer this question affirmatively using the Lemmas 3.5. and 3.6. The second question is that *Whether the converse of the Theorem 4.5 is true, or not?* If isn't true it would be interesting to investigate what conditions have to be hold under which the converse of the theorem makes sense.

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