

A CHARACTERIZATION OF WARPED PRODUCT ON MIXED SUPER QUASI-EINSTEIN MANIFOLD

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ABSTRACT. In this paper we investigate the expressions of the Ricci tensors and scalar curvatures of warped product manifold with respect to the bases and fibres when warped product manifold is a mixed super quasi-Einstein manifold. In some cases we give some obstructions to the existence of such manifolds and in the last section we give an example of warped product on mixed super quasi-Einstein manifold.

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1. INTRODUCTION

A Riemannian manifold (M, g) with dimension $(n \geq 2)$, is said to be an Einstein manifold if the following condition

$$S(X, Y) = \frac{r}{n}g(X, Y),$$

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holds on M , where S and r denote the Ricci tensor and the scalar curvature of (M, g) respectively. The notion of quasi-Einstein manifold was defined in [1]. A non-flat Riemannian manifold (M, g) , $(n \geq 2)$ is defined to be a quasi-Einstein manifold if the condition

$$S(X, Y) = \alpha g(X, Y) + \beta \rho(X) \rho(Y),$$

is fulfilled on M , where α and β are scalars of which $\beta \neq 0$ and ρ is non-zero 1-form such that $g(X, \xi) = \rho(X)$ for all vector field X . We call ρ the associated 1-form and ξ is called the generator of the manifold. In this paper we consider ξ as a unit vector field.

In [2], Chaki introduced super quasi-Einstein manifold, denoted by $S(QE)_n$ and gave an example of a 4-dimensional semi Riemannian super quasi Einstein manifold, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X, Y),$$

where $\alpha, \beta, \gamma, \delta$ are scalars such that β, γ, δ are nonzero and A, B are two nonzero 1-forms such that

$$g(X, \xi_1) = A(X) \text{ and } g(X, \xi_2) = B(X),$$

ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X, \xi_1) = 0, \forall X \in \chi(M).$$

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

In [3], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of mixed super quasi-Einstein manifold, denoted by $MS(QE)_n$ with an example. A non-flat Riemannian manifold (M^n, g) , $(n \geq 3)$ is called mixed super quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X, Y),$$

where $\alpha, \beta, \varrho, \gamma, \delta$ are scalars such that $\beta, \varrho, \gamma, \delta$ are nonzero and A, B are two nonzero 1-forms such that

$$g(X, \xi_1) = A(X) \text{ and } g(X, \xi_2) = B(X),$$

ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is a symmetric (0,2) tensor with zero trace which satisfies the condition

$$D(X, \xi_1) = 0, \forall X \in \chi(M).$$

Here $\alpha, \beta, \varrho, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are called the main and the auxiliary generators and D is called the associated tensor of the manifold.

The notion of warped product generalizes that of a surface of revolution. It was introduced in [4] for studying manifolds of negative curvature. Let $B = (B_m, g_B)$ and $F = (F_k, g_F)$ be two Pseudo-Riemannian manifolds of dimensions $m \geq 1$ and $k \geq 0$, respectively and also let $B \times F$ be the usual product manifold of B and F . Given a smooth function $f \in C_{>0}^\infty(B) = \{f \in C^\infty(B) : f > 0\}$, the warped product $M = B \times_f F = ((B \times_f F)_{m+k} : g_M = g_B + f^2 g_F)$. Here B is called the base of M and F the fiber [5]. A warped product $B \times_f F$ with a constant warping function f can be considered as a Riemannian product.

In this paper we investigate when an warped product manifold is a mixed super

quasi-Einstein manifold and super quasi-Einstein manifold and we give the expressions of Ricci tensors and scalar curvatures for the bases and fibres. In some cases we give some obstructions to the existence of such manifolds. We denote by Ric_B , Ric_F , and H^f the lifts to M of the Ricci curvature of B and F , and the Hessian of f , respectively. Then we have the following from [5,6]

Proposition 1.1. The Ricci curvature Ric of the warped product $M = B \times_f F$ with $k = \dim F$ satisfies

- (1) $Ric(X, Y) = Ric_B(X, Y) - \frac{k}{f} H^f(X, Y)$,
- (2) $Ric(X, V) = 0$,
- (3) $Ric(V, W) = Ric_F(V, W) - g(V, W) f^\#$, $f^\# = \frac{-\Delta f}{f} + \frac{k-1}{f^2} |\nabla f|^2$ for any horizontal vectors X, Y (that is $X, Y \in \tau(TB)$) and any vertical vectors V, W (that is $V, W \in \tau(TF)$), where H^f and Δf denote the Hessian of f and the Laplacian of f given by $\Delta f = -tr(H^f)$, respectively.

Proposition 1.2. Let $M = B \times_f F$ be an warped product. Then the scalar curvature of M is given by

$$\tau_M = \tau_B + \frac{\tau_F}{f^2} - 2k \frac{\Delta f}{f} - k(k-1) \frac{|\nabla f|^2}{f^2}.$$

2. MIXED SUPER QUASI-EINSTEIN WARPED PRODUCTS

Let $M = B \times_f F$ be an warped product manifold with $f \in C_{>0}^\infty(B)$ and the metric $g_M = g_B + f^2 g_F$ which is also a mixed super quasi-Einstein manifold. That means its Ricci tensor satisfies

$$Ric_M(X, Y) = \alpha g_M(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X, Y). \quad (1)$$

Using the above formula we want to compute the Ricci tensors of B and F . To find out this we will consider the following two cases: when ξ_1 is tangent to B and when ξ_1 is tangent to F .

Theorem 2.1. Let $M = B \times_f F$ be an warped product manifold which is also a

mixed super quasi-Einstein manifold. Then the following conditions hold

i) When ξ_1 is tangent to the base B then the Ricci tensors of B and F satisfy the following conditions

$$\begin{cases} Ric_B(X, Y) = \alpha g_B(X, Y) + \frac{k}{f} H^f(X, Y) + \beta g_B(X, U) g_B(Y, U) + \delta D_B(X, Y), \\ Ric_F(X, Y) = g_F(X, Y) [\alpha f^2 - f \Delta f + (k-1) |\Delta f|^2] + \varrho g_F(X, V) g_F(Y, V) \\ + \delta D_F(X, Y). \end{cases} \quad (2)$$

ii) When ξ_1 is tangent to the fibre F then the Ricci tensors of B and F satisfy the following conditions

$$\begin{cases} Ric_B(X, Y) = \alpha g_B(X, Y) + \frac{k}{f} H^f(X, Y) + \varrho g_B(X, V) g_B(Y, V) + \delta D_B(X, Y), \\ Ric_F(X, Y) = g_F(X, Y) [\alpha f^2 - f \Delta f + (k-1) |\Delta f|^2] + \beta f^4 g_F(X, U) g_F(Y, U) \\ + \delta D_F(X, Y). \end{cases} \quad (3)$$

Proof:

i) For $X, Y \in \tau(TB)$, we have from equation (1), $Ric_M(X, Y) = \alpha g_B(X, Y) + \beta g_B(X, U) g_B(Y, U) + \delta D_B(X, Y)$. Then using the equation $Ric_M(X, Y) = Ric_B(X, Y) - \frac{k}{f} H^f(X, Y)$ from proposition (1.1) and applying the above two equations we get the first result of equation (2). For $X, Y \in \tau(TF)$ we have from equation (1), $Ric_M(X, Y) = \alpha f^2 g_F(X, Y) + \varrho g_F(X, V) g_F(Y, V) + \delta D_F(X, Y)$. Then using equation (3) of the proposition (1.1), we can write, $Ric_M(X, Y) = Ric_F(X, Y) - f^2 g_F(X, Y) (\frac{-\Delta f}{f} + \frac{k-1}{f^2} |\nabla f|^2)$. Then from these two equations we get the second result of equation (2).

ii) For $X, Y \in \tau(TB)$, we have from equation (1), $Ric_M(X, Y) = \alpha g_B(X, Y) + \varrho g_B(X, V) g_B(Y, V) + \delta D_B(X, Y)$. Then using the equation $Ric_M(X, Y) = Ric_B(X, Y) - \frac{k}{f} H^f(X, Y)$ from proposition (1.1) and applying the above two equations we get the first result of equation (3). For $X, Y \in \tau(TF)$ we have from equation (1), $Ric_M(X, Y) = \alpha f^2 g_F(X, Y) + \beta f^4 g_F(X, U) g_F(Y, U) + \delta D_F(X, Y)$. Then using equation (3) of the proposition (1.1), we can write, $Ric_M(X, Y) = Ric_F(X, Y) -$

$f^2 g_F(X, Y)(-\frac{\Delta f}{f} + \frac{k-1}{f^2} |\nabla f|^2)$. Then from these two equations we get the second result of equation (3).

From the above theorem we can give the scalar curvatures of M , B and F .

Corollary 2.1. Taking the traces of theorem (2.1) we get the scalar curvature of M , B and F of the different cases. For the first case, we get

i)

$$\begin{cases} \tau_M = (m+k)\alpha + \beta + \varrho, \\ \tau_B = m\alpha - k\frac{\Delta f}{f} + \beta, \\ \tau_F = k[-f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2] + \varrho. \end{cases} \quad (4)$$

For the second case, we get

ii)

$$\begin{cases} \tau_M = (m+k)\alpha + \beta + \varrho, \\ \tau_B = m\alpha - k\frac{\Delta f}{f} + \varrho, \\ \tau_F = k[-f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2] + \beta f^4. \end{cases} \quad (5)$$

3. OBSTRUCTIONS TO THE EXISTENCE OF MIXED SUPER QUASI-EINSTEIN WARPED PRODUCTS

In this section we prove some obstructions to the existence of mixed super quasi-Einstein warped products. Here we consider two cases, first ξ_1 is tangent to the base B and secondly ξ_1 is tangent to the fiber F . Now we prove the following theorem, when ξ_1 is tangent to the base B .

Theorem 3.1. Let $M = B \times_f F$ be an warped product with $f \in C_{>0}^\infty(B)$, $\dim B = m \geq 1$ and $\dim F = k \geq 1$, which is a mixed super quasi-Einstein manifold with ξ_1 as an unitary vector field tangent to F . If $m = 1$ or $k = 1$ then M is a Riemannian product.

Proof: If $m = 1$ then $\tau_B = 0$ and from second equation of (5), we can write

$\alpha - k \frac{\Delta f}{f} + \varrho = 0$, therefore $\Delta f = f[\frac{\alpha}{k} + \frac{\varrho}{k}]$. Then the Laplacian has constant sign and hence f is constant. On the other hand if $k = 1$ then $\tau_F = 0$ and from the third equation of (5) we get, $\alpha f^2 + \beta f^4 - f \Delta f = 0$, therefore $\Delta f = f[\alpha + \beta f^2]$. Here we see that the Laplacian has constant sign and therefore f is constant. Hence the theorem.

Theorem 3.2. Let $M = B \times_f F$ be an warped product with $\dim B = m \geq 2$ and $\dim F = k \geq 2$, which is a mixed super quasi-Einstein manifold with ξ_1 is an unitary vector field tangent to F . If $\beta \neq 0$ then M is a Riemannian product.

Proof: Consider the second equation of (3) that X, Y are orthogonal vector fields tangent to F such that $g(X, \xi_1) \neq 0$ and $g(X, \xi_2) \neq 0$. Now if we take different domains of definition that appear in the second equation of (3), we obtain that f is constant.

Remark 3.1. When ξ_1 is tangent to F for $\beta = 0$ and $\delta = 0$ we obtain that M is an Einstein space. So, we conclude that there does not exist any warped product.

Next we consider ξ_1 is tangent to B .

Theorem 3.3. Let $M = B \times_f F$ be an warped product with $\dim B = m \geq 1$ and $\dim F = k \geq 1$, which is a mixed super quasi-Einstein manifold with ξ_1 is an unitary vector field tangent to B . If $m = 1$ or $k = 1$ then M is a Riemannian product.

Proof: If $m = 1$ then $\tau_B = 0$ and from second equation of (4), we can write $\alpha - k \frac{\Delta f}{f} + \beta = 0$, therefore $\Delta f = f[\frac{\alpha}{k} + \frac{\beta}{k}]$. Then the Laplacian has constant sign and hence f is constant. On the other hand if $k = 1$ then $\tau_F = 0$ and from the third equation of (4) we get, $\alpha f^2 + \varrho - f \Delta f = 0$, therefore $\Delta f = f[\alpha + \frac{\varrho}{f^2}]$. Here we see that the Laplacian has constant sign and therefore f is constant. Hence M is a Riemannian product.

Theorem 3.4. Let $M = B \times_f F$ be an warped product with $\dim B = m \geq 2$

and $\dim F = k \geq 2$, which is a mixed super quasi-Einstein manifold with ξ_1 is an unitary vector field tangent to B . If $\alpha > 0$ then M reduces to a Riemannian product if at least one of the following conditions hold

- a) $\tau_M - \tau_B \leq \varrho$,
- b) $\tau_M \geq \tau_B + \tau_F$ and $c_1 \geq \alpha$ where $c_1 = -f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2$,
- c) $\tau_B \leq 0$ and $\beta \geq 0$,
- d) $\tau_M \geq \tau_B + \frac{\tau_F}{f^2}$.

Proof:

a) From the second equation of (4) we can write, $\tau_B = m\alpha - k\frac{\Delta f}{f} + \beta$. Therefore $\tau_B + \alpha k = \tau_M - (k\frac{\Delta f}{f} + \varrho)$. Hence $\tau_M - \tau_B - \varrho = \alpha k + k\frac{\Delta f}{f} \leq 0$, $\Rightarrow \Delta f \leq -\alpha f < 0$. Thus f is constant.

b) Let us consider $c_1 = -f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 > 0$, therefore we can write from third equation of (4), $\tau_F = kc_1 + \varrho \geq k\alpha + \varrho \Rightarrow \tau_B + \tau_F \geq \tau_M - k\frac{\Delta f}{f}$, hence $k\frac{\Delta f}{f} \geq \tau_M - (\tau_B + \tau_F) \geq 0 \Rightarrow \Delta f \geq 0$. Thus f is constant when $\tau_M \geq \tau_B + \tau_F$.

c) From the second equation of (4) we can write, $\tau_B = m\alpha - k\frac{\Delta f}{f} + \beta \Rightarrow k\frac{\Delta f}{f} \geq m\alpha - \tau_B + \beta \geq 0$. Therefore $\Delta f \geq 0$. Thus f is constant when $\beta \geq 0$.

d) From proposition (1.2) it follows that

$$\tau_M - (\tau_B + \frac{\tau_F}{f^2}) = -2k\frac{\Delta f}{f} - k(k-1)\frac{|\nabla f|^2}{f^2} \geq 0.$$

Therefore $\Delta f \leq 0$. Thus f is constant when $\tau_M \geq \tau_B + \frac{\tau_F}{f^2}$.

Now we see that from the second equation of (2), F is an Einstein space if $\varrho = \delta = 0$. In that case we consider $\alpha f^2 - f\Delta f + (k-1)|\Delta f|^2 = c_1$. For this case we get the following theorems.

Theorem 3.5. Let $M = B \times_f F$ be an warped product with $\dim B = m \geq 2$ and $\dim F = k \geq 2$, which is a mixed super quasi-Einstein manifold with ξ_1 is an unitary vector field tangent to B . If $\alpha > 0$ then M reduces to a Riemannian product if at least one of the following conditions hold

- a) F has negative scalar curvature,
- b) $|\nabla f|^2 \geq \frac{c_1}{k-1}$,
- c) $\tau_B \geq \frac{mc_1}{f^2} + \beta$ and $m \geq k$.

Proof:

a) Using the consideration of c_1 , from the third equation of (4) we get $\tau_F = kc_1$. Since $\tau_F \leq 0$ it follows that $c_1 \leq 0$. Therefore we can write, $-f\Delta f + \alpha f^2 = c_1 - (k-1)|\nabla f|^2 \leq 0$, hence $f\Delta f \geq \alpha f^2 > 0$. Thus f is constant.

b) Let us consider $c_1 > 0$, now it is clear that $-f\Delta f + \alpha f^2 = c_1 - (k-1)|\nabla f|^2 \leq 0$, hence $f\Delta f \geq \alpha f^2 \Rightarrow \Delta f \geq 0$. Thus f is constant only if $|\nabla f|^2 \geq \frac{c_1}{k-1}$.

c) Using the consideration $c_1 = -f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 > 0$ in the second equation of (4), we have

$$\begin{aligned}\tau_B &= m\alpha - k\frac{\Delta f}{f} + \beta \\ \Rightarrow \tau_B f^2 &= m(c_1 + f\Delta f - (k-1)|\nabla f|^2) - kf\Delta f + \beta f^2 \\ \tau_B f^2 - mc_1 - \beta f^2 &= (m-k)f\Delta f - m(k-1)|\nabla f|^2 \geq 0\end{aligned}$$

Now for the case of $m = k$, $\Delta f = 0$ and for the case of $m > k$, $\Delta f \geq 0$. Thus f is constant.

Theorem 3.6. Let $M = B \times_f F$ be an warped product with $\dim B = m \geq 2$ and $\dim F = k \geq 2$, which is a mixed super quasi-Einstein manifold with ξ_1 is an unitary vector field tangent to B . If $\alpha \leq 0$ then M reduces to a Riemannian product.

Proof: We know that $c_1 = -f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 > 0$, let $f(z)$ be the maximum on B , $z \in B$. Then $\nabla f(z) = 0$ and $\Delta f(z) \geq 0$. Therefore $-f(z)\Delta f(z) + \alpha f^2(z) = c_1$. Hence from these two equations we can write

$$\begin{aligned}-f(z)\Delta f(z) + \alpha f^2(z) &= -f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 \\ f\Delta f &= (k-1)|\nabla f|^2 + f(z)\Delta f(z) + \alpha[f^2 - f^2(z)] \geq 0\end{aligned}$$

Therefore $\Delta f \geq 0$, thus f is constant.

4. EXAMPLE OF WARPED PRODUCT ON MIXED SUPER QUASI-EINSTEIN MANIFOLD

Let (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^4)^2 + (e^{x^2})(dx^1)^2 + (x^1 x^3)^2(dx^2)^2 + (dx^3)^2$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M_4 . Then we can show that it is mixed super quasi-Einstein manifold with nonzero and non-constant scalar curvature.

To define warped product on $MS(QE)_4$, we consider the warping function f defined

$$f : \mathbf{R}^3 \longrightarrow (0, \infty)$$

by

$$f(x^1, x^2, x^3) = e^{x^2},$$

here we see that $f = e^{x^2} > 0$ and is a smooth function.

And the line element defined on $\mathbf{R} \times \mathbf{R}^3$ where we consider $\mathbf{R}$ is the fibre and $\mathbf{R}^3$ is the base. For that case we take the line element in the following way

$$ds^2 = g_{ij}dx^i dx^j = (dx^4)^2 + (e^{x^2})((dx^1)^2 + ((x^1 x^3)^2/e^{x^2})(dx^2)^2 + (1/e^{x^2})(dx^3)^2)$$

Note: One can also find out the characterization of warped product on super quasi-Einstein manifolds in a similar way.

REFERENCES

- [1] R. Deszcz, M. Glogowska, M. Hotlos, and Z. Senturk, On certain quasi-Einstein semi-symmetric hypersurfaces, Annales Univ. Sci. Budapest. Eotvos Sect. Math. 41 (1998), 151-164.
- [2] M.C. Chaki, On super quasi-Einstein manifolds, Publ. Math. Debrecen, 64/3-4 (2004), 481-488.
- [3] A. Bhattacharyya, M. Tarafdar and D. Debnath, On mixed super-quasi Einstein manifolds, Diff. Geom. - Dyn. Syst. 10 (2008), 44-57.
- [4] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc. 145 (1969), 1-49.

- [5] B. O'Neill, *Semi-Riemannian Geometry with applications to relativity*, Academic press, New York (1983).
- [6] J.K. Beem and P. Ehrlich, *Lorentzian Geometry*, Markel-Deccer Inc., First Edition, New York (1981).
- [7] A.L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg (1987).