SOME APPLICATIONS OF BANACH FIXED POINT AND LERAY SCHAUDER THEOREMS FOR FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study a boundary value problem of nonlinear fractional differential equations of order $\alpha, 3 < \alpha < 4$. We use Banach fixed point theorem to establish new results on the existence and uniqueness of solutions . Then, we use Leray-Schauder nonlinear alternative to prove some existence results.

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1. Introduction

This paper deals with the existence and uniqueness of solutions for the following fractional boundary value problem:

$$(1.1)\quad \begin{cases} D^{\alpha}x(t) = f\left(t, x(t), x'(t), -x''(t), x'''(t)\right), \quad t \in [0,1], \quad 3 < \alpha < 4, \\ \\ x(0) = x'(1) = x''(1) = x'''(0) = 0, \end{cases}$$

where D^{α} is the Caputo derivative, $f \in C([0,1]\times\mathbb{R}^4, \mathbb{R})$ is an appropriate function that will be specified later.

2. Preliminaries

In this section, we introduce some preliminary facts which are used throughout this paper [2, 5, 6].

Definition 1: A real valued function $f(t), t > 0$ is said to be in the space $C_{\mu}, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1 \in C([0,\infty))$.

Definition 2: A function $f(t), t > 0$ is said to be in the space C_{μ}^n, $n \in \mathbb{N}$, if $f^{(n)} \in C_{\mu}$.

Definition 3: The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_{\mu}, \mu \geq -1$, is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0, \tag{2.1}$$
$$J^0 f(t) = f(t).$$

Definition 4: The fractional derivative of $f \in C_{-1}^n$ in the sense of Caputo is defined as:

$$D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in N^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2.2}$$

We give the following lemmas [3, 4]:

Lemma 5: For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha} x(t) = 0$ is given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.3}$$

where c_i are arbitrary real constants for $i = 0, 1, 2, ..., n$, $n - 1 = [\alpha]$.

Lemma 6: Let $\beta > \alpha > 0$. Then the formula $D^{\alpha} J^{\beta} f(t) = J^{\beta-\alpha} f(t)$ is valid for $f \in L^1[a,b]$.

Lemma 7: Let $\alpha > 0$, $f \in L^1([a,b], \mathbb{R})$. Then, for all $t \in [a,b]$, we have

$$J^{\alpha+1} f(t) \leq \| J^{\alpha} f(t) \|_{L^1}.$$

We give also the following auxiliary Lemma :

Lemma 8: Let $3 < \alpha < 4$. A solution of the problem

$$\left\{ \begin{array}{l} D^{\alpha}x\,(t) = F\,(t, \varkappa)\,, \quad t \in J, \quad 3 < \alpha < 4, \\ \\ x(0) = x^{'}(1) = x^{''}(0) = x^{'''}(1) = 0 \end{array} \right. \tag{2.4}$$

is given by

(2.5)

$$\begin{aligned} x\,(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F\,(\tau, \varkappa)\, d\tau \\ &+ t \left[\frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} F\,(\tau, \varkappa)\, d\tau - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2} F\,(\tau, \varkappa)\, d\tau \right] \\ &- \frac{t^2}{2} \left[\frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3} F\,(\tau, \varkappa)\, d\tau \right]. \end{aligned}$$

Proof: Assume that $D^{\alpha}x\,(t) = F(t, \varkappa)$. Using Lemma 5, we can write

$$\begin{aligned} x\,(t) &= J^{\alpha}F\,(t, \varkappa) + c_0 + c_1 t + c_2 t^2 + c_3 t^3, \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F\,(\tau, \varkappa)\, d\tau + c_0 + c_1 t + c_2 t^2 + c_3 t^3. \end{aligned}$$

Using Lemma 6, we obtain:

$$x^{'}\,(t) = J^{\alpha-1}F\,(t, \varkappa) + c_1 + 2c_2 t + 3c_3 t^2,$$

$$x^{''}\,(t) = J^{\alpha-2}F\,(t, \varkappa) + 2c_2 + 6c_3 t,$$

$$x^{'''}\,(t) = J^{\alpha-3}F\,(t, \varkappa) + 6c_3.$$

Thanks to the conditions $x(0) = x^{'}(1) = x^{''}(1) = x^{'''}(0) = 0$, we obtain:

$$c_1 = \frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} F(\tau,\varkappa)\,d\tau - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} F(\tau,\varkappa)\,d\tau,$$

$$c_2 = -\frac{1}{2}\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} F(\tau,\varkappa)\,d\tau, c_0 = c_3 = 0,$$

which imply (2.5) and end the proof of Lemma 8.

3. Main Results

Let us consider E the Banach space of all functions $x \in C^3([0,1],\mathbb{R})$, with the norm $\|x\| = \sum_{i=0}^{3} \max_{t\in[0,1]} \left|x^{(i)}(t)\right|$ and let us define the integral operator $T : E \to E$ by

(3.1)
$$\begin{aligned}
Tx(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau \\
& + t[\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau \\
& - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau] \\
& - \frac{t^2}{2}\left[\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau\right].
\end{aligned}$$

We have
$$\begin{aligned}
T'x(t) = {} & \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau \\
& - t\left[\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau\right] \\
& + [\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau \\
& - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2} f\left(\tau, x(\tau), x'(\tau), -x''(\tau), x'''(\tau)\right) d\tau],
\end{aligned}$$

$$T^{''}x(t) = \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-\tau)^{\alpha-3} f\left(\tau, x(\tau), x^{'}(\tau), -x^{''}(\tau), x^{'''}(\tau)\right) d\tau$$
$$- \frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3} f\left(\tau, x(\tau), x^{'}(\tau), -x^{''}(\tau), x^{'''}(\tau)\right) d\tau,$$

and

$$T^{'''}x(t) = \frac{1}{\Gamma(\alpha-3)}\int_0^t (t-\tau)^{\alpha-4} f\left(\tau, x(\tau), x^{'}(\tau), -x^{''}(\tau), x^{'''}(\tau)\right) d\tau.$$

Let us now consider the following hypotheses:

$(\mathbf{H}_1)$: Assume that there exist nonnegative functions $\alpha_i \in L^1([0,1], \mathbb{R}_+)$ such that for all $x_i,\ y_i \in \mathbb{R}$, $i = \overline{0,3}$ and $t \in [0,1]$, we have:

$$|f(t, x_0, x_1, x_2, x_3) - f(t, y_0, y_1, y_2, y_3)| \leq \sum_{i=0}^{3} \alpha_i(t)\, |x_i - y_i|.$$

$(\mathbf{H}_2)$: For $\alpha_i \in L^1([0,1], \mathbb{R}_+)$, $i = 0, .., 3$, we have

$$\Lambda + \Delta < 1,$$

where:

$$\Lambda = \left\| J^{\alpha-1}\left(\sum_{i=0}^{3}\alpha_i(t)\right)\right\|_{L^1} + \left\| J^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i(t)\right)\right\|_{L^1}$$
$$+ \left\| J^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(t)\right)\right\|_{L^1} + \left\| J^{\alpha-4}\left(\sum_{i=0}^{3}\alpha_i(t)\right)\right\|_{L^1}$$

$$\Delta = \frac{9}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right) d\tau$$
$$+ \frac{2}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right) d\tau.$$

$(\mathbf{H}_3)$: Assume that $f(t,0,0) \neq 0, t \in [0,1]$, and suppose that there exist nonnegative functions $(\Phi_i)_{i=0,..,3}$ and φ in $L^1([0,1], \mathbb{R}_+)$ and nondecreasing functions

$(\Psi_i)_{i=0,..3}$ in $C\left([0,1], \mathbb{R}_+^*\right)$, such that:

$$|f(t,x_0,x_1,x_2,x_3)| \leq \sum_{i=0}^{3} \Phi_i(t)\, \Psi_i(|x_i|) + \varphi(t), \text{ a.e } (t,x_0,x_1,x_2,x_3) \in [0,1] \times \mathbb{R}^4.$$

$(\mathbf{H}_4)$: The function f is a bounded on $[0,1] \times \mathbb{R}^4$.

$(\mathbf{H}_5)$: There exists $r > 0$, such that

$$\sum_{i=0}^{3} \Psi_i(r)\,(M_i + C_i) + M_\varphi + C_\varphi < r,$$

where,

$$C_0 = \left\|J^{\alpha-1}\Phi_0(t)\right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\Phi_0(\tau)\, d\tau + \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\Phi_0(\tau)\, d\tau,$$

$$C_1 = \left\|J^{\alpha-1}\Phi_1(t)\right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\Phi_1(\tau)\, d\tau + \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\Phi_1(\tau)\, d\tau,$$

$$C_2 = \left\|J^{\alpha-1}\Phi_2(t)\right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\Phi_2(\tau)\, d\tau + \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\Phi_2(\tau)\, d\tau,$$

$$C_3 = \left\|J^{\alpha-1}\Phi_3(t)\right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\Phi_3(\tau)\, d\tau + \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\Phi_3(\tau)\, d\tau,$$

$$C_\varphi = \left\|J^{\alpha-1}\varphi(t)\right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\varphi(\tau)\, d\tau + \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\varphi(\tau)\, d\tau,$$

and

$$M_0 = \left\| J^{\alpha-2}\Phi_0(t) \right\|_{L^1} + \left\| J^{\alpha-3}\Phi_0(t) \right\|_{L^1} + \left\| J^{\alpha-4}\Phi_0(t) \right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2}\Phi_0(\tau)\, d\tau + \frac{3}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3}\Phi_0(\tau)\, d\tau,$$

$$M_1 = \left\| J^{\alpha-2}\Phi_1(t) \right\|_{L^1} + \left\| J^{\alpha-3}\Phi_1(t) \right\|_{L^1} + \left\| J^{\alpha-4}\Phi_1(t) \right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2}\Phi_1(\tau)\, d\tau + \frac{3}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3}\Phi_1(\tau)\, d\tau,$$

$$M_2 = \left\| J^{\alpha-2}\Phi_2(t) \right\|_{L^1} + \left\| J^{\alpha-3}\Phi_2(t) \right\|_{L^1} + \left\| J^{\alpha-4}\Phi_2(t) \right\|_{L^1} \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2}\Phi_2(\tau)\, d\tau + \frac{3}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3}\Phi_2(\tau)\, d\tau,$$

$$M_3 = \left\| J^{\alpha-2}\Phi_3(t) \right\|_{L^1} + \left\| J^{\alpha-3}\Phi_3(t) \right\|_{L^1} + \left\| J^{\alpha-4}\Phi_3(t) \right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2}\Phi_2(\tau)\, d\tau + \frac{3}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3}\Phi_2(\tau)\, d\tau,$$

$$M_\varphi = \left\| J^{\alpha-2}\varphi(t) \right\|_{L^1} + \left\| J^{\alpha-3}\varphi(t) \right\|_{L^1} + \left\| J^{\alpha-4}\varphi(t) \right\|_{L^1} + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-\tau)^{\alpha-2}\varphi(\tau)\, d\tau + \frac{3}{\Gamma(\alpha-2)} \int_0^1 (1-\tau)^{\alpha-3}\varphi(\tau)\, d\tau.$$

Our first result is given by the following theorem:

Theorem 9: Suppose that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ are satisfied. Then, the problem (1.1) has a unique solution on $[0, 1]$.

Proof: We prove that T is a contraction mapping on E : For x and $y \in E$, we have:

$$
\begin{aligned}
Tx(t) - Ty(t) &= J^{\alpha}\left(f\left(t, x(t), x^{'}(t), x^{''}(t), x^{'''}(t)\right) - f\left(t, y(t), y^{'}(t), y^{''}(t), y^{'''}(t)\right)\right) \\
&+ t[\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}(f\left(\tau, x(\tau), x^{'}(\tau), x^{''}(\tau), x^{'''}(\tau)\right) \\
&- f\left(\tau, y(\tau), y^{'}(\tau), y^{''}(\tau), y^{'''}(\tau)\right))d\tau \\
&- \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}(f\left(\tau, x(\tau), x^{'}(\tau), x^{''}(\tau), x^{'''}(\tau)\right) \\
&- f\left(\tau, y(\tau), y^{'}(\tau), y^{''}(\tau), y^{'''}(\tau)\right))d\tau] \\
&- \frac{t^2}{2}\left(\begin{array}{c} \frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}(f\left(\tau, x(\tau), x^{'}(\tau), x^{''}(\tau), x^{'''}(\tau)\right) \\ -f\left(\tau, y(\tau), y^{'}(\tau), y^{''}(\tau), y^{'''}(\tau)\right))d\tau \end{array}\right)
\end{aligned}
\tag{3.2}
$$

where $t \in [0,1]$.
Applying $(\mathbf{H}_1)$, yields

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\le J^{\alpha}\left(\sum_{i=0}^{3}\alpha_i(t)\left|x^{(i)}(t) - y^{(i)}(t)\right|\right) \\
&+ [\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(\tau)\left|x^{(i)}(\tau) - y^{(i)}(\tau)\right|\right)d\tau \\
&+ \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i(\tau)\left|x^{(i)}(\tau) - y^{(i)}(\tau)\right|\right)d\tau] \\
&+ \frac{1}{2}[\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(\tau)\left|x^{(i)}(\tau) - y^{(i)}(\tau)\right|\right)d\tau].
\end{aligned}
\tag{3.3}
$$

Therefore,

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\le \sum_{i=0}^{3}\max_{t\in[0,1]}\left|x^{(i)}(t) - y^{(i)}(t)\right| J^{\alpha}\left(\sum_{i=0}^{3}\alpha_i(t)\right) \\
&+ \sum_{i=0}^{3}\max_{t\in[0,1]}\left|x^{(i)}(t) - y^{(i)}(t)\right| [\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right)d\tau \\
&+ \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right)d\tau] \\
&+ \frac{1}{2}\sum_{i=0}^{3}\max_{t\in[0,1]}\left|x^{(i)}(t) - y^{(i)}(t)\right| [\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\right)d\tau].
\end{aligned}
\tag{3.4}
$$

Hence,

$$\begin{aligned}
|Tx(t) - Ty(t)| \leq \|x-y\| [J^{\alpha}\left(\sum_{i=0}^{3}\alpha_i(t)\right) \\
+ \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right)d\tau \\
+ \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right)d\tau].
\end{aligned} \tag{3.5}$$

By Lemma 7, we get

$$\begin{aligned}
|Tx(t) - Ty(t)| \leq \|x-y\| [\left\|J^{\alpha-1}\left(\sum_{i=0}^{3}\alpha_i(t)\right)\right\|_{L^1} \\
+ \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right)d\tau \\
+ \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i(\tau)\right)d\tau].
\end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned}
T^{'}x(t) - T^{'}y(t) = J^{\alpha-1}\Big[f\left(t, x(t), x^{'}(t), -x^{''}(t), x^{'''}(t)\right) \\
- f\left(t, y(t), y^{'}(t), -y^{''}(t), y^{'''}(t)\right) \\
- t[\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}(f\left(\tau, x(\tau), x^{'}(\tau), -x^{''}(\tau), x^{'''}(\tau)\right) \\
- f\left(\tau, y(\tau), y^{'}(\tau), -y^{''}(\tau), y^{'''}(\tau)\right))d\tau] \\
+ [\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}(f\left(\tau, x(\tau), x^{'}(\tau), -x^{''}(\tau), x^{'''}(\tau)\right) \\
- f\left(\tau, y(\tau), y^{'}(\tau), -y^{''}(\tau), y^{'''}(\tau)\right))d\tau \\
- \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}(f\left(\tau, x(\tau), x^{'}(\tau), -x^{''}(\tau), x^{'''}(\tau)\right) \\
- f\left(\tau, y(\tau), y^{'}(\tau), -y^{''}(\tau), y^{'''}(\tau)\right))d\tau].
\end{aligned}$$

With the help of $(\mathbf{H}_1)$, we obtain

$$
\begin{aligned}
&\left|T^{'}x\left(t\right)-T^{'}y\left(t\right)\right| \\
&\leq\sum_{i=0}^{3}\max_{t\in[0,1]}\left|x^{(i)}\left(t\right)-y^{(i)}\left(t\right)\right|J^{\alpha-1}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right) \\
&+2\sum_{i=0}^{3}\max_{t\in[0,1]}\left|x^{(i)}\left(t\right)-y^{(i)}\left(t\right)\right|[\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau] \\
&+\sum_{i=0}^{3}\max_{t\in[0,1]}\left|x^{(i)}\left(t\right)-y^{(i)}\left(t\right)\right|[\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|T^{'}x\left(t\right)-T^{'}y\left(t\right)\right| \leq \|x-y\|\,[&\left\|J^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1} \\
&+\frac{2}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau \\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau],
\end{aligned}
$$

$$
\begin{aligned}
\left|T^{''}x\left(t\right)-T^{''}y\left(t\right)\right| \leq \|x-y\|\,[&\left\|J^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1} \\
&+\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau],
\end{aligned}
$$

and

$$
\left|T^{'''}x\left(t\right)-T^{'''}y\left(t\right)\right| \leq \|x-y\|\left\|J^{\alpha-4}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1}.
$$

Consequently,

$$\|Tx-Ty\| \leq \|x-y\| \, [\left\| J^{\alpha-1}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1} + \left\| J^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1}$$
$$+ \left\| J^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1} + \left\| J^{\alpha-4}\left(\sum_{i=0}^{3}\alpha_i\left(t\right)\right)\right\|_{L^1}$$
$$+ \frac{9}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau$$
$$+ \frac{2}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\alpha_i\left(\tau\right)\right)d\tau].$$

$$\leq \left(\Delta+\Lambda\right)\left\|x-y\right\|. \tag{3.7}$$

Thanks to $(\mathbf{H}_2)$, the operator T is a contraction on E. Hence, by Banach fixed point theorem, we conclude that there exists a unique fixed point $x \in E$, which is the solution of (1.1).

Our second result is based on the Leray-Schauder nonlinear alternative [1]:

Lemma 10: Let X be a Banach space, $\Omega \subset X$ bounded and open, $0 \in \Omega$, and $A : \Omega \to X$ a completely continuous operator. Then, either there exist $x \in \partial\Omega$ and $\lambda > 1$, such that $Ax = \lambda x$ or A has a fixed point in Ω.

We have:

Theorem 11: Assume that $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$ are satisfied. Then, (1.1) has at least a solution on $[0, 1]$.

Proof: First, let us prove that T is completely continuous:

(1*) Since f is continuous, then T is continuous.

(2*) Let us take $x \in B_r$, where r is given by $(\mathbf{H}_5)$. Using $(\mathbf{H}_3)$, we can write

$$|Tx\left(t\right)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\sum_{i=0}^{3}\Phi_i\left(\tau\right)\Psi_i\left(\left|x^{(i)}\left(\tau\right)\right|\right)+\varphi\left(\tau\right)\right)d\tau \tag{3.8}$$
$$+ \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\Phi_i\left(\tau\right)\Psi_i\left(\left|x^{(i)}\left(\tau\right)\right|\right)+\varphi\left(\tau\right)\right)d\tau$$
$$+ \frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\Phi_i\left(\tau\right)\Psi_i\left(\left|x^{(i)}\right|\left(\tau\right)\right)+\varphi\left(\tau\right)\right)d\tau.$$

Since $(\Psi_i)_{i=0,..3}$ are nondecreasing, then (3.8) implies that

$$|Tx(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau \tag{3.9}$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau$$
$$+\frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau.$$

Using the same arguments as before, we get

$$|Tx(t)| \leq \left\| J^{\alpha-1}\left(\sum_{i=0}^{3}\Phi_i(t)\Psi_i(r)+\varphi(t)\right)\right\|_{L^1} \tag{3.10}$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau$$
$$+\frac{3}{2\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau.$$

Therefore,

$$|Tx(t)| \leq \sum_{i=0}^{3}\Psi_i(r)\,C_i + C_\varphi. \tag{3.11}$$

Moreover, we have

$$|T'x(t)| \leq \left\| J^{\alpha-1}\left(\sum_{i=0}^{3}\Phi_i(t)\Psi_i(r)+\varphi(t)\right)\right\|_{L^1} \tag{3.12}$$
$$+\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-\tau)^{\alpha-2}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau$$
$$+\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau,$$

$$\left|T''x(t)\right| \leq \left\| J^{\alpha-3}\left(\sum_{i=0}^{3}\Phi_i(t)\Psi_i(r)+\varphi(t)\right)\right\|_{L^1} \tag{3.14}$$
$$+\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-\tau)^{\alpha-3}\left(\sum_{i=0}^{3}\Phi_i(\tau)\Psi_i(r)+\varphi(\tau)\right)d\tau$$

and

$$\left|T^{'''}x(t)\right| \leq \left\| J^{\alpha-4}\left(\sum_{i=0}^{3}\Phi_i(t)\Psi_i(r)+\varphi(t)\right)\right\|_{L^1}. \tag{3.15}$$

By simple calculations, we get

$$\|Tx\| \leq \sum_{i=0}^{3}\Psi_i(r)(M_i+C_i)+M_\varphi+C_\varphi. \tag{3.16}$$

Then $T(B_r)$ is uniformly bounded.

(3*) The equicontinuity: Let us take $t_1,\, t_2 \in [0,1]$, $t_1 < t_2$ and $x \in B_r$. We have

$$\begin{aligned}
&|Tx(t_2) - Tx(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}\right]\left|f\left(\tau,x(\tau),x^{'}(\tau),-x^{''}(\tau),x^{'''}(\tau)\right)\right|d\tau \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}\right]\left|f\left(\tau,x(\tau),x^{'}(\tau),-x^{''}(\tau),x^{'''}(\tau)\right)\right|d\tau \\
&+(t_2-t_1)\,[\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}\left|f\left(\tau,x(\tau),x^{'}(\tau),-x^{''}(\tau),x^{'''}(\tau)\right)\right|d\tau \\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}\left|f\left(\tau,x(\tau),x^{'}(\tau),-x^{''}(\tau),x^{'''}(\tau)\right)\right|d\tau] \\
&+\left(\frac{t_2^2}{2}-\frac{t_1^2}{2}\right)\left(\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}\left|f\left(\tau,x(\tau),x^{'}(\tau),-x^{''}(\tau),x^{'''}(\tau)\right)\right|d\tau\right).
\end{aligned} \tag{3.17}$$

Using (H_4), we can write

$$\begin{aligned}
|Tx(t_2)-Tx(t_1)| \leq \frac{L}{\Gamma(\alpha)}&\left[\int_0^{t_2}\left[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}\right]d\tau\right. \\
&+\int_{t_1}^{t_2}\left[(t_2-\tau)^{\alpha-1}-(t_1-\tau)^{\alpha-1}\right]d\tau \\
&+L(t_2-t_1)\,[\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}d\tau- \\
&\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-\tau)^{\alpha-2}d\tau] \\
&+L\left(\frac{t_2^2}{2}-\frac{t_1^2}{2}\right)\left(\frac{1}{\Gamma(\alpha-2)}\int_0^1(1-\tau)^{\alpha-3}d\tau\right).
\end{aligned} \tag{3.18}$$

This implies that

$$|Tx(t_2) - Tx(t_1)| \leq \frac{L}{\Gamma(\alpha)} \left[\int_0^{t_2} \left[(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1} \right] d\tau + \right.$$
$$\int_{t_1}^{t_2} \left[(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1} \right] d\tau$$
$$+ L\left(\frac{1}{\Gamma(\alpha-1)} - \frac{1}{\Gamma(\alpha)} \right)(t_2-t_1) + \frac{L}{\Gamma(\alpha-1)}\left(\frac{t_2^2}{2} - \frac{t_1^2}{2} \right). \tag{3.19}$$

We see that the function $\varphi(x) = x^{\alpha-1} - (\alpha-1)x$ is decreasing on $[0,1]$, therefore $(t_2-\tau)^{\alpha-1} - (t_1-\tau)^{\alpha-1} \leq (\alpha-1)(t_2-t_1)$, from which we deduce that

$$|Tx(t_2) - Tx(t_1)| \leq \frac{L(\alpha-1)(t_2-t_1)(2t_{2-}t_1)}{\Gamma(\alpha)}$$
$$+ L(t_2-t_1)\left(\frac{1}{\Gamma(\alpha-1)} - \frac{1}{\Gamma(\alpha)} \right) + \frac{L(t_2-t_1)(t_2+t_1)}{2\Gamma(\alpha-1)}. \tag{3.20}$$

On the other hand, we have the inequalities

$$\left| T^{'}x(t_2) - T^{'}x(t_1) \right| \leq \frac{L}{\Gamma(\alpha-1)}[\int_0^{t_2} \left[(t_2-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2} \right] d\tau$$
$$+ \int_{t_1}^{t_2} \left[(t_2-\tau)^{\alpha-2} - (t_2-\tau)^{\alpha-2} \right] d\tau]$$
$$+ \frac{L(t_2-t_1)}{\Gamma(\alpha-1)}, \tag{3.21}$$

$$\left| T^{''}x(t_2) - T^{''}x(t_1) \right| \leq \frac{L}{\Gamma(\alpha-2)}[\int_0^{t_2} \left[(t_2-\tau)^{\alpha-3} - (t_1-\tau)^{\alpha-3} \right] d\tau$$
$$+ \int_{t_1}^{t_2} \left[(t_2-\tau)^{\alpha-3} - (t_1-\tau)^{\alpha-3} \right] d\tau] \tag{3.22}$$

and

$$\left| T^{'''}x(t_2) - T^{'''}x(t_1) \right| \leq \frac{L}{\Gamma(\alpha-3)}[\int_0^{t_2} \left[(t_2-\tau)^{\alpha-4} - (t_1-\tau)^{\alpha-4} \right] d\tau$$
$$+ \int_{t_1}^{t_2} \left[(t_2-\tau)^{\alpha-4} - (t_1-\tau)^{\alpha-4} \right] d\tau]. \tag{3.23}$$

Consequently, we get

$$\left| T^{'}x(t_2) - T^{'}x(t_1) \right| \leq \frac{L(\alpha-2)(t_2-t_1)(2t_2-t_1)}{\Gamma(\alpha-1)} + \frac{L(t_2-t_1)}{\Gamma(\alpha-1)}, \tag{3.24}$$

$$\left| T^{''}x(t_2) - T^{''}x(t_1) \right| \leq \frac{L(\alpha-3)(t_2-t_1)(2t_2-t_1)}{\Gamma(\alpha-2)} \tag{3.25}$$

and

$$\left|T^{'''}x(t_2) - T^{'''}x(t_1)\right| \leq \frac{L(\alpha-4)(t_2-t_1)(2t_2-t_1)}{\Gamma(\alpha-3)}. \tag{3.26}$$

It is clear that when $t_1 \to t_2$ the right hand sides of the quantities $(320, 324, 325, 326)$ tend to 0. Therefore, $T(B_r)$ is equicontinuous. So, by (1*,2*,3*) and the Arzela-Ascoli Theorem, we deduce that T is a completely continuous operator.

Now letting $\Omega = \{x \in E, \|x\| < r\}$. For any $x \in \partial\Omega$, such that $\lambda T x = x$, $0 < \lambda < 1$, and with the help of (3.11) the following inequality holds:

$$\|x\| = \|\lambda T x\| \leq \|Tx\| \leq \sum_{i=0}^{3} \Psi_i(r)(M_i + C_i) + M_\varphi + C_\varphi. \tag{3.27}$$

Using (H_5), we deduce that

$$\|x\| \leq \sum_{i=0}^{3} \Psi_i(r)(M_i + C_i) + M_\varphi + C_\varphi < r. \tag{3.28}$$

This contradicts the fact that $x \in \partial\Omega$. Hence, Lemma 10 allows us to conclude that the operator T has a fixed point $x \in \Omega$, and then the problem (1.1) has a solution x on $[0, 1]$.

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