ON $\sigma - C^*-$ DYNAMICS

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Abstract. Let σ be an idempotent bounded linear $*-$ endomorphism on the C^*- algebra A. Introducing the notions of $\sigma-$ densely defined $*-\sigma-$ derivation and $\sigma-C^*-$ dynamics, we find a corresponding between these two notions. We prove that if $\{\varphi_t\}_{t\in\mathbb{R}}$ is a so-called $\sigma - C^*-$ dynamics with the generator d, then d is a $\sigma-$ densely defined $*-\sigma-$ derivation. Conversely, it is shown that each bounded $*-\sigma-$ derivation on A satisfying $d\sigma = \sigma d = d$, produces a $\sigma - C^*-$ dynamics.

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1. Introduction

One parameter groups of bounded linear operators have more importance because of their applications in the theory dynamical systems. Such groups apply widely to describe the dynamical systems appearing in the mathematical physics. For instance, in a quantum field theory, a C^*- dynamics are defined as a strongly continuous one parameter group of $*-$ automorphisms on C^*- algebras.

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On the other hand, the infinitesimal generator of a C^*- dynamics is a class of densely defined linear operators called $*-$ derivations which play essential role in a wide spectrum of scientific branches.

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. In each case of generalization of derivation, a noted point drawing the attention of analysts is trying to demonstrate a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a C^*- dynamical system.

From now on, let A be a $(*-)$ Banach algebra and $\sigma : A \to A$ be a $(*-)$ linear mapping. Following [7], a $(*-)$ linear map $d : A \to A$ is called a $(*-)$ $\sigma-$ derivation if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in A$. In 2006, Mirzavaziri and Moslehian ([7]) introduced the concept of a $\sigma-$ endomorphism, as a generalization of endomorphism, by a linear mapping $\varphi : A \to A$ that for all $a, b \in A$, fulfills

$(\varphi + \sigma - I)(ab) - (\varphi + \sigma - I)(a)(\varphi + \sigma - I)(b) = \sigma(ab) - \sigma(a)\sigma(b)$.

Adjoining this property to a uniformly continuous one parameter group of bounded linear operators on A, they also defined a new dynamical system called, $\sigma-$ dynamics which had a $\sigma-$ derivation as its infinitesimal generator.

However, this dynamical system, in fact, was not an extension of C^*- dynamics. More precisely, by subsituting $\sigma = I$ (the identity operator on A) in the definition of $\sigma-$ dynamics a C^*- dynamics can not be obtained.

In 2008, Janfada [4], among defining the concept of $\sigma-$ one parameter semigroups, compared some properties of semigroups and $\sigma-$ semigroups.

The above concepts (C^*- dynamical systems and $\sigma-$ one parameter groups) motivate us to remove the fault of definition of $\sigma-$ dynamics by defining the $\sigma-C^*-$ dynamical system as follows:

Let A be a C^*- algebra, σ be a linear $*-$ endomorphism and let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a $\sigma-$ one parameter group of bounded linear $*-$ endomorphism on A such that for each $t \in \mathbb{R}$, $\varphi_t(A) = \sigma(A)$ and $ker(\varphi_t) \subseteq ker(\sigma)$. If additionally $\{\varphi_t\}_{t\in\mathbb{R}}$ is strongly continuous, then it is called a $\sigma - C^*-$ dynamical system on A.

In this paper, we introduce the notions of $\sigma-$ densely defined (inner) $\sigma-$ derivations, relative endomorphisms with respect to σ, $\sigma-$ inner endomorphism and $\sigma-C^*-$ dynamics on A. In order to demonstrate a suitable corresponding between $\sigma - C^*-$ dynamics and $* - \sigma-$ derivations we prove that if $\{\varphi_t\}_{t\in\mathbb{R}}$ is a $\sigma - C^*-$

dynamics with the generator d, then d is a $\sigma-$ densely defined $*-\sigma-$ derivation. Also using the computation formula for $d^n(ab)$, we deal with the converse under some conditions. More precisely, if σ is an idempotent bounded linear $*-$ endomorphism and and d is a bounded $*-\sigma-$ derivation on A satisfying $d\sigma = \sigma d = d$, then d induces a $\sigma - C^*-$ dynamics on A.

The reader is referred to [5, 6] and [7] for more details on $\sigma-$ derivations and to [2] and [10] for more information on dynamical systems.

2. Preliminaries

Definition 2.1. A $(*-)$ linear mapping $d : A \to A$ is called a $\sigma-$ *derivation* if $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ for all $a, b \in A$.

Example 2.2. Let σ be an arbitrary $*-$ linear mapping on the C^*- algebra A and suppose that h is a self adjoint element of A satisfying $h\left(\sigma(ab) - \sigma(a)\sigma(b)\right) = \left(\sigma(ab) - \sigma(a)\sigma(b)\right)h$, for all $a, b \in D$. Then the mapping d_h^σ defined by $d_h^\sigma(a) = i[h, \sigma(a)]$ is a $*-\sigma-$ derivation which is called *inner* . Note that if σ is an endomorphism, then h can be any arbitrary self adjoint element of A.

Definition 2.3. A linear mapping $\varphi : A \to A$ is called *relative endomorphism with respect to* σ if for all $a, b \in A$

$$\varphi(ab) - \varphi(a)\varphi(b) = \sigma(ab) - \sigma(a)\sigma(b).$$

Note that if σ is an endomorphism on A, then φ as a relative endomorphism with respect to σ is nothing than an endomorphism in the usual sense.

Example 2.4. Let A and B be C^*- algebras. Then $A \times B$ is also a C^*- algebra by regarding the following structure:

(i) $(a, b) + (c, d) = (a + c, b + d)$

(ii) $\lambda(a, b) = (\lambda a, \lambda b)$

(iii) $(a, b).(c, d) = (ac, bd)$, $(a, b)^* = (a^*, b^*)$

(iv) $\| (a, b) \| = \max\{\| a \|, \| b \|\}$

Now define the maps φ and σ as follows:

$\varphi(a, b) = (0, -2b)$

$\sigma(a, b) = (0, 3b)$

Then φ is a relative endomorphism with respect to σ.

Definition 2.5. A linear mapping φ on the C^*- algebra A is called a $\sigma-$ *inner endomorphism* if there exist a unitary element $u \in A$ such that for each $a, b \in A$

(i)$\varphi(a) = u\sigma(a)u^*$

(ii)$u\left(\sigma(ab) - \sigma(a)\sigma(b)\right) = \left(\sigma(ab) - \sigma(a)\sigma(b)\right)u.$

Example 2.6. Let $\sigma : A \to A$ be an arbitrary $*-$ linear endomorphism and h be a self adjoint element of the C^*- algebra A. Then the mapping $\varphi : A \to A$ given by $\varphi(a) = e^{ih}\sigma(a)e^{-ih}$ is a $* - \sigma-$ inner endomorphism.

Lemma 2.7. *Each $\sigma-$ inner endomorphism is indeed a relative endomorphism with respect to σ.*

Proof. Suppose φ is a $\sigma-$ inner endomorphism, so there exist a unitary element $u \in A$ such that for each $a, b \in A$

(i)$\varphi(a) = u\sigma(a)u^*$

(ii)$u\left(\sigma(ab) - \sigma(a)\sigma(b)\right)u^* = \left(\sigma(ab) - \sigma(a)\sigma(b)\right).$

Therefore

$$\begin{aligned}\varphi(ab) - \varphi(a)\varphi(b) &= (u\sigma(ab)u^*) - (u\sigma(a)u^*)\,(u\sigma(b)u^*)\\ &= u\left(\sigma(ab) - \sigma(a)\sigma(b)\right)u^*\\ &= \sigma(ab) - \sigma(a)\sigma(b).\end{aligned}$$

Hence φ is a relative endomorphism with respect to σ. □

The remainder of this section is devoted to introduction of $\sigma-$ semigroups of bounded linear operators and their related concepts. This notion was introduced by Janfada in 2008. We refer the reader to [4] for more details.

Definition 2.8. Let σ be a bounded operator on the Banach algebra A. A one parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on A is called a $\sigma-$ *one parameter group* if

$(i)\varphi_0 = \sigma$ and

$(ii)\varphi_{t+s} = \varphi_t\varphi_s \quad (s, t \in \mathbb{R}).$

An operator $d : D(d) \subseteq A \to A$ is called the *infinitesimal generator* of $\{\varphi_t\}_{t\in\mathbb{R}}$ if $d(a) = \lim_{t\to 0} \dfrac{\varphi_t(a) - \sigma(a)}{t}$ where $D(d) = \{a \in A \text{ such that } \lim_{t\to 0} \dfrac{\varphi_t(a) - \sigma(a)}{t} exists\}$. The $\sigma-$ one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called

(i) *uniformly continuous* if $\lim_{t\to 0} \parallel \varphi_t - \sigma \parallel = 0$.

(ii) *strongly continuous* (or $C_0 - \sigma-$ *group*) if for each $a \in A$, $\lim_{t\to 0} \varphi_t(a) = \sigma(a)$.

If $\{\varphi_t\}_{t\in\mathbb{R}}$ is a is a $\sigma-$ one parameter group with the generator d, then one can easily see that

(i) $\sigma^2 = \sigma$ and for each $t \in \mathbb{R}$, $\sigma\varphi_t = \varphi_t\sigma = \varphi_t$.

(ii) for each $a \in D(d)$, $\sigma d = d\sigma = d$.

Example 2.9. Let M be a closed subspace of Hilbert space H, $M^{\perp}$ be the set

$$\{x \in H : < x, m >= 0 \ for \ every \ m \in M\}$$

and let $\{\psi_t\}_{t\in\mathbb{R}}$ be a continuous group on H. If σ is the first projection operator on M, then for $x = y + z \in M \oplus M^{\perp} = H$, the one parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ defined by

$$\varphi_t(x) = \psi_t(y)$$

is a $\sigma-$ group on H with the same continuity of $\{\psi_t\}_{t\in\mathbb{R}}$.

Definition 2.10. The infinitesimal generator d of the $\sigma-$ one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called $\sigma-$ *densely defined* if $\overline{\sigma\big(D(d)\big)} = \sigma(A)$.

3. $\sigma - C^*-$ Dynamics

From now on, A is considered as a C^*- algebra.

Definition 3.1. Let σ be a linear $*-$ endomorphism and $\{\varphi_t\}_{t\in\mathbb{R}}$ be a $\sigma-$ one parameter group of bounded linear $*-$ endomorphism on A such that for each $t \in \mathbb{R}$, $\varphi_t(A) = \sigma(A)$ and $ker(\varphi_t) \subseteq ker(\sigma)$. If additionally $\{\varphi_t\}_{t\in\mathbb{R}}$ is strongly continuous, then it is called a $\sigma - C^*-$ *dynamics* on A.

In the case that $\sigma = I_A$ (the identity operator on A), a $\sigma - C^*-$ dynamics $\{\varphi_t\}_{t\in\mathbb{R}}$ is in fact a C^*- dynamical system in the usual sense.

Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a $\sigma - C^*-$ dynamics on A. We define the *infinitesimal generator* d of $\{\varphi_t\}_{t\in\mathbb{R}}$ as a mapping $d : D(d) \subseteq A \to A$ such that

$$d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - \sigma(a)}{t}$$

where

$$D(d) = \{a \in A \ such \ that \ \lim_{t\to 0} \frac{\varphi_t(a) - \sigma(a)}{t} exists\}.$$

Lemma 3.2. *Let σ be a linear $*-$ endomorphism and $\{\varphi_t\}_{t\in\mathbb{R}}$ be a $\sigma - C^*-$ dynamics on A with the infinitesimal generator d. Then the one parameter family $\{\psi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\sigma(A)$ defined by $\psi_t(\sigma(a)) = \varphi_t(a)$ is a C^*- dynamics and the mapping $\delta : \sigma\big(D(d)\big) \subseteq \sigma(A) \to \sigma(A)$ defined by $\delta(\sigma(a)) = d(a)$ is its generator.*

Proof. Straightforward. □

The followings give us a suitable corresponding between $\sigma - C^*-$ dynamics and $* - \sigma-$ derivations. First, we correspond to each $\sigma - C^*-$ dynamics a $* - \sigma-$ derivation named as its infinitesimal generator.

Theorem 3.3. *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a $\sigma - C^*-$ dynamics on A with the infinitesimal generator d. Then d is a $\sigma-$ densely defined $* - \sigma-$ derivation.*

Proof. First note that by the above lemma, the mapping $\delta : \sigma\big(D(d)\big) \subseteq \sigma(A) \to \sigma(A)$ defined by $\delta(\sigma(a)) = d(a)$ is the generator of the C^*- dynamics $\{\psi_t\}_{t\in\mathbb{R}}$ on $\sigma(A)$ defined by $\psi_t(\sigma(a)) = \varphi_t(a)$ and is a densely defined operator by [9] which follows that d is $\sigma-$ densely defined. Now let $a, b \in D(d)$. We have

$$\begin{aligned}\lim_{t\to 0}\frac{\varphi_t(ab)-\sigma(ab)}{t} &= \lim_{t\to 0}\frac{\varphi_t(a)-\sigma(a)}{t}\varphi_t(b) + \sigma(a)\lim_{t\to 0}\frac{\varphi_t(b)-\sigma(b)}{t}\\ &= d(a)\sigma(b) + \sigma(a)d(b).\end{aligned}$$

Therefore, $ab \in D(d)$ and

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b).$$

Furthermore,

$$\varphi_t(a^*) - \sigma(a^*) = \big(\varphi_t(a) - \sigma(a)\big)^*$$

and since the conjugation operation is norm continuous, so

$$\lim_{t\to 0}\frac{\varphi_t(a^*)-\sigma(a^*)}{t} = \lim_{t\to 0}(\frac{\varphi_t(a)-\sigma(a)}{t})^* = d(a)^*.$$

Therefore $a^* \in D(d)$ and $d(a^*) = d(a)^*$ which shows that d is a $* - \sigma-$ derivation. □

We are going to establish some conditions making the converse of the above theorem be held. More precisely, we like to investigate some restrictions under which a bounded $* - \sigma-$ derivation induces a $\sigma - C^*-$ dynamics. For, we need the following which can be found in [7]:

Lemma 3.4. *Suppose that $\sigma : A \to A$ is an idempotent linear operator and d is a $\sigma-$ derivation such that $d\sigma = \sigma d = d$. Then*

$$d^n(ab) = \sum_{k=0}^{n} \binom{n}{k} d^{n-k}(\sigma(a)) d^k(\sigma(b)) \qquad (a, b \in A \text{ and } n \in \mathbb{N}).$$

Theorem 3.5. *Let σ be an idempotent bounded linear $*-$ endomorphism and d be a bounded $* - \sigma-$ derivation on A such that $d\sigma = \sigma d = d$. Then d induces a $\sigma - C^*-$ dynamics on A.*

Proof. For each $t \in \mathbb{R}$, $a \in A$, define $\varphi_t(a) = e^{td}(\sigma(a))$. Clearly $\{\varphi_t\}_{t\in\mathbb{R}}$ is a $\sigma-$ one parameter group satisfying $\varphi_t(A) = \sigma(A)$ and $ker(\varphi_t) \subseteq ker(\sigma)$. Also the method has been used in the proof of Theorem 1.2.1 of [9] shows that, $\{\varphi_t\}_{t\in\mathbb{R}}$ is uniformly continuous with the generator d. Finally, we have

$$\begin{aligned}
\varphi_t(ab) &= e^{td}(\sigma(ab)) \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} d^k(\sigma(ab)) \\
&= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{t^{(k-r)+r}}{k!} \frac{k!}{r!(k-r)!} d^r(\sigma(a)) d^{k-r}(\sigma(b)) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^m . t^n}{n!m!} d^n(\sigma(a)) d^m(\sigma(b)) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} d^n(\sigma(a)) . \sum_{m=0}^{\infty} \frac{t^m}{m!} d^m(\sigma(b)) \\
&= \varphi_t(a) . \varphi_t(b).
\end{aligned}$$

That is $\{\varphi_t\}_{t\in\mathbb{R}}$ is a $\sigma - C^*-$ dynamics on A. □

The followings investigate the relationship between inner $* - \sigma-$ derivations and $\sigma - C^*-$ dynamics of $* - \sigma-$ inner endomorphisms.

Theorem 3.6. *Let $\{u_t\}_{t\in\mathbb{R}}$ be a uniformly continuous one parameter group of unitary elements in A and $\varphi_t : A \to A$ be the $\sigma - C^*-$ dynamics $\varphi_t(a) = u_t \sigma(a) u_t^*$ of $* - \sigma-$ inner endomorphisms with the generator d. Then d is an inner $* - \sigma-$ derivation.*

Proof. By Stone's theorem [9], there is a self- adjoint element $h \in A$ such that $u_t = e^{ith}$. Also applying the L'Hopital rule we have

$$\begin{aligned} d(a) &= \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} \\ &= \lim_{t\to 0} \frac{e^{ith}\sigma(a)e^{-ith} - \sigma(a)}{t} \\ &= \lim_{t\to 0}(ihe^{th}\sigma(a)e^{-ith} - ie^{th}\sigma(a)ae^{-ith}) \\ &= i[h, \sigma(a)] \end{aligned}$$

therefore by Example 2.2, d is an inner $\sigma-$ derivation. Finally, since σ is a $*-$ linear operator, so d is a $*-$ $\sigma-$ derivation. □

Now we show that the inner $*-$ $\sigma-$ derivation $d_h^\sigma-$ under the following limitations induces a $\sigma - C^*-$ dynamics $\{\varphi_t\}_{t\in\mathbb{R}}$ of $\sigma-$ inner endomorphisms.

Theorem 3.7. *Let $\sigma : A \to A$ be an idempotent bounded linear $*-$ endomorphism on A, h be a self- adjoint element in A satisfying $\sigma(ha) = h\sigma(a)$ and $\sigma(ah) = \sigma(a)h$, and $d_h^\sigma : A \to A$ be the inner $*-\sigma-$ derivation $d_h^\sigma(a) = i[h, \sigma(a)]$. Then there exists a $\sigma - C^*-$ dynamics $\{\varphi_t\}_{t\in\mathbb{R}}$ of $*-\sigma-$ inner endomorphisms on A such that d_h^σ is its generator.*

Proof. First note that we have by induction $\sigma(h^n a) = \sigma(h^n)a$ and $\sigma(ah^n) = \sigma(a)h^n$.

Also the continuity of σ implies that for each $s \in \mathbb{R}$, $\sigma(e^{ish}a) = e^{ish}\sigma(a)$ and $\sigma(ae^{ish}) = \sigma(a)e^{ish}$.

Define $\varphi_t(a) = e^{ith}\sigma(a)e^{-ith}$. Then it is easy to see that $\varphi_t(A) = \sigma(A)$, $ker(\varphi_t) \subseteq ker(\sigma)$. Also, $\varphi_0 = \sigma$ and

$$\begin{aligned} \varphi_t(\varphi_s(a)) &= e^{ith}\sigma(e^{ish}\sigma(a)e^{-ish})e^{-ith} \\ &= e^{ith}.e^{ish}\sigma(\sigma(a)e^{-ish})e^{-ith} \\ &= e^{i(t+s)h}\sigma(\sigma(a))e^{-ish}.e^{-ith} \\ &= e^{i(t+s)h}\sigma(a)e^{-i(t+s)h} \\ &= \varphi_{t+s}(a). \end{aligned}$$

Taking $u_t = e^{ith}$, it follows that $\{u_t\}_{t\in\mathbb{R}}$ is a uniformly continuous one parameter groups of unitaries and

$$\| \varphi_t(a) - \sigma(a) \| \leq 2 \| u_t - I \| \| \sigma \| \| a \| \to 0 \ (as\ t \to 0).$$

Since

$$\begin{aligned} \| \varphi_t(a) - \sigma(a) \| &= \| u_t\sigma(a)u_t^* - \sigma(a) \| \\ &= \| (u_t\sigma(a) - \sigma(a)u_t)u_t^* \| \\ &\leq \| u_t\sigma(a) - \sigma(a)u_t \| \\ &\leq \| u_t\sigma(a) - \sigma(a) \| + \| \sigma(a) - \sigma(a)u_t \| \\ &\leq (\| u_t - I \| + \| I - u_t \|) \| \sigma \| \| a \| . \end{aligned}$$

Therefore $\{\varphi_t\}_{t\in\mathbb{R}}$ is a $\sigma - C^*-$ dynamics of $* - \sigma-$ inner endomorphisms on A and the method has been used in the proof of Theorem 3.6, implies that d_h^σ is its generator. □

Before the next remark, we need the following useful lemma which can be found in [7].

Lemma 3.8. *Let $d : A \to A$ be the inner $\sigma-$ derivation $d(a) = i[h, \sigma(a)]$. If $\sigma^2 = \sigma$, $\sigma(ha) = h\sigma(a)$ and $\sigma(ah) = \sigma(a)h$, then*

$$(-1)^k d^k(a) = i^k \sum_{r=0}^{k} (-1)^r \binom{k}{r} h^r \sigma(a) h^{k-r}$$

for all $x \in M$, $0 \leq k \leq r$ and $r \geq 1$. (See [7], Lemma 2.13.)

Remark 3.9. It is recalled from one parameter group theory, that a bounded $*-$ derivation d on A induces the uniformly continuous one parameter group $\alpha_t(a) = e^{td}(a)$ of $*-$ automorphisms on A. In particular, if h is a self- adjoint element in A and $d(a) = i(ha - ah)$, then it is the generator of the uniformly continuous one parameter group $\beta_t(a) = e^{th}ae^{-th}$ of inner $*-$ automorphism and by uniqueness theorem [9], we have $e^{td}(a) = e^{th}ae^{-th}$. Now under assumption of Theorem 3.7, as we show below a direct calculation applying the formula in Lemma 3.8, gives us the formula $e^{td}\sigma(a) = e^{ith}\sigma(a)e^{-ith}$ for $\sigma-$ one parameter groups similar to the form as stated above for one parameter case.

$$\begin{aligned}
e^{td}(\sigma(a)) &= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}(-1)^k d^k(\sigma(a)) \\
&= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\left[i^k\sum_{r=0}^{k}(-1)^r\tbinom{k}{r}h^r\sigma(a)h^{k-r}\right] \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(-it)^{(k-r)+r}}{k!}(-1)^r\frac{k!}{r!(k-r)!}h^r\sigma(a)h^{k-r} \\
&= \sum_{k=0}^{\infty}\sum_{r=0}^{k}\frac{(-it)^{k-r}.(it)^r}{(k-r)!r!}h^r\sigma(a)h^{k-r} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-it)^m.(it)^n}{m!n!}h^n\sigma(a)h^m \\
&= (\sum_{n=0}^{\infty}\frac{(it)^n}{n!}h^n)\sigma(a)(\sum_{m=0}^{\infty}\frac{(-it)^m}{m!}b^m) \\
&= e^{ith}\sigma(a)e^{-ith}.
\end{aligned}$$

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