HOMOTHETIC MOTIONS AND LIE GROUPS IN $\mathbb{R}_2^4$

FERDAG KAHRAMAN AKSOYAK [1], YUSUF YAYLI [2]
[1]ERCIYES UNIVERSITY, DEPARTMENT OF MATHEMATICS, KAYSERI, TURKEY
[2]ANKARA UNIVERSITY, DEPARTMENT OF MATHEMATICS, ANKARA, TURKEY
E-MAILS: FERDA@ERCIYES.EDU.TR, YAYLI@SCIENCE.ANKARA.EDU.TR

(Received: 29 January 2013, Accepted: 20 February 2013)

Abstract. In this paper, we determine the surfaces by means of the homothetic motions in $\mathbb{R}_2^4$ and reparametrize these surfaces with bicomplex numbers. Also, by using curves and surfaces which are obtained by homothetic motions, we give some special subgroups of the Lie group Q.

AMS Classification: 43A80, 30G35.
Keywords: Lie group, Bicomplex number, Surfaces in semi-Euclidean Space, Homothetic Motion

1. Introduction

In the semi-Euclidean space $\mathbb{R}_2^4$, a one-parameter homothetic motion of a rigid body is generated by the transformation

$$\begin{bmatrix} X' \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} \tag{1}$$

in which X' and X are the position vectors of the same point with respect to the rectangular coordinate frames of the fixed space R' and the moving space R, respectively. Also, h, A and C are continuously differentiable functions of a real parameter t, where A is an orthonormal $n \times n$ matrix that satisfies the property

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 11, NUMBER 1-2 (2013) 23-38.

$A^T \varepsilon A = \varepsilon$, C is a translation vector and h is the homothetic scale of the motion. If an arbitrary position vector of a curve instead of the point X at one-parameter homothetic motion equation which is given by (1) is taken then a surface is obtained in the semi-Euclidean space $\mathbb{R}^4_2$. Yaylı [10] showed that the Hamilton motions are the homothetic motions in four dimensional Euclidean space. Kula and Yaylı [6] showed that the Hamilton motions in semi-Euclidean space $\mathbb{R}^4_2$ are the homothetic motions.

In mathematics, a Lie group is a group which is also a differentiable manifold with the property that the group operations are differentiable [4]. A manifold M carrying n linearly independent non-vanishing vector fields is called parallelisable and a Lie group is parallelisable. The spheres that admit the structure of a Lie group are the 0-sphere S^0 (real numbers with absolute value 1), the circle S^1 (complex numbers with absolute value 1), the 3-sphere S^3 (the set of quaternions of unit form) and S^7. For even $n > 1$ S^n is not a Lie group because it can not be parallelisable as a differentiable manifold. Thus S^n is parallelisable if and only if $n = 0, 1, 3, 7$.

In [1], the authors determined a homothetic motion by using a rotation matrix which is given by Moore in [7] and obtained a surface M by means of this homothetic motion in $\mathbb{R}^4$. They reparametrized this surface M with bicomplex numbers. Also, in the same paper, by using curves and surfaces which are obtained by homothetic motion, they gave some special subgroups of a Lie group P.

Karakuş and Yaylı [3] obtained Lie group structure of a hyperquadric Q in $\mathbb{R}^4_2$ by using bicomplex number product. They changed the tensor product rule and defined a new tensor product rule in $\mathbb{R}^4_2$. By using the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve according to this new tensor product rule they determined some special subgroups of this Lie group Q.

In this paper, we determine two different types of homothetic motions by using two different orthonormal matrices in $\mathbb{R}^4_2$ and obtain the surfaces M_1 and M_2 by means of these homothetic motions. If we take as the homothetic scale $h(t) = 1$ and the translation vector $C(t) = 0$, we obtain rotational surfaces in $\mathbb{R}^4_2$ [5]. Even if, in special cases, we get some tensor product surfaces which is given by Karakuş and Yaylı in [3]. We reparametrize these surfaces M_1 and M_2 with bicomplex numbers. Since to establish group structure on a surface is quite difficult, we ask that question:" How should we choose the position vector of the curve at homothetic motions given by (2) and (3) that the surfaces M_1 and M_2 be a Lie subgroup of the hyperquadric Q. In this study, we answer this question and obtain some special Lie subgroups of this Lie group Q by using surfaces M_1 and M_2 which are obtained by homothetic motions.

Furthermore we mention $C^\infty-$action of the Lie group Q onto the manifold $\mathbb{R}_2^4$ and define actions of $\mathbb{R}$ on Q by using orthonormal matrices at homothetic motions. Also we determine some special subgroups of Q by means of the actions of $\mathbb{R}$ on Q too.

2. Preliminaries

A bicomplex number is a number of the form

$$x = x_1 1 + x_2 i + x_3 j + x_4 ij,\ \ x_1, x_2, x_3, x_4 \in \mathbb{R},$$

where $1, i, j, ij$ are arbitrary units which satisfy the relations $i^2 = -1$, $j^2 = -1$, $ij = ji$. Let us denote the set of bicomplex numbers by C_2. For any $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 1 + y_2 i + y_3 j + y_4 ij$ in C_2, the bicomplex number addition and the scalar multiplication are given by

$$x + y = (x_1 + y_1) + (x_2 + y_2)\, i + (x_3 + y_3)\, j + (x_4 + y_4)\, ij$$

and

$$\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij,$$

respectively. So the set of bicomplex numbers C_2 is a vector space over the real numbers with above the addition and scalar multiplication operations. Let us denote the bicomplex number product over C_2 by $\times$. The product of two bicomplex numbers is defined by

$$\begin{aligned} x \times y &= (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4) + (x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3)\, i \\ &\quad + (x_1y_3 + x_3y_1 - x_2y_4 - x_4y_2)\, j + (x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2)\, ij. \end{aligned}$$

Since the bicomplex number product is associative, commutative and it distributes over vector addition, the set of bicomplex numbers C_2 is a real algebra with the bicomplex product $\times$.

Any bicomplex number $x \in C_2$ can be expressed as 4×4 real matrix as follows:

$$P = \left\{ M_x = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} ; \qquad x_i \in \mathbb{R} \ , \quad 1 \leq i \leq 4 \right\}$$

The mapping

$$g : C_2 \to P$$

defined by

$$g\left(x = x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

transforms a bicomplex number into a 4×4 real matrix. This mapping is one to one and onto. Furthermore $\forall x, y \in C_2$ and $\lambda \in \mathbb{R}$, it satisfies the properties given by

$$\begin{aligned} g\left(x+y\right) &= g\left(x\right) + g\left(y\right) \\ g\left(\lambda x\right) &= \lambda g\left(x\right) \\ g\left(xy\right) &= g\left(x\right) g\left(y\right). \end{aligned}$$

Thus the algebras C_2 and P are isomorphic.

Let $x \in C_2$. Then x can be expressed as $x = \left(x_1 + x_2 i\right) + \left(x_3 + x_4 i\right) j$. In this case, there are three different conjugations of bicomplex numbers as follows:

$$\begin{aligned} x^{t_1} &= \left[\left(x_1 + x_2 i\right) + \left(x_3 + x_4 i\right) j\right]^{t_1} = \left(x_1 - x_2 i\right) + \left(x_3 - x_4 i\right) j \\ x^{t_2} &= \left[\left(x_1 + x_2 i\right) + \left(x_3 + x_4 i\right) j\right]^{t_2} = \left(x_1 + x_2 i\right) - \left(x_3 + x_4 i\right) j \\ x^{t_3} &= \left[\left(x_1 + x_2 i\right) + \left(x_3 + x_4 i\right) j\right]^{t_3} = \left(x_1 - x_2 i\right) - \left(x_3 - x_4 i\right) j \end{aligned}$$

and we can write

$$\begin{aligned} xx^{t_1} &= \left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right) + 2\left(x_1 x_3 + x_2 x_4\right) j \\ xx^{t_2} &= \left(x_1^2 - x_2^2 + x_3^2 - x_4^2\right) + 2\left(x_1 x_2 + x_3 x_4\right) i \\ xx^{t_3} &= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) + 2\left(x_1 x_4 - x_2 x_3\right) ij \end{aligned}$$

For further bicomplex number concepts see [9].

3. Homothetic Motions and Lie Groups in E_2^4

In this section, we define two different types of surfaces by using the homothetic motions. Let $\varphi_1 : M_1 \to E_2^4$ and $\varphi_2 : M_2 \to E_2^4$ be immersions of the surfaces M_1 and M_2 in semi-Euclidean space in E_2^4, respectively.

$$\varphi_1(t,s) = h(t)\begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}\begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \\ \alpha_3(s) \\ \alpha_4(s) \end{pmatrix} + \begin{pmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \end{pmatrix} \tag{2}$$

(3)

$$\varphi_2(t,s) = h(t)\begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & -\sinh t & 0 \\ 0 & -\sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix}\begin{pmatrix} \alpha_1(s) \\ \alpha_2(s) \\ \alpha_3(s) \\ \alpha_4(s) \end{pmatrix} + \begin{pmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \end{pmatrix}$$

where $h(t)$ is the homothetic scale of the motion, $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ is the translation vector and $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ is the profile curve.

Now, we can reparametrize these surfaces by using bicomplex product and addition.

Proposition 3.1. *Let $\varphi_1 : M_1 \to E_2^4$ be an immersion of a surface M_1 in the semi Euclidean space E_2^4. If M_1 is a surface in E_2^4 given by the parametrization* (2), *then M_1 can be reparametrized by $\varphi(t,s) = \beta(t) \times \alpha(s) + C(t)$, where " $\times$ " bicomplex product, " $+$ " bicomplex addition, $\beta(t) = (h(t)\cos t, h(t)\sin t, 0, 0)$, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ are the curves and $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ is the translation vector.*

Proof. We can consider the curves β, α and the translation vector $C(t)$ as bicomplex numbers. Then we can rewrite them as follows:

$$\begin{aligned} \beta(t) &= h(t)\cos t + (h(t)\sin t)\, i \\ \alpha(s) &= \alpha_1(s) + \alpha_2(s)i + \alpha_3(s)j + \alpha_4(s)ij \\ C(t) &= C_1(t) + C_2(t)\, i + C_3(t)\, j + C_4(t)\, ij. \end{aligned}$$

By using the bicomplex product and addition, the surface M_1 can be written as $\varphi(t,s) = \beta(t) \times \alpha(s) + C(t)$. $\square$

Proposition 3.2. *Let $\varphi_2 : M_2 \to E_2^4$ be an immersion of a surface M_2 in the semi Euclidean space E_2^4. If M_2 is a surface in E_2^4 given by the parametrization* (3)*, then M_2 can be reparametrized by $\varphi(t,s) = \delta(t) \times \alpha(s) + C(t)$, where " $\times$ " bicomplex product, " $+$ " bicomplex addition, $\delta(t) = (h(t)\cosh t, 0, 0, h(t)\sinh t)$, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ are the curves and $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ is the translation vector.*

Proof. We can consider the curves δ, α and the translation vector $C(t)$ as bicomplex numbers. Then we can rewrite them as follows:

$$\begin{aligned} \beta(t) &= h(t)\cosh t + (h(t)\sinh t)\, ij \\ \alpha(s) &= \alpha_1(s) + \alpha_2(s)i + \alpha_3(s)j + \alpha_4(s)ij \\ C(t) &= C_1(t) + C_2(t)\, i + C_3(t)\, j + C_4(t)\, ij. \end{aligned}$$

By using the bicomplex product and addition, the surface M_2 can be written as $\varphi(t,s) = \delta(t) \times \alpha(s) + C(t)$. $\square$

Corollary 3.1. *Let $M_{\beta(t)}$ be the matrix representation of bicomplex $\beta(t) = h(t)\cos t + (h(t)\sin t)\, i$. Then we get the surface M_1 given by the parametrization* (2) *as $\varphi(t,s) = M_{\beta(t)}\alpha(s) + C(t)$.*

Corollary 3.2. *Let $M_{\delta(t)}$ be the matrix representation of bicomplex $\delta(t) = h(t)\cosh t + (h(t)\sinh t)\, ij$. Then we get the surface M_2 given by the parametrization* (3) *as $\varphi(t,s) = M_{\delta(t)}\alpha(s) + C(t)$.*

The surfaces M_1 and M_2 given by the parametrization (2) and (3) are reparametrized as bicomplex product of two curves in four dimensional semi-Euclidean space E_2^4. Now we can reparametrize the surfaces M_1 and M_2 as bicomplex product of a curve and a surface.

Corollary 3.3. *Let $\varphi_1 : M_1 \to E_2^4$ be an immersion of a surface M_1 in the semi Euclidean space E_2^4. If M_1 is a surface in E_2^4 given by the parametrization* (2)*, then M_1 can be reparametrized by $\varphi(t,s) = \gamma(t) \times r(t,s) + C(t)$ or $\varphi(t,s) = M_{\gamma(t)} r(t,s) + C(t)$, where $\gamma(t) = (\cos t, \sin t, 0, 0)$ is a circle, $M_{\gamma(t)}$*

is the matrix representation of the curve γ, $r(t,s) = h(t)\alpha(s)$ *is a surface and* $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ *is the translation vector.*

Corollary 3.4. *Let* $\varphi_2 : M_2 \to E_2^4$ *be an immersion of a surface* M_2 *in the semi Euclidean space* E_2^4. *If* M_2 *is a surface in* E_2^4 *given by the parametrization* (3), *then* M_2 *can be reparametrized by* $\varphi(t,s) = \eta(t) \times r(t,s) + C(t)$ *or* $\varphi(t,s) = M_{\eta(t)} r(t,s) + C(t)$, *where* $\eta(t) = (\cosh t, 0, 0, \sinh t)$ *is a circle,* $M_{\eta(t)}$ *is the matrix representation of the curve* η, $r(t,s) = h(t)\alpha(s)$ *is a surface and* $C(t) = (C_1(t), C_2(t), C_3(t), C_4(t))$ *is the translation vector.*

Remark 3.5. *The matrices* $M_{\gamma(t)}$ *and* $M_{\eta(t)}$ *are the representing bicomplex number multiplication in matrix form of the curves* $\gamma(t) = (\cos t, \sin t, 0, 0)$ *and* $\eta(t) = (\cosh t, 0, 0, \sinh t)$, *respectively. The matrices* $M_{\gamma(t)}$ *and* $M_{\eta(t)}$ *are the orthonormal matrices in* E_2^4, *that is, they satisfy*

$$M_{\gamma(t)}^T \varepsilon M_{\gamma(t)} = \varepsilon \text{ and } M_{\eta(t)}^T \varepsilon M_{\eta(t)} = \varepsilon \qquad \varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where ε *is signature matrix in* E_2^4.

4. Lie Groups, C^∞ Action of the Lie Groups and Some Special Lie Subgroups

4.1. **Lie Groups.** Karakuş and Yaylı [3] obtained Lie group structure of a hyperquadric Q that is defined as follows in $\mathbb{R}_2^4$ by using bicomplex number product.

Let the the hyperquadric Q be given by

$$Q = \left\{ x = (x_1, x_2, x_3, x_4) \neq 0 \qquad x_1x_3 + x_2x_4 = 0,\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0 \right\}.$$

We can write Q as the set of bicomplex number as follows:

$$Q = \{ x = x_1.1 + x_2 i + x_3 j + x_4 ij \ ;\ x_1x_3 + x_2x_4 = 0 \}.$$

The matrix representation of Q is given by

$$\tilde{Q} = \left\{ M_x = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} ;\ x_1x_3 + x_2x_4 = 0,\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0 \right\}$$

Theorem 4.1. ([3]) *The set of Q together with the bicomplex number product is a Lie group.*

Proof. $\tilde{Q}$ is a differentiable manifold and at the same time a group with group operation given by matrix multiplication. The group function

$$. : \tilde{Q} \times \tilde{Q} \to \tilde{Q}$$

defined by $(x, y) \to x.y$ is differentiable. So, $(Q, .)$ can be made a Lie group so that g is a isomorphism. □

We denote the set of all unit bicomplex numbers on Q by Q_1. In that case Q_1 is defined as

$$Q_1 = \left\{ x \in Q \ ; \quad \|x\|_{t_1} = 1 \right\}.$$

And we denote the matrix form of the group Q_1 by $\tilde{Q}_1$.

$$\tilde{Q}_1 = \left\{ x \in \tilde{Q} \ ; \ \|x\|_{t_1} = 1 \right\}$$

Q_1 is a subgroup of Q with the group operation of bicomplex multiplication.

Lemma 4.1. ([3]) *Q_1 is 2-dimensional Lie subgroup of Q .*

4.2. **$\mathbf{C^\infty}$ Action of the Lie Groups.** Özkaldı and Yaylı in [8] showed that a hyperquadric $\tilde{P}$ in $\mathbb{R}^4$ together with bicomplex number product is a Lie group and they gave $C^\infty-$action of the Lie group $\tilde{P}$ onto the manifold $\mathbb{R}^4$. Similarly, in this subsection, we mention $C^\infty-$action of the Lie groups $\tilde{Q}$ and $\tilde{Q}_1$ onto the manifold $\mathbb{R}^4_2$ and define the actions of $\mathbb{R}$ on $\tilde{Q}$ and $\tilde{Q}_1$ Lie groups. We obtain some special Lie subgroups of Q and Q_1 by means of these actions of $\mathbb{R}$ on $\tilde{Q}$ and $\tilde{Q}_1$.

Let $\phi : \tilde{Q} \times \mathbb{R}^4_2 \to \mathbb{R}^4_2$ be the mapping that is defined by

$$(A, X) \to \phi(A, X) = AX.$$

for $\forall A \in \tilde{Q}$ and $\forall X \in \mathbb{R}^4_2$.

Theorem 4.2. *The mapping ϕ is a $C^\infty-$action that is effective and transitive of the Lie group $\tilde{Q}$ onto the manifold $\mathbb{R}^4_2$.*

Proof. It can be easily seen that the mapping ϕ satisfes the following equality for $A, B \in \tilde{Q}$ and $X \in \mathbb{R}^4_2$

$$\phi(A, \phi(B, X)) = \phi(AB, X)$$

For all $X \in \mathbb{R}_2^4$, I_4 is the only element of $\tilde{Q}$ such that $\phi(B, X) = X$ and given any two elements $X, Y \in \mathbb{R}_2^4$ there is an element $A \in \tilde{Q}$ such that $\phi(A, X) = Y$. Hence this action is effective and transitive. □

Theorem 4.3. *Let $f : \mathbb{R} \to \tilde{Q}$ and $\tilde{f} : \mathbb{R} \to \tilde{Q}$ be the mappings which are defined by*

$$t \to f(t) = e^{bt} \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix}$$

and

$$t \to \tilde{f}(t) = e^{bt} \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & -\sinh t & 0 \\ 0 & -\sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix},$$

respectively and the mappings ψ and $\tilde{\psi}$ be given by

$$\psi : \mathbb{R} \times \tilde{Q} \to \tilde{Q}$$

$$(t, A) \to \psi(t, A) = Af(t)$$

and

$$\tilde{\psi} : \mathbb{R} \times \tilde{Q} \to \tilde{Q}$$

$$(t, A) \to \tilde{\psi}(t, A) = A\ \tilde{f}(t).$$

Then both mappings ψ and $\tilde{\psi}$ are the actions of $\mathbb{R}$ on $\tilde{Q}$ Lie group.

Proof. Let the mapping $\psi : \mathbb{R} \times \tilde{Q} \to \tilde{Q}$ be given by

$$\psi(t, A) = Af(t).$$

Since f is a homomorphism, it can be easily seen that ψ satisfies

i) $\psi(0, A) = A$

ii) $\psi(t_1 + t_2, A) = \psi(t_1, \psi(t_2, A))$.

Hence, the mapping ψ is an action of $\mathbb{R}$ on $\tilde{Q}$. Similarly $\tilde{\psi}$ is an action of $\mathbb{R}$ on $\tilde{Q}$ Lie group. □

Corollary 4.4. *The images of the mappings f and $\tilde{f}$ determine one-parameter Lie-subgroups of Q.*

Proof. Since the mapping $f:\mathbb{R}\to\tilde{Q}$ is a homomorphism, homomorphic image $H=f(\mathbb{R})$ is a subgroup of $\tilde{Q}$ and since g is a isomorphism, H is a spiral curve as $\beta(t)=e^{bt}(\cos t,\sin t,0,0)$ in hyperquadric Q. So it is a one-parameter Lie-subgroup of Q. Similarly, since the mapping $\tilde{f}:\mathbb{R}\to\tilde{Q}$ is a homomorphism, homomorphic image $K=f(\mathbb{R})$ is a subgroup of $\tilde{Q}$ and K is a curve as $\gamma(t)=e^{bt}(\cosh t,0,0,\sinh t)$ in hyperquadric Q. So the curve γ is a one-parameter Lie-subgroup of Q. □

Corollary 4.5. $\theta_{\tilde{Q}_1}:\tilde{Q}_1\times\mathbb{R}^4_2\to\mathbb{R}^4_2$ *defines a* C^∞ *action of* $\tilde{Q}_1$ *on* $\mathbb{R}^4_2$.

Proof. Since $\tilde{Q}_1$ is a Lie-subgroup of $\tilde{Q}$ and the inclusion map $i:\tilde{Q}_1\to\tilde{Q}$ is C^∞. The restriction $\theta_{\tilde{Q}_1}=\theta\circ i:\tilde{Q}_1\times\mathbb{R}^4_2\to\mathbb{R}^4_2$ defines a C^∞ action of $\tilde{Q}_1$ on $\mathbb{R}^4_2$. □

Theorem 4.6. *Let* $f_1:\mathbb{R}\to\tilde{Q}_1$ *and* $\tilde{f}_1:\mathbb{R}\to\tilde{Q}_1$ *be the mappings which are defined by*

$$t\to f_1(t)=\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}$$

and

$$t\to\ \tilde{f}_1(t)=\begin{pmatrix}\cosh t & 0 & 0 & \sinh t\\ 0 & \cosh t & -\sinh t & 0\\ 0 & -\sinh t & \cosh t & 0\\ \sinh t & 0 & 0 & \cosh t\end{pmatrix},$$

respectively and the mappings ψ_1 *and* $\tilde{\psi}_1$ *be given by*

$$\psi_1:\mathbb{R}\times\tilde{Q}_1\to\tilde{Q}_1$$
$$(t,A)\to\psi_1(t,A)=Af_1(t)$$

and

$$\tilde{\psi}_1:\mathbb{R}\times\tilde{Q}_1\to\tilde{Q}_1$$
$$(t,A)\to\tilde{\psi}_1(t,A)=A\ \tilde{f}_1(t).$$

Then both mappings ψ *and* $\tilde{\psi}$ *are the actions of* $\mathbb{R}$ *on* $\tilde{Q}_1$ *Lie group.*

Proof. Since f_1 is a homomorphism, it can be easily seen that ψ_1 satisfies

i) $\psi_1(0,A)=A$

ii) $\psi_1(t_1+t_2,A)=\psi_1(t_1,\psi_1(t_2,A))$.

Hence, the mapping ψ_1 is an action of $\mathbb{R}$ on $\tilde{Q}_1$. Similarly $\tilde{\psi}_1$ is an action of $\mathbb{R}$ on $\tilde{Q}_1$ Lie group. □

Corollary 4.7. *The images of the mappings f_1 and $\tilde{f}_1$ determine one parameter Lie-subgroups of Q_1*

Proof. Since the mapping $f_1 : \mathbb{R} \rightarrow \tilde{Q}_1$ is a homomorphism, homomorphic image $H_1 = f_1(\mathbb{R})$ is a subgroup of $\tilde{Q}_1$. and H_1 is a circle as $\delta(t) = (\cos t, \sin t, 0, 0)$ in Q_1. So it is a one-parameter Lie-subgroup of Q_1. Similarly since the mapping $\tilde{f}_1 : \mathbb{R} \rightarrow \tilde{Q}_1$ is a homomorphism, homomorphic image $K_1 = f(\mathbb{R})$ is a subgroup of $\tilde{Q}_1$ and K_1 is a curve as $\eta(t) = (\cosh t, 0, 0, \sinh t)$ in Q_1. So the curve η is a one-parameter Lie-subgroup of Q_1. □

4.3. **Some Special Lie Subgroups.** Ö. Karakuş and Yaylı [3] showed that a hyperquadric Q in $\mathbb{R}^4_2$ is a Lie group by using bicomplex number product. Also they gave a new tensor product definition in $\mathbb{R}^4_2$ as follows:

$$(\alpha \otimes \beta)(t, s) = (\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), -\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s))$$

where $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2_1$ and $\beta : \mathbb{R} \rightarrow \mathbb{R}^2$ is respectively a Lorentzian plane curve and a Euclidean plane curve. Hence, by means of the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, they determined some special subgroups of this Lie group Q.

Since to establish group structure on a surface is quite difficult, we ask that question:" How should we choose the position vector of the curve at homothetic motions given by (2) and (3) that the surfaces M_1 and M_2 be a Lie subgroup of the hyperquadric Q. We answer this question. If we take the profile curve $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s), \alpha_4(s))$ such that $\alpha_1(s)\alpha_3(s) + \alpha_2(s)\alpha_4(s) = 0$ and the translation vector $C(t) = 0$, then the surfaces M_1 and M_2 are given by the parametrizations (2) and (3) are subsets of Q. Even if, in special cases, the surfaces M_1 and M_2 coincide above new tensor product surface. Thus, by using the surfaces M_1 and M_2 which are obtained with homothetic motions, we determine some special subgroups of this Lie group Q.

Theorem 4.8. *Let γ be a curve which is obtained by using the homothetic motion given by* (3) *with the homothetic function $h(t) = e^{at}$ and the profile curve $\alpha(t) =$*

$e^{bt}(\cosh t, 0, 0, \sinh t)$, *where* a, b *are real constants. Then the curve* γ *is a one-parameter Lie subgroup in Lie group* Q.

Proof. We can write the curve γ as follows:

$$\begin{aligned}\gamma(t) &= e^{at}\begin{pmatrix}\cosh t & 0 & 0 & \sinh t\\ 0 & \cosh t & -\sinh t & 0\\ 0 & -\sinh t & \cosh t & 0\\ \sinh t & 0 & 0 & \cosh t\end{pmatrix}\begin{pmatrix}e^{bt}\cosh t\\ 0\\ 0\\ e^{bt}\sinh t\end{pmatrix}\\ &= e^{(a+b)t}(\cosh 2t, 0, 0, \sinh 2t).\end{aligned}$$

Since $\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$, the closure property is satisfied. The identity element of curve γ is $\gamma(0) = (1, 0, 0, 0)$ and for all $t \in \mathbb{R}$ the inverse element of $\gamma^{-1}(t) = \gamma(-t)$. Hence γ is a one parameter Lie subgroup of Q. □

Theorem 4.9. *Let* γ *be a curve which is obtained by using the homothetic motion given by* (2) *with the homothetic function* $h(t) = e^{at}$ *and the profile curve* $\alpha(t) = e^{bt}(\cos t, \sin t, 0, 0)$, *where* a, b *are real constants. Then the curve* γ *is a one-parameter Lie subgroup in Lie group* Q.

Proof. We can write the curve γ as follows:

$$\begin{aligned}\gamma(t) &= e^{at}\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}e^{bt}\cos t\\ e^{bt}\sin t\\ 0\\ 0\end{pmatrix}\\ &= e^{(a+b)t}(\cos 2t, \sin 2t, 0, 0).\end{aligned}$$

Since $\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$ for all $t_1, t_2 \in \mathbb{R}$, the closure property is satisfied. The identity element of curve γ is $\gamma(0) = (1, 0, 0, 0)$ and for all $t \in \mathbb{R}$ the inverse element of $\gamma^{-1}(t) = \gamma(-t)$. Hence γ is a one parameter Lie subgroup of Q. □

Theorem 4.10. *Let* γ *be a curve which is obtained by using the homothetic motion given by* (2) *with the homothetic function* $h(t) = e^{at}$ *and the profile curve* $\alpha(t) = e^{bt}(\cosh t, 0, 0, \sinh t)$, *where* a, b *are real constants. Then the curve* γ *is a one-parameter Lie subgroup in Lie group* Q.

Proof. We can write the curve γ as follows:

$$\begin{aligned}\gamma(t) &= e^{at}\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}e^{bt}\cosh t\\ 0\\ 0\\ e^{bt}\sinh t\end{pmatrix}\\ &= e^{(a+b)t}\left(\cosh t\cos t, \cosh t\sin t, -\sinh t\sin t, \sinh t\cos t\right).\end{aligned}$$

Since $\gamma(t_1)\times\gamma(t_2)=\gamma(t_1+t_2)$ for all $t_1,t_2\in\mathbb{R}$, the closure property is satisfied. The identity element of curve γ is $\gamma(0)=(1,0,0,0)$ and for all $t\in\mathbb{R}$ the inverse element of $\gamma^{-1}(t)=\gamma(-t)$. Hence γ is a one parameter Lie subgroup of Q. □

Remark 4.11. *The above curve γ can be obtained by using the homothetic motion given by* (3) *with the homothetic function $h(t)=e^{at}$ and the profile curve $\alpha(t)=e^{bt}(\cos t,\sin t,0,0)$. Also, the same curve γ can be expressed as tensor product of two spirals with the same parameter, that is, let $\beta:\mathbb{R}\to\mathbb{R}^2_1$, $\beta(t)=e^{at}(\cosh t,\sinh t)$ and $\delta:\mathbb{R}\to\mathbb{R}^2$ $\delta(t)=e^{bt}(\cos t,\sin t)$ be two spirals. Then the curve γ can be written as $\gamma(t)=\beta(t)\otimes\delta(t)$.*

Remark 4.12. *From Corollary(*4.4*), we know that $\beta(t)=e^{bt}(\cos t,\sin t,0,0)$ and $\gamma(t)=e^{bt}(\cosh t,0,0,\sinh t)$ are one-parameter Lie subgroups of Q. In Theorem (*4.8*), in Theorem (*4.9*) and in Theorem (*4.10*) we show that the trajectors of the curves β and δ under the homothetic motions given by* (2) *and* (3) *are one-parameter Lie subgroups of Q too.*

Corollary 4.13. *Let γ be a curve which is obtained by using the homothetic motion given by* (3) *with the homothetic function $h(t)=1$ and the profile curve $\alpha(t)=(\cosh t,0,0,\sinh t)$. Then the curve γ is a one-parameter Lie subgroup in Lie group Q_1.*

Proof. By using homothetic motion is given by (3) for $h(t)=1$ and the profile curve $\alpha(t)=(\cosh t,0,0,\sinh t)$, we get

$$\gamma(t)=(\cosh 2t,0,0,\sinh 2t).$$

Since $\|\gamma(t)\|_{t_1}=1$, it implies that $\gamma(t)\subset Q_1$. So it is a one-parameter Lie subgroup of Q_1. □

Corollary 4.14. *Let γ be a curve which is obtained by using the homothetic motion given by* (2) *with the homothetic function $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, \sin t, 0, 0)$. Then curve γ is a one-parameter Lie subgroup in Lie group Q_1.*

Proof. By using homothetic motion is given by (2) for $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, \sin t, 0, 0)$, we get

$$\gamma(t) = (\cos 2t, \sin 2t, 0, 0).$$

Since $\|\gamma(t)\|_{t_1} = 1$, it implies that $\gamma(t) \subset Q_1$. So it is a one-parameter Lie subgroup of Q_1. □

Corollary 4.15. *Let γ be a curve which is obtained by using the homothetic motion given by* (2) *with the homothetic function $h(t) = 1$ and the profile curve $\alpha(t) = (\cosh t, 0, 0, \sinh t)$. Then the curve γ is a one-parameter Lie subgroup in Lie group Q_1.*

Proof. For $h(t) = 1$ and the profile curve $\alpha(t) = (\cosh t, 0, 0, \sinh t)$, by using (2) we get

$$\gamma(t) = (\cosh t \cos t, \cosh t \sin t, -\sinh t \sin t, \sinh t \cos t).$$

Since $\|\gamma(t)\|_{t_1} = 1$, it follows that $\gamma(t) \subset Q_1$. So it is a one-parameter Lie subgroup of Q_1. □

Remark 4.16. *The above curve γ can be obtained by using the homothetic motion given by* (3) *with the homothetic function $h(t) = 1$ and the profile curve $\alpha(t) = (\cos t, \sin t, 0, 0)$. Also, the same curve γ can be expressed as tensor product of two circles with the same parameter, that is, let $\beta : \mathbb{R} \to \mathbb{R}^2_1$, $\beta(t) = (\cosh t, \sinh t)$ be a Lorentzian circle and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(t) = (\cos t, \sin t)$ be a circle. Then the curve γ can be written as $\gamma(t) = \beta(t) \otimes \delta(t)$.*

Theorem 4.17. *Let M be a surface which is obtained by using the homothetic motion given by* (2) *with the homothetic function $h(t) = e^{at}$ and the profile curve $\alpha(s) = e^{bs}(\cosh s, 0, 0, \sinh s)$. Then the surface M is 2-dimensional Lie subgroup of Q.*

Proof. We obtain

$$\begin{aligned}\varphi(t,s) &= e^{at}\begin{pmatrix}\cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & -\sin t\\ 0 & 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}e^{bs}\cosh s\\ 0\\ 0\\ e^{bs}\sinh s\end{pmatrix}\\ &= e^{at+bs}\left(\cosh s\cos t, \cosh s\sin t, -\sinh s\sin t, \sinh s\cos t\right).\end{aligned}$$

Since $\varphi(t_1,s_1)\times\varphi(t_2,s_2)=\varphi(t_1+t_2,s_1+s_2)$ for all $t_1,t_2,s_1,s_2\in\mathbb{R}$, the closure property is satisfied. The identity element of M is $\varphi(0,0)=(1,0,0,0)$ and for all $t,s\in\mathbb{R}$ and the inverse element of M $\varphi^{-1}(t,s)=\varphi(-t,-s)$. Then M is a subgroup of Q. On the other hand, since M is a submanifold of Q, it is a 2-dimensional Lie subgroup of Q. □

Remark 4.18. *The surface M can be obtained by using the homothetic motion given by* (3) *with the homothetic function $h(t)=e^{at}$ and the profile curve $\alpha(s)=e^{bs}(\cos s,\sin s,0,0)$. Also, this surface M can be expressed as tensor product surface of two spirals, that is, let $\beta:\mathbb{R}\to\mathbb{R}^2_1$, $\beta(s)=e^{as}(\cosh s,\sinh s)$ be a hyperbolic spiral and $\delta:\mathbb{R}\to\mathbb{R}^2$ $\delta(t)=e^{bt}(\cos t,\sin t)$ be a spiral. Then the surface M can be written as $\varphi(t,s)=\beta(s)\otimes\delta(t)$.*

Corollary 4.19. *Let M be a surface which is obtained by using the homothetic motion is given by* (2) *with the homothetic function $h(t)=1$ and the profile curve $\alpha(s)=(\cosh s,0,0,\sinh s)$. Then surface M is 2-dimensional Lie subgroup of Q_1.*

Proof. By using homothetic motion is given by (2) for $h(t)=1$ and the profile curve $\alpha(s)=(\cosh s,0,0,\sinh s)$, we get

$$\varphi(t,s)=\left(\cosh s\cos t,\cosh s\sin t,-\sinh s\sin t,\sinh s\cos t\right).$$

Since $\|\varphi(t,s)\|_{t_1}=1$, it follows that $\varphi(t,s)\subset Q_1$. Hence the surface M is a 2-dimensional Lie subgroup of Q_1. □

References

[1] F. K. Aksoyak, Y. Yaylı, Homothetic motions and surfaces in $\mathbb{E}^4$, Bull. Malays. Math. Sci. Soc. (accepted)

[2] F. Bricklell, R.S. Clark, Differentiable Manifolds, Van Nostrand Reinhold Company London, 1970.

[3] S. Ö. Karakuş, Y. Yaylı, Bicomplex number and tensor product surfaces in $\mathbb{R}_2^4$, Ukrainian Mathematical Journal, 64 (3) (2012), 344-355

[4] A. Karger, J. Novak , Space kinematics and Lie groups, Gordan and Breach Publishers, 1985

[5] Y. H. Kim, D.W. Yoon, Classifications of rotation surfaces in pseudo- Euclidean space, J. Korean Math. Soc., 41 (2004), 379-396

[6] L. Kula, Y. Yaylı, Homothetic motions in semi-Euclidean space, Proceeding of the Royal Irish Academy, 105A (1) (2005), 9-15.

[7] C.L.E. Moore, Surfaces of rotation in a space of four dimensions, Ann. Math., 21 (2) (1919), 81-93.

[8] S. Özkaldı, Y. Yaylı, Tensor product surfaces and Lie groups, Bull. Malays. Math. Sci. Soc., 33 (2) (2010), 69-77.

[9] G. B. Price, An introduction to multicomplex spaces and functions, Marcel Dekker Inc., (1990).

[10] Y. Yaylı , 1992 Homothetic Motions at $\mathbb{E}^4$. Mech. Mach. Theory., 27 (3) (1992), 303-305.