

h -AVERAGE-SHADOWING PROPERTY AND CHAOS

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ABSTRACT. Let X be a compact metric space and $f : X \rightarrow X$ be a homeomorphism map. In this paper we are going to introduce the h -average shadowing property for discrete dynamical system (X, f) . It is shown that f has the h -average shadowing property if and only if f^{-1} has the h -average shadowing property. We also prove that if f has the h -average shadowing property, then every point x is chain recurrent. The relation between chaos and shadowing property in discrete dynamical system is considered.

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1. INTRODUCTION

Let (X, d) be a metric space and $f : X \rightarrow X$ be a homeomorphism map. For every positive integer n , we define f^n inductively by $f^n = f \circ f^{n-1}$ and f^0 is the identity map on X . Also $f^{-n} = (f^{-1})^n$ where f^{-1} is inverse of f . If $x \in X$ then the orbit of x under f is the sequence $x = \{f^n(x)\}_{n \in \mathbb{Z}}$. We refer (X, f) as a discrete dynamical system. A subset A in X is said to be f -invariant if $f(A) \subseteq A$.

The concept of shadowing was investigated by many authors, for example see

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[1, 2, 3, 9, 13]. In this direction the concept of average shadowing property for discrete dynamical system (X, f) , where f is an onto continuous map, was considered by some researchers, see [4, 5, 8, 10]. Especially, we have the works of Kazuhiro Sakai [11, 12, 13]. We use these works in the next section and we thank him for that.

Chaos is a complex dynamical behavior that occurs in many nonlinear systems, such as cryptography [14]. The term chaos in connection with a map was firstly used by Li and Yorke, *A system is chaotic if it contains infinitely many periodic orbits whose periods are arbitrary large; moreover some property like "expansiveness" holds* [7]. Although this concept was without giving any formal definition. Today there are various definitions of what it means for a system to be chaotic. A common idea of them is to show the complexity and unpredictability of behavior of the orbits of a system. In this article the dynamical system (X, f) is called chaotic if some positive iteration of f is semi-conjugated to a shift map. It seems very attractive to study the relation between chaos and other notion of dynamical systems. In some recent works, Niu [8], Koscielniak and Mazur [6], discussed the relation between shadowing property and chaotic dynamical systems. In this paper we are going to investigate the relation between the average-shadowing property and chaotic dynamical systems. In Section 3 we study average shadowing property for homeomorphism maps and proved that if f has average shadowing property then f^{-1} has also average shadowing property. In Section 4, some relations between average shadowing property and chaotic dynamical systems are considered.

2. PRELIMINARIES

In [5, 8, 10] the average shadowing property is defined and discussed for continuous maps. Let $f : X \rightarrow X$ be a continuous map. For $\delta > 0$, a sequence $\{x_i\}_{i \geq 0} \subset X$ is called a δ -average-pseudo-orbit of f if there is a number $N = N(\delta)$ such that for all $n \geq N$ and $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

We say that f has the average shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ -average-pseudo-orbit $\{x_i\}_{i \geq 0} \subset X$ is ϵ -shadowed in average by

some point $y \in X$, that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon.$$

For $\delta > 0$, a sequence $\{x_i\}_{i=-\infty}^{\infty}$ of points in X is called a δ -average pseudo-orbit of f if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

The function f as a homeomorphism map has the average shadowing property, if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ -average-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ -shadowed in average by some point $y \in X$; that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y), x_i) < \epsilon.$$

We use the notion of chain recurrent points to study chaotic dynamical systems.

The set $CR(f)$ consisting of all chain recurrent points, i.e., such points $x \in X$ that for any $\delta > 0$, there exists a periodic δ -pseudo-orbit through x , is called the chain recurrent set of the discrete dynamical system (X, f) .

A point $x \in A \subset X$ is said to be a stable point for homeomorphism map $f : X \rightarrow X$ in A if for any $\epsilon > 0$, there exists $\delta > 0$ such that for every $y \in A$ with $d(x, y) < \delta$ we have $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{N}$. Otherwise we call such a point unstable in A [6].

Suppose Σ denote the compact shift space with $\{0, 1\}^{\mathbb{Z}}$, topologized by the metric $\rho(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^{|i|}}$. It is clear that (Σ, ρ) is a bounded metric space with diameter 3. Let $\sigma : \Sigma \rightarrow \Sigma$ be the corresponding shift map, i.e. $(\sigma(x))_i = x_{i+1}$, for all $i \in \mathbb{Z}$.

The discrete dynamical system (X, f) is said to be semi-conjugated to (Σ, σ) , if there exist a closed, f -invariant set $\Lambda \subseteq X$ and a continuous surjection $\phi : \Lambda \rightarrow \Sigma$ such that $\phi \circ f = \sigma \circ \phi$.

The dynamical system (X, f) is called chaotic if some positive iterations of f is semi-conjugated to (Σ, σ) .

3. h -average shadowing property

In all of this paper $H(X)$ denotes the set of all homeomorphisms of X and f is a function from X onto itself.

Now we define average-shadowing property for homeomorphism maps.

Definition 3.1. For $\delta > 0$. The sequence $\{x_i\}_{i \in \mathbb{Z}} \subset X$ is called a $\delta - h$ -average-pseudo-orbit of f , if $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f(x_i), x_{i+1}) < \delta$.

Definition 3.2. A map f is said to has the h -average-shadowing property if for every $\epsilon > 0$, there is $\delta > 0$ such that, every $\delta - h$ -average-pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}}$ is ϵ -shadowed in average by some point $z \in X$, that is,

$$\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Similar to shadowing property we show that the homeomorphism map f has the h -average-shadowing property if and only if f^{-1} has the h -average-shadowing property.

In order to prove the Proposition 3.4, we need the following lemma.

Lemma 3.3. Suppose X is a compact metric space. For every $\delta > 0$ there exist $\gamma > 0$ such that if $\{y_i\}_{i \in \mathbb{Z}}$ is a $\gamma - h$ -average-pseudo-orbit for f then $\{x_i\}_{i \in \mathbb{Z}}$ is a $\delta - h$ -average-pseudo-orbit for f^{-1} , where $x_i = y_{-i}$ for all $i \in \mathbb{Z}$.

Proof. Fix δ as a positive number. Since f^{-1} is a continuous map then there exists $0 < \delta' < \delta$ such that if $d(x, y) < \delta'$ then $d(f^{-1}(x), f^{-1}(y)) < \delta$, for all $x, y \in X$. Now, suppose $\{y_i\}_{i \in \mathbb{Z}}$ is a $\frac{(\delta')^2}{L} - h$ -average-pseudo-orbit for f , where L is diameter of X .

Suppose $A = \{i \in \mathbb{Z} | d(f(y_i), y_{i+1}) \geq \delta'\}$, $A_n = A \cap \{-n+1, \dots, 0, \dots, n-1\}$, $B_n = A^c \cap \{-n+1, \dots, 0, \dots, n-1\}$ and $\limsup_{n \rightarrow \infty} \frac{\text{card}(A_n)}{2n} = t$. Please note that $\text{card}(A_n)$ and $\text{card}(A^c)$ denote the cardinality of A_n and cardinality of complement of A , respectively.

Since $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f(y_i), y_{i+1}) < \frac{(\delta')^2}{L}$. Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i \in A_n} \delta' &\leq \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i \in A_n} d(f(y_i), y_{i+1}) \\ &< \frac{(\delta')^2}{L}. \end{aligned}$$

So $t\delta' < \frac{(\delta')^2}{L}$ and consequently $t < \frac{\delta'}{L}$. Now consider the sequence $\{x_i\}_{i \in \mathbb{Z}}$ where by hypothesis of lemma $x_i = y_{-i}$.

Then

$$\frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^{-1}(x_i), x_{i+1}) = \frac{1}{2n} \left[\sum_{i \in A_n} d(f^{-1}(x_i), x_{i+1}) + \sum_{i \in B_n} d(f^{-1}(x_i), x_{i+1}) \right].$$

Please note that, $x_{i+1} = f^{-1}(f(x_{i+1})) = f^{-1}(f(y_{-i-1}))$ and $f^{-1}(x_i) = f^{-1}(y_{-i})$. So $d(f^{-1}(x_i), x_{i+1}) = d(f^{-1}(y_{-i}), f^{-1}(f(y_{-i-1}))) = d(f^{-1}(f(y_{-i-1})), f^{-1}(y_{-i}))$.

Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^{-1}(x_i), x_{i+1}) &\leq \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i \in A_n} d(f^{-1}(x_i), x_{i+1}) \\ &+ \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i \in B_n} d(f^{-1}(x_i), x_{i+1}) \\ &\leq tL \\ &+ \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i \in B_n} d(f^{-1}(f(y_{-i-1})), f^{-1}(y_{-i})) \\ &< \delta' + \delta < 2\delta. \end{aligned}$$

So $\{x_i\}_{i \in \mathbb{Z}}$ is a $2\delta - h$ -pseudo-orbit for f^{-1} . □

Proposition 3.4. *Suppose X is a compact metric space. If $f \in H(X)$ has the h -average-shadowing property, then f^{-1} has the same property.*

Proof. Suppose $\epsilon > 0$, since f has the h -average shadowing property then there is $\delta > 0$ such that for any $\delta - h$ -average-pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}}$ we can find $z \in X$ which $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^i(z), x_i) < \epsilon$. By Lemma 3.3, there exist a $\delta' > 0$ such that if $\{y_i\}_{i \in \mathbb{Z}}$ is $\delta' - h$ -average-pseudo-orbit for f^{-1} then $\{x_i\}_{i \in \mathbb{Z}}$ is a $\delta - h$ -average-pseudo-orbit for f , where $x_i = y_{-i}$ for all $i \in \mathbb{Z}$. Therefore if $\{y_i\}_{i \in \mathbb{Z}}$ is $\delta' - h$ -average-pseudo-orbit for f^{-1} , then there exist a $z \in X$ such that $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^i(z), x_i) < \epsilon$. This implies that $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d((f^{-1})^{-i}(z), y_{-i}) < \epsilon$. So f^{-1} has the h -average shadowing property. □

Lemma 3.5. *If f has the h -average shadowing property then has average shadowing property as a homeomorphism map.*

Proof. Let ϵ be an arbitrary positive number and $\delta > 0$ be an ϵ -modulus of h -average shadowing property. Suppose $\{x_i\}_{i=-\infty}^{\infty}$ is an δ -average-pseudo-orbit, so there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

This implies that $\frac{1}{2n} \sum_{i=0}^{2n-1} d(f(x_{i-n}), x_{i+1-n}) < \delta$, for all $n \geq N$.

Then $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{j=-n}^n d(f(x_j), x_{j+1}) < \delta$. Therefore $\{x_i\}_{i=-\infty}^{\infty}$ is a $\delta - h$ -average-pseudo-orbit. By h -average shadowing property there exist $z \in X$ such that $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < \epsilon$. Then $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) \leq 2 \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(f^i(z), x_i) < 2\epsilon$. $\square$

Theorem 3.6. [10] *If a continuous surjective map f on a compact metric space X has the average shadowing property, then every point x is chain recurrent. Moreover f has only one chain component which is the whole space.*

By Lemma 3.5 and Theorem 3.6 we have the following theorem.

Theorem 3.7. *If a homeomorphism map f on a compact metric space X has the h -average shadowing property, then every point x is chain recurrent. Moreover f has only one chain component which is the whole space.*

Example 3.8. Let $\{a_i\}_{i=-\infty}^{\infty}$ and $\{b_i\}_{i=-\infty}^{\infty}$ are two sequences in Euclidean metric space $(\mathbb{R}^2, d)$ such that $d(a_i, a_{i+1}) = d(b_i, b_{i+1}) = d(a_i, b_i) = |i|$, for all $i \in \mathbb{Z}$. Assume that $X = \{a_i\}_{i=-\infty}^{\infty} \cup \{b_i\}_{i=-\infty}^{\infty}$ and $f : X \rightarrow X$ be a map that $f(a_i) = a_{i+1}$ and $f(b_i) = b_{i+1}$, for all $i \in \mathbb{Z}$ (see figure 3.8). This is clear that f is a homeomorphism on X , respect to Euclidean metric. We show that $f : X \rightarrow X$ has the average shadowing property but does not have the h -average shadowing property. Let $\epsilon > 0$ and $\{x_i\}_{i=-\infty}^{\infty}$ is a δ -average-pseudo-orbit, where $\delta = \min\{\frac{\epsilon}{2}, 1\}$. We claim that there exist $n_0 \in \mathbb{Z}$ that $f(x_i) = x_{i+1}$, for all $i \geq n_0$. It is clear that if there exist an increasing sequence $A = \{n_k\}_{k \geq 1}$ of positive integer numbers that $f(x_{n_k}) \neq x_{n_k+1}$, then $\frac{1}{n_k} \sum_{i=0}^{n_k} d(f(x_i), x_{i+1}) \geq 1$, for all $n_k \in A$. That is contradict with this fact that, there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta < 1.$$

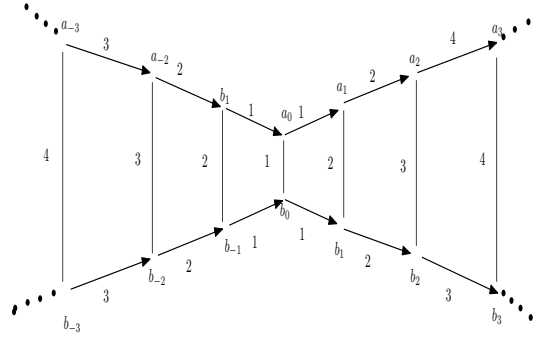


FIGURE 1. The graph of the function f from Example 3.8

So there exist $n_0 \in \mathbb{Z}$ that $f(x_i) = x_{i+1}$, for all $i \geq n_0$. Without loss of generality, we may assume that there exist $j_0 \in \mathbb{Z}$ such that $x_{n_0} = a_{j_0}$. This implies that $x_{n_0+i} = a_{j_0+i}$, for all $i \geq 0$. Therefore $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n d(f^i(a_{j_0-n_0}), x_i) < \epsilon$.

Now we show that f does not have the h -average shadowing property. Let $\epsilon = 1$ and $\{x_i\}_{i=-\infty}^{\infty}$ is a sequence that $x_i = a_i$ for all $i \geq 0$ and $x_i = b_i$ for all $i < 0$. This is clear that for every $\delta > 0$, $\{x_i\}_{i=-\infty}^{\infty}$ is a δ - h -average-pseudo-orbit for f , but for every $a \in X$, $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^i(a), x_i) > 1$.

4. h -AVERAGE SHADOWING PROPERTY AND STABILITY

Now we investigate the relation between chaotic dynamical system and h -average shadowing property for discrete dynamical system (X, f) where $f \in H(X)$ and (X, d) is a compact metric space.

As following we define h -stable and h -average stable for homeomorphism maps.

Definition 4.1. Let $f \in H(X)$, a point $x \in A \subset X$ is said to be h -stable point for f in A if for any $\epsilon > 0$, there is a $\delta > 0$ such that for every $y \in A$ satisfying $d(x, y) < \delta$, we have $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$.

Now we define the notion of stability in average.

Definition 4.2. Let $f \in H(X)$, a point $x \in A \subset X$ is said to be h -average-stable point for f in A if for any $\epsilon > 0$, there is a $\delta > 0$ such that for every $y \in A$ satisfying

$d(x, y) < \delta$, we have $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^i(x), f^i(y)) < \epsilon$. Otherwise we call such a point h -average unstable in A .

Based on the above notions and definitions, we prove the following two lemmas. We use these lemmas to prove the Theorem 4.7.

Lemma 4.3. *Let $f \in H(X)$ and A be a subset of X .*

1. *Every stable point for f in A , is an h -average-stable point for f in A .*
2. *Suppose m is a nonzero integer number. If x is an h -average-unstable point for f^m in A , then x is an h -average-unstable point for f in A .*

Proof. 1. Suppose $x \in A$ is a stable point for f in A and $\epsilon > 0$. There exist a number $\delta > 0$ such that for every $y \in A$ satisfying $d(x, y) < \delta$, $d(f^i(x), f^i(y)) < \epsilon$ for all $i \in \mathbb{Z}$, therefore $\sum_{|i|=0}^{n-1} d(f^i(x), f^i(y)) < 2n\epsilon$. This implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^i(x), f^i(y)) < \epsilon.$$

Thus x is an h -average-stable point for f in A .

2. By the definition of h -average-unstable point, we can find $\epsilon > 0$ such that for any $\delta > 0$ there exists $y \in X$ such that $d(x, y) < \delta$ and $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d((f^m)^i(x), (f^m)^i(y)) > \epsilon$. Obviously, we have

$$\frac{1}{2n |m|} \sum_{|i|=0}^{n-1} d(f^{mi}(x), f^{mi}(y)) \leq \frac{1}{2n |m|} \sum_{|i|=0}^{|m|n-1} d(f^i(x), f^i(y)).$$

Hence

$$\begin{aligned} & \frac{1}{|m|} \left[\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d((f^m)^i(x), (f^m)^i(y)) \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{2n |m|} \sum_{|i|=0}^{|m|n-1} d(f^i(x), f^i(y)) \end{aligned}$$

So $\limsup_{n \rightarrow \infty} \frac{1}{2n|m|} \sum_{|i|=0}^{mn-1} d(f^i(x), f^i(y)) > \frac{\epsilon}{|m|}$. □

Lemma 4.4. *All points of Σ are h -average-unstable point for σ in Σ .*

Proof. Suppose $x \in (\Sigma, \sigma)$ and $\epsilon = \frac{1}{3}$, if δ is an arbitrary positive number, then there exists $N = N(\delta)$, such that $\sum_{|i| > N(\delta)} \frac{1}{2^{|i|}} < \delta$ and $\sum_{|i|=0}^{N(\delta)} \frac{1}{2^{|i|}} > \frac{1}{2}$. Now we

construct $y_\delta \in \Sigma$ as follows:

$$y_i = \begin{cases} x_i & \text{if } |i| \leq N(\delta) \\ 1 - x_i & \text{if } |i| > N(\delta) \end{cases},$$

Obviously $\rho(x, y_\delta) < \delta$. On the other hand $(\sigma^n(x))_i \neq (\sigma^n(y_\delta))_i$ for all $|n| > N(\delta)$ and $|i| < N(\delta)$. So $\rho(\sigma^n(x), \sigma^n(y_\delta)) > \frac{1}{2}$, for all n which $|n| > N(\delta)$. This implies that $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} \rho(\sigma^n(x), \sigma^n(y_\delta)) \geq \frac{1}{2} > \frac{1}{3}$. Therefore all points of Σ are h -average-unstable point for σ , in Σ . $\square$

The next theorem is one of the main results of this paper.

Theorem 4.5. *Assume that $f \in H(X)$ has the h -average-shadowing-property. If f is chaotic, there is an infinite number of h -average chain recurrent points which are h -average-unstable in $CR(f)$.*

The original statement of the proof of this theorem comes from [6].

Proof. Suppose f^k , with some $k \in \mathbb{N}$, is semi-conjugate to (Σ, σ) and $\phi : \Lambda \rightarrow \Sigma$ denote a semi-conjugate map. Because of Theorem 3.7 we can assume that $\Lambda \subset CR(f)$. By Lemma 4.3, it suffices to prove that for all $x \in \Sigma$ there exists a point $x \in \phi^{-1}(x)$ which is h -average-unstable for f^k in Λ .

By Lemma 4.4, suppose $x \in \Sigma$ is an h -average-unstable point with constant 1. For every $m \in \mathbb{N}$, there exist $y_m \in \Sigma$ such that $\rho(x, y_m) < \frac{1}{m}$ and $\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} \rho(\sigma^i(x), \sigma^i(y_m)) > 1$. Since the map ϕ is surjection, then for every $m \in \mathbb{N}$ we can find $y_m \in \Lambda$ that $\phi(y_m) = y_m$. Without loss of generality suppose $\lim_{m \rightarrow \infty} y_m = x$, and so $\phi(x) = x$ for some $x \in X$.

Assume that x is an h -average-stable point for f^k in $CR(f)$. Continuity of ϕ imply that there exists a $\delta > 0$ such that for all $y \in X$, if $d(x, y) < \delta$, then $\rho(\phi(x), \phi(y)) < \frac{1}{3}$. The h -average stability of x implies that there exists a $\delta' > 0$, such that for all $y \in CR(f)$ satisfying $d(x, y) < \delta'$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^{ik}(x), f^{ik}(y)) < \frac{\delta}{3}.$$

Let m be a number that $d(x, y_m) < \delta'$. Then

$$\limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} d(f^{ik}(x), f^{ik}(y_m)) < \frac{\delta}{3}.$$

Consider the set K as the set of all $i \in \mathbb{Z}$. that $d(f^{ik}(x), f^{ik}(y)) \geq \frac{\delta}{3}$, $K_n = K \cap \{-n+1, \dots, 0, \dots, n-1\}$ and $K_n^c = K^c \cap \{-n+1, \dots, 0, \dots, n-1\}$ where K^c denote the complement of the set K . Therefore, $\lim_{n \rightarrow \infty} \frac{1}{2n} (K \cap \{i : |i| \leq n-1\}) \leq \frac{1}{3}$, on the other hand, $\phi(f^{ik}(x)) = \sigma^{ik}(\phi(x)) = \sigma^{ik}(x)$.

Thus if i belong to complement of K then $d(f^{ik}(x), f^{ik}(y)) \leq \frac{\delta}{3}$, so $\rho(\phi(f^{ik}(x)), \phi(f^{ik}(y))) \leq \frac{1}{3}$. In conclusion $\rho(\phi(x), \phi(y_m)) \leq \frac{1}{3}$. Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{2n} \sum_{|i|=0}^{n-1} \rho(\sigma^{ik}(x), \sigma^{ik}(y_m)) \\ = \limsup_{n \rightarrow \infty} \left[\frac{1}{2n} \sum_{i \in K_n} \rho(\sigma^{ik}(x), \sigma^{ik}(y_m)) \right. \\ \left. + \left(\frac{1}{2n} \sum_{i \in K_n^c} \rho(\sigma^{ik}(x), \sigma^{ik}(y_m)) \right) \right] \\ < \frac{2}{3} + \frac{1}{3}. \end{aligned}$$

That is contradict with h -average-instability of $x \in \Sigma$.

Then for every point, $x \in \Sigma$ there exists a point $x \in X$ such that $\phi(x) = x$ and x is an h -average-unstable point for f^k in $CR(f)$. $\square$

5. CONCLUSION

In this paper some (average) shadowing properties for homeomorphism maps are considered. For further research in this direction, the following problems should be considered:

1. Investigating equivalent conditions in Theorem 3.6, when $f : X \rightarrow X$ has h -average shadowing property.
2. Classification of compact subsets X of real number set R such that $f : X \rightarrow X$ can have h -average shadowing property when f is homeomorphic.

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