

CONDITIONS FOR INFINITELY GENERATED KLEINIAN GROUPS WITH SMALL LIMIT SETS

ANDREW LAZOWSKI

DEPARTMENT OF MATHEMATICS, SACRED HEART UNIVERSITY,
5151 PARK AVENUE, FAIRFIELD, CT 06825, USA
E-MAILS: LAZOWSKIA@SACREDHEART.EDU

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ABSTRACT. We provide conditions for an infinitely generated Kleinian group to have its exponent of convergence arbitrarily small and the Hausdorff dimension of the limit set less than two. We use theory developed by Patterson to show that the exponent of convergence is small. To obtain the conclusion about the Hausdorff dimension of the limit set of the group we use porosity.

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1. INTRODUCTION

Consider a non-elementary Kleinian group. In studying such groups, interesting aspects include the Hausdorff dimension of their limit sets and the value of their critical exponents. The work of Patterson [12], Sullivan [16], and Bishop and Jones [3] provides us with the knowledge that for geometrically finite Kleinian groups, the Hausdorff dimension of the limit set of the group is equal to the critical exponent. In the case of non-geometrically finite Kleinian groups, since this relationship may not be true, literature exists providing details for Kleinian groups to have critical exponent strictly less than a given $\varepsilon > 0$ and the Hausdorff dimension of the limit

set two (see [13]). More recently, in [15] conditions for Kleinian groups to have critical exponent strictly less than a given $\varepsilon > 0$ and the Hausdorff dimension of the limit set less than two are provided. In this paper we will provide another set of conditions for a Kleinian group to have critical exponent smaller than a given $\varepsilon > 0$ while the Hausdorff dimension of the limit set is less than two. The methods used here differ from that of [15] where pseudo-Markov systems were used. In order to show that the critical exponent is less than a given $\varepsilon > 0$ we use theory created by Patterson in [13]. Then, to show that the Hausdorff dimension of the limit set is less than two we will use the notion of porosity which was used in [17] in order to show that the Hausdorff dimension of the limit set of a geometrically finite Kleinian group is strictly less than two. The purpose of the approach used here is that it uses the theory of Patterson and Sullivan. This very theory was used extensively to study the critical exponents and Hausdorff dimensions of limit sets for Kleinian groups. The proceeding work is intended to feature an appreciation and application for such endeavors. In addition, since porosity was used originally in [17] to show Hausdorff dimension less than two, using it here is yet another celebration of tradition.

2. BASICS

For the fundamentals of Kleinian groups, we refer the reader to [7], [8], or [9]. Let $\text{Isom}^+(\mathbb{B}^n)$ denote the group of orientation preserving isometries of hyperbolic n -space. A *Kleinian group* Γ is a discrete subgroup of $\text{Isom}^+(\mathbb{B}^n)$. The *limit set* $\Lambda(\Gamma)$ of a Kleinian group Γ is the subset of $\partial\mathbb{B}^n$ defined for any $x \in \mathbb{B}^n$ by $\Lambda(\Gamma) = \overline{\Gamma(x)} \cap \partial\mathbb{B}^n$. An *elementary* Kleinian group Γ is a group such that $\Lambda(\Gamma)$ contains at most two points. A *non-elementary* Kleinian group has a limit set with more than two points, which is uncountably big. The *domain of discontinuity* is the complement of the limit set and is denoted by $\Omega(\Gamma)$.

The group $\text{Isom}^+(\mathbb{B}^n)$ can be identified with the group of conformal homeomorphisms on $\partial\mathbb{B}^n$. We use this connection to help classify elements of Kleinian groups. An element of a Kleinian group is called *elliptic* if it has a fixed point in $\mathbb{B}^n$. On the other hand, if the element has exactly one fixed point in $\partial\mathbb{B}^n$ then it is called *parabolic*. All other elements are called *loxodromic*. Such elements have two fixed points in $\partial\mathbb{B}^n$ and fix the hyperbolic line between them. This line is called the *axis* of the loxodromic element. For any point x on this axis, the images of increasing

positive powers of the loxodromic element move x towards one of its fixed points. This fixed point is called the *attracting fixed point* while the other is called the *repelling fixed point*.

There are certain Kleinian groups that have beneficial properties. In order to introduce these groups we must first provide some information on the classification of limit points. The *conical limit set* $\Lambda_c(\Gamma)$ of a Kleinian group Γ acting on $\mathbb{B}^n$ is the set of points $\zeta \in \Lambda(\Gamma)$ where there exists a sub-orbit of $\Gamma(x)$ that converges to ζ within a bounded hyperbolic distance from a geodesic ray ending at ζ . An example of a conical limit point is a loxodromic fixed point: take x to be a point on the axis of a loxodromic element γ . Then the images, $\gamma^m(x)$ for $m = 1, 2, \dots$ converge to the attracting fixed point along this axis and hence converge within a bounded distance of the geodesic. On the other hand, a parabolic fixed point is not a conical limit point. See [1] for a clear explanation of this fact. Parabolic fixed points have their own distinction in the limit set. Denote the stabilizer of ζ by $\Gamma_\zeta = \{\gamma \in \Gamma : \gamma(\zeta) = \zeta\}$ and let ζ be a parabolic fixed point. Let Π_ζ be the minimal plane preserved by Γ_ζ and whose quotient by Γ_ζ is compact. Then ζ is a *bounded parabolic fixed point* if the Euclidean distance between ξ and Π_ζ is bounded for all $\xi \in \Lambda(\Gamma) - \zeta$. A Kleinian group Γ is called *geometrically finite* if the limit set consists entirely of conical limit points and bounded parabolic fixed points. There are multiple equivalent definitions of a geometrically finite Kleinian group, see [9]. We provide the reader with this definition because the Hausdorff dimension of the limit set is the focus of this paper and results concerning the Hausdorff dimension of the limit set of a geometrically finite Kleinian group have been confirmed in this context, see [11].

The Hausdorff dimension of the limit set of a Kleinian group is a number that can be used to identify geometric properties of the limit set of a Kleinian group. We refer the reader to [3] and [7] for examples of this. We begin by defining the Hausdorff dimension in a general setting. Consider U a subset of a metric space (X, ρ) and let $\text{diam}(U)$ denote the diameter of U . Given a Borel subset E of X and $\alpha \geq 0$ and $\varepsilon > 0$,

$$(2.1) \quad \mathcal{H}_\varepsilon^\alpha(E) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(U_i)^\alpha : E \subseteq \bigcup_{i=1}^{\infty} U_i; \text{diam}(U_i) \leq \varepsilon \right\}.$$

The α -dimensional Hausdorff measure of E is $\mathcal{H}^\alpha(E) = \lim_{\varepsilon \rightarrow 0} \mathcal{H}_\varepsilon^\alpha(E)$ and the Hausdorff dimension of a set E , denoted $\dim(E)$, is defined by

$$(2.2) \quad \dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha : \mathcal{H}^\alpha(E) = +\infty\}.$$

One may use different tools to determine the Hausdorff dimension of the limit set of a Kleinian group. An example is the critical exponent of the group. Note that the *Poincaré series* for a Kleinian group Γ , given that $x, y \in (\mathbb{B}^n, d)$, and $s \geq 0$, is defined by

$$(2.3) \quad P_s(x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma(y))}.$$

The *critical exponent* or *exponent of convergence* $\delta = \delta(\Gamma)$ of a Kleinian group Γ is

$$(2.4) \quad \delta = \inf \left\{ s : \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma(y))} < \infty \right\}.$$

If $P_\delta(x, y)$ converges then Γ is of *convergence type*. If $P_\delta(x, y)$ diverges then Γ is of *divergence type*.

Theorem 1. [3, Theorem 1.1] *If Γ is a non-elementary Kleinian group, then the critical exponent of Γ is equal to the Hausdorff dimension of the conical limit set of Γ .*

For geometrically finite Kleinian groups the Hausdorff dimension of the limit set is equal to the Hausdorff dimension of the conical limit set. We refer the reader to [11, Chapter 9] for the technical details of this fact. The following corollary is now immediate.

Corollary 1. *If Γ is a non-elementary geometrically finite Kleinian group then $\dim(\Lambda(\Gamma)) = \delta(\Gamma)$.*

Also, the following lemma is of importance in our main result.

Lemma 1. [16, Corollary 20] *Geometrically finite Kleinian groups are of divergence type.*

In later sections we will provide examples of Kleinian groups. An example will be constructed from Schottky groups. A *Schottky group* is a Kleinian group Γ that is

generated by transformations $\gamma_1, \dots, \gamma_k$, where there are $2k$ ($k \geq 2$) disjoint Jordan curves, $C_1, C'_1, \dots, C_k, C'_k$, which bound a common domain D in $\partial\mathbb{B}^3 = \widehat{\mathbb{C}}$ where $\gamma_i(C_i) = C'_i$ and $\gamma_i(D) \cap D = \emptyset$ for each i . We call the set of Jordan curves, $C_1, C'_1, \dots, C_k, C'_k$ *defining loops* for our given *Schottky generators* $\gamma_1, \dots, \gamma_k$. A *rank k Schottky group* is a Schottky group with k Schottky generators. Take note of the following property of Schottky groups.

Proposition 1. [8, X.H.2] *If Γ is a Schottky group on the generators $\gamma_1, \dots, \gamma_k$ acting on $\widehat{\mathbb{C}}$, then Γ is purely loxodromic, geometrically finite, and free on the generators $\gamma_1, \dots, \gamma_k$.*

3. INFINITELY GENERATED GROUPS

The first order of business is to construct infinitely generated Kleinian groups. Such groups are not geometrically finite so the critical exponents will not automatically provide information about the Hausdorff dimension of their limit sets. In this section we will examine the critical exponents of these groups and in Section 4 we will discuss the Hausdorff dimension of their limit sets. We begin with a collection of Kleinian groups under a certain topology.

Let X be a topological space and denote the family of all closed subsets of X by $\mathcal{C}(X)$. The *Chabauty topology* on $\mathcal{C}(X)$ is the topology with a sub-basis given by $\mathcal{V} = \{A : A \cap K = \emptyset\}$ where K is compact and $\mathcal{U} = \{A : A \cap U \neq \emptyset\}$ where U is open. Relevant facts pertaining to the Chabauty topology will be listed as needed. For an in-depth study of this topology see [2, E.1] or [4, I.3]. Note the following.

Lemma 2. [2, E.1.2] *Suppose that X is a locally compact metrizable space. A sequence $\{C_i\} \subseteq \mathcal{C}(X)$ converges to $C \in \mathcal{C}(X)$ in the Chabauty topology if and only if the following conditions are satisfied:*

- (1) *if $x \in C$ then there exists $x_i \in C_i$ for all i such that $x_i \rightarrow x$;*
- (2) *if $x \in X$ and there exists a subsequence $\{C_{i_j}\}$ of $\{C_i\}$ and $x_j \in C_{i_j}$ with $x_j \rightarrow x$, then $x \in C$.*

Now apply the Chabauty topology to Kleinian groups. In the compact-open topology, Kleinian groups are closed subgroups in $\text{Isom}^+(\mathbb{B}^n)$. A sequence of Kleinian groups $\{\Gamma_i\}$ is said to *converge geometrically* to Γ_∞ if it converges in the Chabauty topology. That is (1) for every element $\gamma \in \Gamma_\infty$, there exists a sequence of elements

$\gamma_i \in \Gamma_i$ such that $\gamma = \lim \gamma_i$ and (2) if there exists a subsequence $\{\gamma_{i_j}\}$ of $\{\gamma_i\}$ with $\lim \gamma_{i_j} = \hat{\gamma}$ then $\hat{\gamma} \in \Gamma_\infty$.

Here is a useful fact that we will take advantage of.

Lemma 3. [4, I.3.1.2] *Suppose X is an arbitrary topological space, then the set $\mathcal{C}(X)$ with the Chabauty topology is compact.*

Corollary 2. *If $\{\Gamma_i\}$ is a sequence of Kleinian groups then there exists a subsequence which converges geometrically to a group Γ_∞ .*

Notice in the above corollary that we do not mention any information about Γ_∞ , it might not even be a Kleinian group. Make note of the following.

Lemma 4. [6, Section 3.2] *If $\{\Gamma_i\}$ is a sequence of Kleinian groups that converges geometrically to Γ_∞ , then Γ_∞ is either a non-elementary Kleinian group, or it is elementary.*

In our construction we will begin with a loxodromic cyclic group and add generators. The following lemma will give an indication of our approach.

Lemma 5. *If $\{\Gamma_i\}$ is a sequence of free, purely loxodromic groups such that $\Gamma_i \subseteq \Gamma_{i+1}$ and $\Gamma_i \rightarrow \Gamma_\infty$ geometrically, then Γ_∞ is also free.*

Proof. Let $\mathcal{G}$ be a set of generators for Γ_∞ . Suppose to the contrary that Γ_∞ is not free, i.e. there is a relation $\gamma_k \gamma_{k-1} \cdots \gamma_1 = id$ such that $\gamma_j \in \mathcal{G}$ for all $j \in \{1, 2, \dots, k\}$. By the definition of geometric convergence, for any $\gamma_j \in \mathcal{G}$, $\gamma_j = \lim_{i \rightarrow \infty} \gamma_{j_i}$ where $\gamma_{j_i} \in \Gamma_i$. Furthermore, each $\gamma_{j_i} \in \Gamma_\infty$ because the sequence $\{\Gamma_i\}$ is nested. Since Γ_∞ is discrete and closed, for sufficiently large i we have that $\gamma_1, \gamma_2, \dots, \gamma_k \in \Gamma_i$, and hence Γ_i has the relation $\gamma_k \gamma_{k-1} \cdots \gamma_1 = id$. But this violates the assumption that each Γ_i is free. Hence, Γ_∞ is free. $\square$

In order to construct Kleinian groups with our desired properties we must make use of Klein-combination. See [8] for an in-depth description of this notion. The following theorem is sufficient for the constructions in the upcoming sections. More specifically, for our purposes, the hypothesis of this theorem is what we mean for groups to be Klein combinable.

Theorem 2. [8, VII.A.13] *Suppose Γ_1 and Γ_2 are Kleinian groups such that D_i is a fundamental polyhedron for Γ_i , $D_1 \cup D_2 = \partial \mathbb{B}^n$ and $D_1 \cap D_2 \neq \emptyset$. Then $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle = \Gamma_1 * \Gamma_2$ is also a Kleinian group.*

Suppose that Γ is a Kleinian group constructed from Γ_1 and Γ_2 so that $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$. The following three theorems describe circumstances where one may compare the critical exponents of Γ to the critical exponents of each Γ_i . The first theorem makes no mention of critical exponents explicitly, however, it indicates when Γ is geometrically finite which is of importance because of Lemma 1.

Theorem 3. [8, VII.C.2] *Suppose Γ, Γ_1 , and Γ_2 are Kleinian groups where $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$. The group Γ is geometrically finite if and only if Γ_1 and Γ_2 are geometrically finite.*

Theorem 4. [5, Theorem 1] *Suppose Γ_1 and Γ_2 are Kleinian groups that satisfy the requirements of Theorem 2 so that $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ is a Kleinian group. If $\delta(\Gamma_1) \geq \delta(\Gamma_2)$ and Γ_1 is of divergence type, then $\delta(\Gamma) > \delta(\Gamma_1)$.*

Theorem 5. [13, Theorem 1] *Suppose $d_E(S_1, S_2)$ denotes the Euclidean distance between two subsets S_1 and S_2 of $\mathbb{B}^n$, $H = \{h_i : i \in I\} \subseteq \text{Isom}^+(\mathbb{B}^n)$, and Γ_1, Γ_2 are Kleinian groups with convex fundamental polyhedrons F_1, F_2 respectively such that $F_1^c \cap h(F_2^c) = \emptyset$ ($h \in H$) and $\gamma(F_i) \cap F_j = \emptyset$ ($\gamma \in \Gamma_i - \{id\}$). If $h_i \in H$ and*

$$(3.1) \quad \frac{\sup_{w \in F_2^c} \left(\frac{1 - |h_i(w)|^2}{1 - |w|^2} \right)}{d_E(F_1^c, h_i(F_2^c))} \rightarrow 0$$

*as $i \rightarrow \infty$, then $\delta(\Gamma_1 * (h_i \Gamma_2 h_i^{-1})) \rightarrow \max\{\delta(\Gamma_1), \delta(\Gamma_2)\}$.*

Clearly the denominator is bounded above by two so the fraction approaches zero if the numerator does. It is shown in [11, Theorem 1.3.3] that $\frac{1 - |h_i(w)|^2}{1 - |w|^2} = |h'_i(w)|$. An example of a set H that would satisfy the conditions of the theorem would be $H = \langle h \rangle$ for h a loxodromic element in $\text{Isom}^+(\mathbb{B}^n)$ where the isometric spheres of h (the set of points where $|h'(z)| = 1$) and h^{-1} are disjoint and $|h'(w)| < 1$. Then the derivatives of higher powers of h tend to zero by the chain rule. Therefore, the above theorem states that as the distance between F_1^c and $h_i(F_2^c)$ increases, the critical exponent of $\Gamma_1 * (h_i \Gamma_2 h_i^{-1})$ converges to the critical exponent of either Γ_1 or Γ_2 .

Now consider a Kleinian group Γ_∞ as in Lemma 5. The following result provides information on the critical exponent for the group Γ_∞ with regards to the Kleinian groups in the sequence that create it.

Theorem 6. [10, Theorem 7.7] *If a sequence of Kleinian groups $\{\Gamma_i\}$ converges to Γ_∞ geometrically then $\delta(\Gamma_\infty) \leq \liminf \delta(\Gamma_i)$.*

Now we will go through the construction of an infinitely generated Kleinian group whose exponent of convergence is arbitrarily small.

Theorem 7. *Given an $\varepsilon > 0$ there exists a free, purely loxodromic, infinitely generated Kleinian group $\Gamma_\infty(\varepsilon)$ such that $\delta(\Gamma_\infty(\varepsilon)) \leq \varepsilon$.*

Proof. Fix $\varepsilon > 0$, let $\varepsilon_j = \frac{\varepsilon}{2^j}$, and let Γ_0 be an elementary Kleinian group generated by a loxodromic element. Let γ_1 be a loxodromic element of $\text{Isom}^+(\mathbb{B}^n)$ such that $\langle \gamma_1 \rangle$ is Klein combinable with Γ_0 . Consider fundamental polyhedrons for Γ_0 and $\langle \gamma_1 \rangle$ so that their union is all of $\mathbb{B}^n$. Let h be another loxodromic element so that the isometric spheres of h and h^{-1} are in the interior of the intersection of the fundamental polyhedrons for Γ_0 and $\langle \gamma_1 \rangle$. Let $H = \langle h \rangle$. Now we can apply Theorem 5 by letting $h_i = h^i$. Therefore we can pick I_1 sufficiently large so that for $\Gamma_1 = \Gamma_0 * \langle h_{I_1} \gamma_1 h_{I_1}^{-1} \rangle$ the critical exponent $\delta(\Gamma_1) < \varepsilon - \varepsilon_1$. Since the Kleinian groups whose free product is Γ_1 are both geometrically finite, by Theorem 3 Γ_1 is geometrically finite as well. Therefore, by Lemma 1 and Theorem 4 it follows that $\delta(\Gamma_0) < \delta(\Gamma_1)$. The group Γ_1 is actually a rank two Schottky group.

Now continue with this construction. Given Γ_j take a loxodromic element γ_{j+1} of $\text{Isom}^+(\mathbb{B}^n)$ so that $\langle \gamma_{j+1} \rangle$ is Klein combinable with Γ_j , let h be another loxodromic element of Γ so that the isometric spheres of h and h^{-1} are in the interior of the intersection of the fundamental polyhedrons for Γ_j and $\langle \gamma_{j+1} \rangle$. Note that this is a different h from above, we will find a distinct γ_{j+1} and distinct h for each Γ_j . Again consider $H = \langle h \rangle$ and apply Theorem 5 where $h_i = h^i$. Therefore we can pick I_{j+1} sufficiently large so that for $\Gamma_{j+1} = \Gamma_j * \langle h_{I_{j+1}} \gamma_{j+1} h_{I_{j+1}}^{-1} \rangle$ the critical exponent $\delta(\Gamma_{j+1}) < \varepsilon - \varepsilon_{j+1}$ and $\delta(\Gamma_j) < \delta(\Gamma_{j+1})$. By this construction $\delta(\Gamma_j) < \varepsilon - \varepsilon_j$ for all $j > 0$, each Γ_j is a Schottky group, and since $\{\varepsilon_j\} \rightarrow 0$, $\lim \delta(\Gamma_j) \rightarrow \delta \leq \varepsilon$.

Since we have an infinite sequence of Kleinian groups, by Corollary 2 there is a subsequence that has a geometric limit $\Gamma_\infty(\varepsilon)$. From Lemma 4 the geometric limit $\Gamma_\infty(\varepsilon)$ is a non-elementary Kleinian group since Γ_1 is non-elementary. Also, each Γ_i is a Schottky group so each Γ_i is purely loxodromic by Proposition 1. Then from Lemma 5, $\Gamma_\infty(\varepsilon)$ is free. Furthermore, $\Gamma_\infty(\varepsilon)$ is purely loxodromic. For if it was not then it would contain a non-loxodromic element, however, $\Gamma_\infty(\varepsilon)$ is the

limit of a nested sequence so it would contain each element of the sequence and the non-loxodromic limiting element. This would violate the discreteness of $\Gamma_\infty(\varepsilon)$ and hence it must be purely loxodromic. It is infinitely generated by construction. By Theorem 6 and the fact that $\Gamma_i \subset \Gamma_\infty(\varepsilon)$ for all i , $\delta(\Gamma_\infty(\varepsilon)) = \delta$. Therefore, $\delta(\Gamma_\infty(\varepsilon)) \leq \varepsilon$. $\square$

Example 1. Consider $\Gamma_\infty(\varepsilon)$ as constructed in Theorem 7. It was constructed from Schottky groups. The positioning of the Jordan curves which define these Schottky groups in the sequence affects $\dim(\Lambda(\Gamma_\infty(\varepsilon)))$. If the Jordan curves are Euclidean circles all with centers on S^1 then $\Lambda(\Gamma) \subseteq S^1$ and it follows that $\dim(\Lambda(\Gamma_\infty(\varepsilon))) \leq 1$. This is an example of a group which satisfies the type of Kleinian group we are after.

4. POROSITY OF LIMIT SETS

We now wish to expand on this construction. What happens, for instance, if we change Example 1 so that the Jordan curves are not Euclidean circles or are not centered on S^1 ? We will provide more conditions so that Kleinian groups constructed as in Theorem 7 will have Hausdorff dimension of their limit sets strictly less than 2.

As stated earlier, the tool we use here is porosity. A bounded set $A \subset \partial\mathbb{B}^n$ is *porous* if there exists a constant $p > 0$ such that for any open ball $B(a, \varepsilon)$ centered at a point $a \in A$ and of radius $\varepsilon \in (0, 1]$, there is an open ball $B(y, p\varepsilon)$ contained in $B(a, \varepsilon)$ that is disjoint from A . The value p is called a *porosity constant* of A . A Kleinian group is porous if its limit set is porous. For a given collection of Kleinian groups $\{\Gamma_i\}$, the collection is *uniformly porous* with porosity constant p if each Γ_i in the collection has porosity constant p . We use porosity in the following way.

Theorem 8. [14] *If E is a porous subset of $\partial\mathbb{B}^n$, then $\dim(E) < n - 1$.*

For $\varepsilon > 0$ the following theorem along with Theorems 7 and 8 can be used to construct an infinitely generated Kleinian group Γ such that $\delta(\Gamma) < \varepsilon$ and $\dim(\Lambda(\Gamma)) < 2$.

Theorem 9. *Suppose $\{\Gamma_i\}$ is a sequence of Kleinian groups such that $\Gamma_i \subseteq \Gamma_{i+1}$ and the sequence is uniformly porous with porosity constant p . Let Γ be the geometric limit of the sequence. Then Γ is porous with porosity constant p .*

Proof. Suppose $\Lambda(\Gamma)$ is not porous, then for every $p > 0$ there exists a ball $B(\zeta, \varepsilon)$ with $\zeta \in \Lambda(\Gamma)$ such that for every $B(y, p\varepsilon) \subset B(\zeta, \varepsilon)$ there is an element $\xi \in \Lambda(\Gamma) \cap B(y, p\varepsilon)$. Let $\mathcal{S} = \{B(y, p\varepsilon) : B(y, p\varepsilon) \subset B(\zeta, \varepsilon)\}$. The center of any ball in $\mathcal{S}$ is contained in $\overline{B(\zeta, \varepsilon - p\varepsilon)}$.

Since each ball from $\mathcal{S}$ contains some $\xi \in \Lambda(\Gamma)$ and loxodromic fixed points are dense in the limit set, each ball contains a loxodromic fixed point of some loxodromic element in Γ . Now cover $\overline{B(\zeta, \varepsilon - p\varepsilon)}$ by balls of the form $B(\eta, p\varepsilon)$ where η is a loxodromic fixed point of an element of Γ . Such a covering exists because if not, say $y \in \overline{B(\zeta, \varepsilon - p\varepsilon)}$ is not covered by one of these balls then $B(y, p\varepsilon)$ contains an η and we can add $B(\eta, p\varepsilon)$ to the cover which will contain y . Since $\overline{B(\zeta, \varepsilon - p\varepsilon)}$ is compact we may find a finite subcover and hence there is a finite collection of loxodromic fixed points $\{\eta_j\}$ such that any point $y \in \overline{B(\zeta, \varepsilon - p\varepsilon)}$ is within a distance of $p\varepsilon$.

Therefore, there is a finite collection of loxodromic fixed points $\{\eta_j\}$ of loxodromic elements $\{\gamma_{\eta_j}\} \in \Gamma$ such that any ball in $\mathcal{S}$ will contain an η_j . Each of these γ_{η_j} are in Γ_i for $i \geq \mathcal{I}$ where $\mathcal{I}$ is sufficiently large because $\Gamma_i \subseteq \Gamma_{i+1}$. But then each Γ_i for $i \geq \mathcal{I}$ is not porous with a porosity constant p which is a contradiction. Therefore, Γ is porous with porosity constant p . $\square$

Corollary 3. *If a sequence of Kleinian groups is constructed as in the proof of Theorem 7 so that it is uniformly porous, then the Hausdorff dimension of the limit set of the geometric limit $\Gamma_\infty(\varepsilon)$ is less than two.*

Proof. The result is immediate from Theorems 8 and 9. $\square$

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