

ON THE CHARACTERIZATION OF NORMAL CLOSED RANGE OPERATORS

MARYAM KHOSRAVI

DEPARTMENT OF MATHEMATICS, SHAHID BAHONAR UNIVERSITY OF KERMAN
KERMAN, IRAN.

E-MAIL: KHOSRAVI_M@UK.AC.IR

(Received: 26 February 2012, Accepted: 26 October 2012)

ABSTRACT. In this note, we found a characterization of normal closed range operators via some inequalities related to Corach-Porta-Recht inequality.

AMS Classification: 47A30, 15A09, 47B15

Keywords: Moore-Penrose inverse; operator inequality; closed range operators; normal operators; norm ideal.

1. INTRODUCTION AND PRELIMINARIES

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space $\mathcal{H}$ and we denote the operator norm by $\|\cdot\|$.

In [3], Corach-Porta-Recht gave the norm inequality

$$(1.1) \quad \forall X \in \mathbb{B}(\mathcal{H}) : \|SXS^{-1} + S^{-1}XS\| \geq 2\|X\|,$$

where S is an invertible self-adjoint operator in $\mathbb{B}(\mathcal{H})$, as a key of their theory on differential geometry. They proved this inequality by using the integral representation of a self-adjoint operator with respect to a spectral measure. J. I. Fujii, M. Fujii,

Furuta and Nakamoto [4], showed that this inequality is equivalent to Heinz inequality, Arithmetic-geometric-Mean inequality and some other inequalities. They present an easy proof for one of them and therefore gave a elementary proof for all these inequalities.

Recently in [10, 11], it was shown that if S is an invertible operator then the inequality (1.1) and the inequalities

$$(1.2) \quad \forall X \in \mathbb{B}(\mathcal{H}) : \|SXS^{-1} + S^{-1}XS\| \geq \|S^*XS^{-1} + S^{-1}XS^*\|,$$

$$(1.3) \quad \forall X \in \mathbb{B}(\mathcal{H}) : \|SXS^{-1} + S^{-1}XS\| = \|S^*XS^{-1} + S^{-1}XS^*\|,$$

are in fact a characterization of $\mathbb{C}^*\mathfrak{S}_0(\mathcal{H}) = \{\lambda M : \lambda \in \mathbb{C} \setminus \{0\}, M \in \mathfrak{S}_0(\mathcal{H})\}$, where $\mathfrak{S}_0(\mathcal{H})$ is the class of invertible self-adjoint operators. Note that this class of operators is the class of all invertible normal operators in $\mathbb{B}(\mathcal{H})$ the spectrum of which is included in a straight line passing through the origin.

Also he could characterize the normal invertible operators using some related inequalities. These results was generalized to the class of Moore-Penrose invertible operators in [6, 7].

We recall that if $\mathcal{A}$ is an algebra with involution and $a \in \mathcal{A}$ such that there exists an element $x \in \mathcal{A}$ satisfied the following four equations

$$\begin{aligned} axa &= a & xax &= x \\ (ax)^* &= ax & (xa)^* &= xa, \end{aligned} \tag{*}$$

then x is called a Moore-Penrose inverse of a and denoted by $a^\dagger$.

It is easy to show that the Moore-Penrose inverse of an element a is unique.

If $a \in \mathcal{A}$ is Moore-Penrose invertible, then

- (a) $a^{\dagger\dagger} = a$,
- (b) a^* is Moore-Penrose invertible and $a^{*\dagger} = a^{\dagger*}$,
- (c) If a is invertible, then $a^\dagger = a^{-1}$,
- (d) $(aa^*)^\dagger = a^{*\dagger}a^\dagger$.

Harte and Mbekhta in [5] proved that if $\mathcal{H}$ is a Hilbert space and $T \in \mathbb{B}(\mathcal{H})$ then the following conditions are equivalent:

- i) T has a generalized inverse (that is there exists an operator $S \in \mathbb{B}(\mathcal{H})$ for which $TST = T$ and $STS = S$).
- ii) $\mathcal{R}(T)$ is closed.

iii) T has a Moore-Penrose inverse.

In this case $TT^\dagger$ is the projection on $\mathcal{R}(T)$ and $T^\dagger T$ is the projection on $\mathcal{R}(T^*)$.

In [?], Using the arithmetic-geometric mean inequality, the author prove that for invertible operator R, S ,

$$\|R^*XS^{-1} + R^{-1}XS^*\| \geq 2\|X\|,$$

for all unitarily invariant norm $\|\cdot\|$.

In [9], the authors ask whether the same characterizations obtained by Seddik is valid true for unitarily invariant norm instead of operator norm. Some positive answers to this questions can be found in [9, 1].

The aim of this work is to give an answer to this question for the Moore-Penrose invertible operators.

2. MAIN RESULTS

One of the versions of the Arithmetic-Geometric mean inequality for operators is presented as follows:

For all operators $A, X, B \in \mathbb{B}(\mathcal{H})$,

$$\|A^*AX + XBB^*\| \geq 2\|AXB\|,$$

for all unitarily invariant norm $\|\cdot\|$. Using this inequality we can state the following result.

Theorem 2.1. *Let S, T be operators on Hilbert space $\mathcal{H}$ such that $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are closed. Then for all unitarily invariant norm,*

$$\|S^*XT^\dagger + S^\dagger XT^*\| \geq 2\|PXQ\|,$$

where $P = SS^\dagger$ and $Q = T^\dagger T$.

Proof. Since $PS = S$ and P is a projection, we have $S^* = S^*P$ and similarly $T^* = QT^*$. So

$$\begin{aligned} \|S^*XT^\dagger + S^\dagger XT^*\| &= \|S^*S(S^\dagger XT^\dagger) + (S^\dagger XT^\dagger)TT^*\| \\ &\geq 2\|S(S^\dagger XT^\dagger)T\| = 2\|PXQ\|. \end{aligned}$$

□

In [9], the authors could proved the following generalization of Seddik result for normal operators.

Theorem 2.2. [9, Corollary 2.2] *All invertible operators $S \in \mathbb{B}(\mathcal{H})$ satisfying the condition*

$$|||SXS^{-1}||| + |||S^{-1}XS||| \geq 2|||X|||,$$

for all $X \in \mathbb{B}(\mathcal{H})$ of rank 1 are normal.

Now we prove a version of this theorem for Moore-Penrose invertible operators.

Theorem 2.3. *Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$ and P be the projection on $\mathcal{K}$. If $S \in \mathbb{B}(\mathcal{H})$ with $\mathcal{R}(S) = \mathcal{R}(S^*) = \mathcal{K}$ and*

$$|||SXS^\dagger||| + |||S^\dagger XS||| \geq 2|||PXP|||,$$

for all rank 1 operator $X \in \mathbb{B}(\mathcal{H})$, then S is normal.

Proof. From the hypothesis, we can consider $S_{\mathcal{K}} = S|_{\mathcal{K}} \in \mathbb{B}(\mathcal{K})$. Note that $S_{\mathcal{K}}$ is invertible in $\mathbb{B}(\mathcal{K})$ and $S_{\mathcal{K}}^{-1} = S^\dagger|_{\mathcal{K}}$. For all $X \in \mathbb{B}(\mathcal{K})$, there exists an extension of X to $\mathbb{B}(\mathcal{H})$ which is zero on $\mathcal{K}^\perp$, name it $\tilde{X}$. Since

$$|||S\tilde{X}S^\dagger||| + |||S^\dagger\tilde{X}S||| \geq 2|||P\tilde{X}P|||,$$

we have

$$|||S_{\mathcal{K}}XS_{\mathcal{K}}^{-1}||| + |||S_{\mathcal{K}}^{-1}XS_{\mathcal{K}}||| \geq 2|||X|||.$$

So from Theorem 2.2, the operator $S_{\mathcal{K}}$ is normal and therefore S is normal. $\square$

We recall that $\mathcal{I}$ is a norm ideal if $\mathcal{I}$ is an ideal of $\mathbb{B}(\mathcal{H})$ and a Banach space with respect to $\|\cdot\|_{\mathcal{I}}$ and satisfies the following properties:

- $\|AXB\|_{\mathcal{I}} \leq \|A\|\|X\|_{\mathcal{I}}\|B\|$ for all $X \in \mathcal{I}$ and $A, B \in \mathbb{B}(\mathcal{H})$.
- $\|X\|_{\mathcal{I}} = \|X\|$ for all rank one operator $X \in \mathbb{B}(\mathcal{H})$.

It is an immediate consequence of this definition that every ideal norm is unitarily invariant. The Schatten p -norms are the most well-known examples of ideal norms.

Lemma 2.4. *If $S \in \mathbb{B}(\mathcal{H})$ is normal and $\mathcal{I}$ is a norm ideal, then for all $X \in \mathcal{I}$,*

$$\|SX\|_{\mathcal{I}} = \|S^*X\|_{\mathcal{I}} \quad \text{and} \quad \|XS\|_{\mathcal{I}} = \|XS^*\|_{\mathcal{I}}.$$

Proof. First we prove it for a normal partial isometry U . Observe that

$$\|UX\|_{\mathcal{I}} = \|UU^*UX\|_{\mathcal{I}} \leq \|U^*UX\|_{\mathcal{I}} = \|UU^*X\|_{\mathcal{I}} \leq \|U^*X\|_{\mathcal{I}}.$$

Replacing U with U^* , we have $\|UX\|_{\mathcal{I}} = \|U^*X\|_{\mathcal{I}}$.

The second relation is proved in a similar way.

Now let S be a normal operator and $S = U|S|$ be the polar decomposition of S . Then U is a normal partial isometry and U commutes with $|S|$. So

$$\|SX\|_{\mathcal{I}} = \|U|S|X\|_{\mathcal{I}} = \|U^*|S|X\|_{\mathcal{I}} = \|S^*X\|_{\mathcal{I}}.$$

Similarly, $\|XS\|_{\mathcal{I}} = \|XS^*\|_{\mathcal{I}}$. □

Combining Theorem 2.1, Theorem 2.3 and Lemma 2.4, we can deduce the following theorem.

Theorem 2.5. *Let $\mathcal{I}$ be a norm ideal and $\mathcal{K}$ be a closed subspace of $\mathcal{H}$ and P be the projection on $\mathcal{K}$. If $S \in \mathbb{B}(\mathcal{H})$ with $\mathcal{R}(S) = \mathcal{R}(S^*) = \mathcal{K}$, then the following statements are equivalent:*

1. S is normal.
2. $\|SXS^\dagger\|_{\mathcal{I}} + \|S^\dagger XS\|_{\mathcal{I}} = \|S^*XS^\dagger\|_{\mathcal{I}} + \|S^\dagger XS^*\|_{\mathcal{I}}$ for all $X \in \mathcal{I}$.
3. $\|SXS^\dagger\|_{\mathcal{I}} + \|S^\dagger XS\|_{\mathcal{I}} \geq \|S^*XS^\dagger + S^\dagger XS^*\|_{\mathcal{I}}$ for all $X \in \mathcal{I}$.
4. $\|SXS^\dagger\|_{\mathcal{I}} + \|S^\dagger XS\|_{\mathcal{I}} \geq 2\|PXP\|_{\mathcal{I}}$ for all $X \in \mathcal{I}$.
5. $\|SXS^\dagger\|_{\mathcal{I}} + \|S^\dagger XS\|_{\mathcal{I}} \geq 2\|PXP\|_{\mathcal{I}}$ for all $X \in \mathcal{I}$, with rank 1.

REFERENCES

- [1] C. Conde, Normal operators and inequalities in norm ideals, Linear Algebra Appl., Vol. 431, 1774-1777 (2009).
- [2] C. Conde, A. Seddik and M.S. Moslehian, Operator inequalities related to the Corach-Porta-Recht inequality, Linear Algebra Appl., Vol. 436, Number 9, 3008-3012 (2012).
- [3] G. Corach, R. Porta and L. Recht, An operator inequality, Linear Algebra Appl., Vol. 142, 153-158 (1990).
- [4] J.I. Fujii, M. Fujii, T. Furuta and R. Nakamoto, Norm inequalities equivalent to Heinz inequality, Proc. Amer. Math. Soc., Vol. 118, 827-830 (1993).
- [5] R. Harte and M. Mbekhta, On generalized inverses in C^* -algebras, Studia Mathematica, Number 103, 71-77 (1992).
- [6] M. Khosravi, Corach-Porta-Recht inequality for closed range operators, Math. Inequal. Appl., Vol. 16, Number 2, 477-481 (2013).
- [7] M. Khosravi, A characterization of the class of partial isometries, Vol. 437, 1300-1304 (2012).
- [8] F. Kittaneh, On some operator inequalities, Linear Algebra Appl. Vol. 208/209, 19-28 (1994).
- [9] B. Magajna, M. Petkovsek, A. Turnsek, An operator inequality and selfadjointness, Linear Algebra Appl., Vol. 377, 181-194 (2004).
- [10] A. Seddik, Some results related to the Corach-Porta-Recht inequality, Proc. Amer. Math. Soc., Vol. 129, Number 10, 3009-3015 (1987).

- [11] A. Seddik, On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities, J. Math. Anal. Appl., Vol. 351, 277-284 (2009).