ON THE BOUR'S THEOREM WITH RESPECT TO CONFORMAL MAP IN MINKOWSKI SPACE E_1^3

[1]ZEHRA BOZKURT, [2]İSMAİL GÖK, [3]F. NEJAT EKMEKCİ, [4]YUSUF YAYLI
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF ANKARA, ANKARA, TURKEY
E-MAILS: [1]ZBOZKURT@ANKARA.EDU.TR, [2]IGOK@SCIENCE.ANKARA.EDU.TR, [3]EKMEKCI@SCIENCE.ANKARA.EDU.TR, [4]YAYLI@SCIENCE.ANKARA.EDU.TR

(Received: 11 May 2012, Accepted: 28 October 2012)

Abstract. A generalized helicoid and a rotational surface have a isometric relation by Bour's theorem. It is that "A generalized helicoid is isometric to a rotational surface. Hence, helices on the helicoid correspond to parallel circles on the rotational surface under the isometric transformation.

In this study, we give a conformal relation between a generalized helicoid and a spiral surface in 3 dimensional Minkowski space E_1^3. In this sitiutation, we can say that helices on the helicoid correspond to spirals on the spiral surface under the conformal transformation. Also, some related figures are given.

AMS Classification: 53A05, 53C10S

Keywords: Bour's theorem; Gauss map; minimal; spiral surface; rotational surface, helicoid.

1. Introduction

Surface theory in 3-dimensional Euclidean space has been studied for a long time. In classical differential geometry, rotational surfaces with constant curvature and the right helicoid (resp catenoid) which is the only ruled minimal surface have been known. Also, a pair of these two surfaces has interesting properties namely, they

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 10, NUMBER 2 (2012) 149-172.

are both members of a one parameter family of isometric minimal surfaces and they have the same Gauss map.

Moreover, Bour [1] showed that a helicoidal surface and a rotational surface are isometric in 3 dimensional Euclidean space. In this generalization, the orginal properties such as minimality and preservation of the Gauss map are not generaly maintained. In [12], Ikawa showed that a helicoidal surface and a rotational surface are isometric according to Bour's theorem in 3-dimensional Euclidean space. He determined the pairs of surfaces with an additional condition that they have the same Gauss map by Bour's theorem. Also, Ikawa gave a classification of the surfaces by types of axis and the profile curves named as (axis's type, profile curve's type)-type. For example, the (S, L)-type which means that the surface has a spacelike axis and a lightlike profile curve. Then in [11], he showed that Bour's theorem is true in 3-dimensional Minkowski space. After these studies, Güler considered the null (lightlike) profile curves of helicoidal and rotational surfaces in Bour's theorem and he showed that Bour's theorem is true in 3-dimensional Minkowski space.

Carmo and Dajczer [4] found that there exist a two-parameter family of helicoidal surfaces which is isometric to a given helicoidal surface by using a result of Bour's theorem. Furthermore, with the help of this parametrization, they characterized helicoidal surfaces which have constant mean curvature.

Spiral curves and surfaces are most fascinating objects. Because they have important properties such as the size increases without altering the shape. Their properties are seen on different objects around us in differential geometry, in science and in the nature. Let us, we mention some phenomena seen curves which are similar to the spirals. For examples, the approach of a hawk to its prey, the approach of an insect to a light source (see for details [3]), the arms of a spiral galaxy, the arms of the tropical cyclones, the nerves of the cornea and several biological structures, e.g. Romanesco broccoli, Convallaria majalis, some spiral roses, sunflower heads, Nautilus shells and so on because of the fact that these curves are named also growth spirals.

In this paper, we give the relations of between Bour's theorem and the conformal map in Lorentz-Minkowski space E_1^3 with spacelike, timelike and null(lightlike) axis. We showed that a spiral surface and a generalized helicoid have a conformal relation in E_1^3. So, helices on the helicoid can be transformed to spirals on the spiral surface.

When the conformal map is isometry we can easily see that this case of the Bour's theorem is Ikawa's study. So, this paper is generalizations of his study.

2. Preliminaries

Let E_1^3 be the 3-dimensional pseudo-Euclidean space which endowed with the standart flat metric given by

$$g(x,y) = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are the usual coordinate system in E_1^3. Due to semi-Riemannian metric there are three different kind of vectors, namely spacelike, timelike and lightlike (null) depending on the properties $g(x,x) > 0$, $g(x,x) < 0$ and $g(x,x) = 0$, respectively, for any vector x in $E_1^3 - \{0\}$. These can be generalized for curves depending on the casual character of their tangent vectors, that is, the curve α is called a spacelike (resp. timelike and lightlike) if its velocity vector $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for any $t \in I$ [13].

In particular, the norm (length) of a vector x is given by $\|x\| = \sqrt{|g(x,x)|}$. Two vectors x and y are orthogonal, if $g(x,y) = 0$. The Lorentzian vector product is defined by

$$x \times y = (x_2y_3 - y_2x_{3,}y_3x_1 - x_3y_{1,}y_1x_2 - x_1y_2)$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in E_1^3$ [13].

Let $X(u,v)$ be a parametrization of a surface in E_1^3. Then we introduce the following traditional notation for coefficients of the first and second fundamental forms with regard to the natural bases $\{X_u, X_v\}$:

$$\begin{aligned} E &= g(X_u, X_u) \qquad & L &= g(X_{uu}, e) \\ F &= g(X_u, X_v) \qquad & M &= g(X_{uv}, e) \\ G &= g(X_v, X_v) \qquad & N &= g(X_{vv}, e) \end{aligned}$$

where the line element of the surface $X(u,v)$ is defined by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The Gauss map is given by

$$e = \frac{X_u \times X_v}{\sqrt{|g(X_u \times X_v, X_u \times X_v)|}}.$$

Let $\gamma : I \subset \mathbb{R} \to \pi$ be a curve in a plane π in E_1^3 and l be a straight line in π which has no any common point of the curve γ and the straight line l. A rotational surface in E_1^3 is obtained by rotating a curve γ around a line l (they are called the profile curve and the axis, respectively) [11].

Suppose that when a profile curve γ rotates around the axis l, it simultaneously moves parallel to line l so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the generalized helicoid or helicoidal surface with the axis l [11].

A spiral surface is the locus of two different positions of any curve which both rotated around an axis and subjected to a homothetic transformation with respect to a point of the axis. Moreover, the tangent vectors of the curve make a constant angle with the rotation axis at any points of the curve [17].

In this paper, we give the relations of between Bour's theorem and the conformal map in Lorentz-Minkowski space E_1^3. We showed that a spiral surface and a generalized helicoid have a conformal relation in Lorentz-Minkowski space E_1^3. So, helices on the helicoid can be transformed to spirals on the spiral surface. When the conformal map is isometry we can easily see that this case of the Bour's theorem is Ikawa's study in [11]. So, this paper is generalizations of his study.

3. Spiral, Helicoidal and rotational Surfaces in Minkowski space E_1^3

We can suppose that l is the line spanned by spacelike vector $(1,0,0)$, timelike vector $(0,0,1)$ or lightlike vector $(0,1,1)$. Then the semi-orthogonal matrices of the above vectors can be given by

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh v & \sinh v \\ 0 & \sinh v & \cosh v \end{bmatrix}, T_1 = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & -v & v \\ v & 1-\frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2} \end{bmatrix}$$

$M_i.l = l$, $M_i^t.\varepsilon.M_i = \varepsilon$, $\varepsilon =$diag $(1,1,-1)$ and $\det M_i = +1$ $(i = 1,2)$ where $M_i = S_1$, $M_i = T_1$ or $M_i = L_i$ [11].

Hence, the parametrizations of the surface $X(u,v)$ is given by

$$X(u,v) = M_i.\gamma.$$

Case 3.1. *If the profile curve is on the x_1x_2-plane, then the profile curve γ has a notation $\gamma(u) = (\phi(u), u, 0)$. In this case, it can be only spacelike. İf the profile curve is on the x_1x_3-plane, then the profile curve γ has a notation $\gamma(u) = (\phi(u), 0, u)$. In this case, it can be spacelike or timelike. So, if we consider that the axis l is spacelike line then there exist a Lorentzian transformation which transform l into x_1-axis and the spiral surface $S(u,v)$ can be calculated as*

$$S(u,v) = (f(v)\phi(u), uf(v)\cosh v, uf(v)\sinh v)$$

or

$$S(u,v) = (f(v)\phi(u), uf(v)\sinh v, uf(v)\cosh v)$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function for all $v \in I$, the function $f(v)$ is equal to $e^{g'(v)}$ and $g''(v) = b$ is a constant function. The rotational surface $R(u,v)$ and the generalized helicoid $H(u,v)$ can be calculated as

$$R(u,v) = (\phi(u), u\cosh v, u\sinh v) \text{ or } R(u,v) = (\phi(u), u\sinh v, u\cosh v)$$

and

$$H(u,v) = (\phi(u) + av, u\cosh v, u\sinh v) \text{ or } H(u,v) = (\phi(u) + av, u\sinh v, u\cosh v)$$

where a is a non-zero constant.

Case 3.2. *If the profile curve is on the x_2x_3-plane, then the profile curve γ has a notation $\gamma(u) = (0, u, \phi(u))$. If the profile curve is on the x_1x_3-plane, then the profile curve γ has a notation $\gamma(u) = (u, 0, \phi(u))$. In these case, it can be spacelike or timelike. So, if we consider that the axis l is timelike line. Then there exist a Lorentzian transformation which transform l into x_3-axis and the spiral surface $S(u,v)$ can be calculated as*

$$S(u,v) = (uf(v)\cos v, uf(v)\sin v, f(v)\phi(u))$$

or

$$S(u,v) = (uf(v)\sin v, uf(v)\cos v, f(v)\phi(u))$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function for all $v \in I$, the function $f(v)$ is equal to $e^{g'(v)}$ and $g''(v) = b$ is a constant function. The rotational surface $R(u,v)$ and the generalized helicoid $H(u,v)$ can be calculated as

$$R(u,v) = (u\cos v, u\sin v, \phi(u)) \text{ or } R(u,v) = (u\sin v, u\cos v, \phi(u))$$

and

$$H(u,v) = (u\cos v, u\sin v, \phi(u) + av) \text{ or } H(u,v) = (u\sin v, u\cos v, \phi(u) + av)$$

where a is a non-zero constant.

Case 3.3. *In this case, we consider that the profile curve γ has a notation*

$$\gamma(u) = (0, \phi(u) + u, \phi(u) - u)$$

and the axis l is lightlike line. Then there exist a Lorentzian transformation which transform l into $Sp\{\xi = (0,1,1)\}$ in E_1^3 and the spiral surface $S(u,v)$ can be calculated as

$$S(u,v) = (2uvf(v), f(v)(\phi(u) + u - uv^2), f(v)(\phi(u) - u - uv^2))$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function for all $v \in I$, the function $f(v)$ is equal to $e^{g'(v)}$ and $g''(v) = b$ is a constant function. The rotational surface $R(u,v)$ and the generalized helicoid $H(u,v)$ can be calculated as

$$R(u,v) = (2uv, \phi(u) + u - uv^2, \phi(u) - u - uv^2)$$

and

$$H(u,v) = (2uv, \phi(u) + u - uv^2 + av, \phi(u) - u - uv^2 + av)$$

where a is a non-zero constant.

Remark 3.4. *If $\phi(u)$ is a constant function then we can easily see that the surface $H(u,v)$ is a right helicoid in E_1^3.*

Definition 3.5. *Let X and Y denote the surfaces and $f : X \to Y$ be a mapping. Then f is conformal provided there exist a real-valued function $\lambda > 0$ on X such that*

$$g(f_* v_p, f_* v_p) = \lambda(p)\, g(v_p, v_p)$$

for all tangent vectors to X.

i) If λ is constant function then f is named as a homothetic function.

ii) If λ is equal to 1 then we say f is an isometric function.

where the function λ is called the *scale factor* of f [16].

Definition 3.6. *Let X and Y denote the surfaces and $f : X \to Y$ be a mapping. Then f is a conformal map iff*

$$E = \lambda^2 \bar{E},\ F = \lambda^2 \bar{F},\ G = \lambda^2 \bar{G}$$

where E, F, G and $\bar{E}$, $\bar{F}$, $\bar{G}$ are the coefficients of the first fundamental form of X and Y, respectively [14, 15]

Theorem 3.7. *(*Bour's theorem*) A generalized helicoid is isometric to a rotational surface so that helices on the helicoid can be transformed to parallel circles on the rotational surface* [1].

Theorem 3.8. *(*Ikawa's theorem*) A generalized helicoid with spccelike (resp. timelike and lightlike) axis is isometric to a rotational surface with spccelike (resp. timelike and lightlike) axis so that helices on the helicoid can be transformed to parallel circles on the rotational surface in Lorentz-Minkowski space E_1^3* [11].

In this section, we give the *Bour's theorem* under the conformal map with spacelike, timelike or lighlike axis l and some related figures are given in Minkowski space E_1^3.

4. Bour's theorem under the conformal map in Minkowski space E_1^3

In the following section, we obtain the *Bour's theorem* under the conformal map with the spacelike axis l and the spacelike profile curve γ which are given in the Case 1. We assume that the profile curve is on the x_1x_2-plane.

Theorem 4.1. *Let $H(u, v)$ and $S(u, v)$ be (S, S)-type helicoidal surface and (S, S)-type spiral surface which are given in the Case 1, respectively. If $H(u, v)$ and $S(u, v)$ have a conformal relation then the spiral surface must be given by the* *following equation*

$$S(u,v)=\begin{bmatrix} e^{g'(v)}\phi_s(u) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\cosh(v) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\sinh(v) \end{bmatrix},\ b^2\neq 1$$

where

$$\phi_s=\frac{b(b^2-1)a\phi'^2\mp\sqrt{B}}{b^4+b^2(b^2-1)\phi'^2}$$

$$B=b^2(b^2-1)^2a^2\phi'^4+(b^4+b^2(b^2-1)\phi'^2)((1-b^2)a\phi'^2+b^2a^2-b^2u^2)$$

and $a\neq 0$, $b\neq 0$ are constants. Consequently, we can easily say that helices on the (S,S)-type helicoidal surface can be transformed to spirals on the (S,S)-type spiral surface.

Proof. The coefficients of the first fundamental form of the (S,S)-type generalized helicoid

$$H(u,v)=(\phi(u)+av,u\cosh v,u\sinh v),a=constant.$$

are given as

$$E=g\left(H_u,H_u\right)=1+\phi'^2$$
$$F=g\left(H_u,H_v\right)=a\phi'$$
$$G=g\left(H_v,H_v\right)=a^2-u^2.$$

Then the line element of the generalized helicoid is

$$ds^2=\left(1+\phi'^2\right)du^2+2a\phi'dudv+\left(a^2-u^2\right)dv^2. \tag{1}$$

On the other hand, the coefficients of the first fundamental form of the (S,S)-type spiral surface

$$S(u_s,v_s)=(e^{g'(v)}\phi(u_s),e^{g'(v)}u_s\cosh v_s,e^{g'(v)}u_s\sinh v_s),\ \ g''(v)=b=constant$$

are given as

$$E=g\left(S_{u_s},S_{u_s}\right)=\left(e^{g'(v)}\right)^2\left(1+\phi_s'^2\right)$$
$$F=g\left(S_{u_s},S_{v_s}\right)=\left(e^{g'(v)}\right)^2(bu_s+b\phi_s\phi_s')$$
$$G=g\left(S_{v_s},S_{v_s}\right)=\left(e^{g'(v)}\right)^2(b^2(u_s^2+\phi_s^2)-u_s^2).$$

Then the line element of the spiral surface is

(2)
$$ds_s^2 = \left(e^{g'(v)}\right)^2 \left[\left(1+\phi_s'^2\right) du_s^2 + 2(bu_s + b\phi_s\phi_s')du_s dv_s + (b^2(u_s^2+\phi_s^2) - u_s^2)dv_s^2\right].$$

Comparing the line element of the helicoid in eq. (1) with the line element of the spiral surface in eq. (2). We obtain the function ϕ_s of the spiral surface as follows

$$\phi_s = \frac{b(b^2-1)a\phi'^2 \mp \sqrt{B}}{b^4 + b^2(b^2-1)\phi'^2}$$

where

$$B = b^2(b^2-1)^2a^2\phi'^4 + (b^4 + b^2(b^2-1)\phi'^2)((1-b^2)a\phi'^2 + b^2a^2 - b^2u^2).$$

Thus,

$$E_s = \left(e^{g'(v)}\right)^2 E, F_s = \left(e^{g'(v)}\right)^2 F, G_s = \left(e^{g'(v)}\right)^2 G.$$

The function $\left(e^{g'(v)}\right)^2$ is positive. So, we have a conformal relation between $H(u,v)$ and $S(u_s, v_s)$. Therefore a generalized helicoid

$$H(u,v) = \begin{bmatrix} \phi(u) + av \\ u\cosh v \\ u \sinh v \end{bmatrix}$$

and the spiral surface

$$S(u,v) = \begin{bmatrix} e^{g'(v)}\phi_s(u) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\cosh(v) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\sinh(v) \end{bmatrix}, \ b^2 \neq 1$$

have an conformal relation. So, helices on the generalized helicoid can be transformed to spirals on the spiral surface. Where

$$EG - F^2 = -u^2(1+\phi'^2) + a^2$$

if $0 < u^2 < a^2/(1+\phi'^2)$, then the surfaces are spacelike, if $u^2 > a^2/(1+\phi'^2)$, then the surfaces are timelike.

Also, a helix on the generalized helicoid is u =constant curve. To give a curve orthogonal to helix, we consider the orthogonal case,

$$a\phi' du + \left(a^2 - u^2\right) dv = 0.$$

Then we obtain

$$v=-\int\frac{a\phi'}{a^2-u^2}du+c$$

where c is constant. Hence, if we consider that

$$\bar{v}=v-\int\frac{a\phi'}{a^2-u^2}du$$

So, the orthogonal curve is given as $\bar{v}$ =constant. If we differentiate the last equation we obtain

$$d\bar{v}=dv-\frac{a\phi'}{u^2-a^2}du.$$

Substituting this equation into the line element, we have

$$ds^2=\left(1+\frac{u^2\phi'^2}{u^2-a^2}\right)du^2-\left(u^2-a^2\right)d\bar{v}^2.$$

Case (i) $u^2-a^2>0$

If we consider that

$$\bar{u}=\int\sqrt{1+\frac{u^2\phi'^2}{u^2-a^2}},\ \text{and}\ f(\bar{u})=\sqrt{u^2-a^2}$$

then the line element reduces to

$$ds^2=d\bar{u}^2-f(\bar{u})^2d\bar{v}^2. \tag{3}$$

If $e^{g'(v)}$ is constant, i.e, $b=0$ then the spiral surface reduces to (S,S)-type rotational surface has the line element

$$ds_s^2=\left(e^{g'(v)}\right)^2\left[\left(1+\phi_s'^2\right)du_s^2-d\bar{v}_s^2\right].$$

So, if we consider that

$$\bar{u}_s=\int\sqrt{1+\phi_s'^2}du_s,\ f_s(\bar{u}_s)=u_s,\ \bar{v}_s=v_s,$$

then the line element of the rotational surface is rewritten as

$$ds_s^2=\left(e^{g'(v)}\right)^2\left[d\bar{u}_s^2-f_s^2(\bar{u}_s)d\bar{v}_s^2\right]. \tag{4}$$

Comparing the line element of the helicoidal surface in eq. (3) and the rotational surface in eq. (4) we consider that

$$\bar{u}=\bar{u}_s\ ,\ \bar{v}=v_s\ \text{and}\ f(\bar{u})=f_s(\bar{u}_s)$$

then we obtain the function ϕ_s of the rotational surface as

$$\phi_s(u) = \int \sqrt{\frac{-a^2 + u^2\phi'^2}{u^2 - a^2}} du.$$

So, we obtain a homothetic relation between the helicoidal surface and the rotational surface. Therefore a generalized helicoid

$$H(u,v) = \begin{bmatrix} \phi(u) + av \\ u\cosh v \\ u\sinh v \end{bmatrix}$$

and the rotational surface

$$R(u,v) = \begin{bmatrix} e^{g'(v)} \int \sqrt{\frac{-a^2+u^2\phi'^2}{u^2-a^2}} du \\ e^{g'(v)} \sqrt{u^2 - a^2} \cosh(v - \int \frac{a\phi'}{u^2-a^2}) \\ e^{g'(v)} \sqrt{u^2 - a^2} \sinh(v - \int \frac{a\phi'}{u^2-a^2}) \end{bmatrix}$$

have a homothetic relation. So, helices on the helicoid can be transformed to circles on the rotational surface.

İf $e^{g'(v)} = 1$ then it is clearly seeng that we obtain a generalized helicoid

$$H(u,v) = \begin{bmatrix} \phi(u) + av \\ u\cosh v \\ u\sinh v \end{bmatrix}$$

and the rotational surface

$$R(u,v) = \begin{bmatrix} \int \sqrt{\frac{-a^2+u^2\phi'^2}{u^2-a^2}} du \\ \sqrt{u^2 - a^2} \cosh(v - \int \frac{a\phi'}{u^2-a^2}) \\ \sqrt{u^2 - a^2} \sinh(v - \int \frac{a\phi'}{u^2-a^2}) \end{bmatrix}$$

have an isometric relation, i.e, we obtain the Ikawa's theorem in [11]. $\square$

In the following section, we obtain the *Bour's theorem* under the conformal map with (S,S)-type helicoidal surface and (S,T)-type spiral surface which are given in the Case 1.

Theorem 4.2. *Let $H(u,v)$ and $S(u,v)$ be (S,S)-type helicoidal surface and (S,T)-type spiral surface which are given in the Case 1, respectively. If $H(u,v)$*

and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equation

$$S(u,v)=\begin{bmatrix} e^{g'(v)}\phi_s(u) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\sinh(v) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\cosh(v) \end{bmatrix},\ b^2\neq 1$$

where

$$\phi_s=\frac{b(b^2-1)a\sqrt{2+\phi'^2}\pm\sqrt{C}}{b^4-b^2(b^2-1)\left(2+\phi'^2\right)}$$

$$C=b^2(b^2-1)^2a^2\phi'^2\left(2+\phi'^2\right)+(-b^4-b^2(b^2-1)\left(2+\phi'^2\right))((b^2-1)a^2\phi'^2+b^2u^2-b^2a^2)$$

and $a\neq 0$, $b\neq 0$ are constants. Consequently, we can easily say that helices on the (S,T)-type helicoidal surface can be transformed to spirals on the (S,T)-type spiral surface.

Proof. We recall the (S,T)-type spiral surface is of the form

$$S(u,v)=\begin{bmatrix} e^{g'(v)}\phi_s(u_s) \\ e^{g'(v)}u_s\sinh(v_s) \\ e^{g'(v)}u_s\cosh(v_s) \end{bmatrix},\ g''(v)=b=\text{constant.}\ (\forall v\in I)$$

Thus, the line element of the (S,T)-type spiral surface is

(5)

$$ds_s^2=\left(e^{g'(v)}\right)^2\left[\left(\phi_s'^2-1\right)du_s^2+2(-bu_s-b\phi_s\phi_s')du_sdv_s+(-b^2(u_s^2+\phi_s^2)+u_s^2)dv_s^2\right].$$

If we compare the line element of the helicoidal surface in eq. (1) with the spiral surface in eq. (5) then we obtain the (S,T)-type spiral surface function ϕ_s as follows

$$\phi_s=\frac{b(b^2-1)a\sqrt{2+\phi'^2}\pm\sqrt{C}}{b^4-b^2(b^2-1)\left(2+\phi'^2\right)}$$

where

$$B=b^2(b^2-1)^2a^2\phi'^2\left(2+\phi'^2\right)+(-b^4-b^2(b^2-1)\left(2+\phi'^2\right))((b^2-1)a^2\phi'^2+b^2u^2-b^2a^2).$$

Therefore

$$E_s=\left(e^{g'(v)}\right)^2E,F_s=\left(e^{g'(v)}\right)^2F,G_s=\left(e^{g'(v)}\right)^2G.$$

The function $\left(e^{g'(v)}\right)^2$ is positive so we have a conformal relation between $H(u,v)$ and $S(u_s,v_s)$. Thus, a generalized helicoid

$$H(u,v)=\begin{bmatrix}\phi(u)+av\\ u\cosh v\\ u\sinh v\end{bmatrix}$$

and the spiral surface

$$S(u,v)=\begin{bmatrix}e^{g'(v)}\phi_s(u)\\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\sinh(v)\\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\cosh(v)\end{bmatrix},\ b^2\neq 1$$

have an conformal relation. So, helices on the generalized helicoid can be transformed to the spirals on the spiral surface.

Case (ii) $a^2/(1+\phi'^2)<u^2<a^2$. In this case if we consider that

$$\bar{u}=\int\sqrt{\frac{u^2\phi'^2}{a^2-u^2}-1},\ \text{and}\ f(\bar{u})=\sqrt{a^2-u^2}$$

then the line element of the generalized helicoid reduces to

$$ds^2=-d\bar{u}^2+f(\bar{u})^2d\bar{v}^2. \tag{6}$$

On the other hand, if $e^{g'(v)}$ is constant, i.e, $b=0$ then the spiral surface reduces to (S,S)-type rotational surface has the line element

$$ds_s^2=\left(e^{g'(v)}\right)^2\left[\left(\phi_s'^2-1\right)du_s^2+u_s^2d\bar{v}_s^2\right].$$

So, if we consider that

$$\bar{u}_s=\int\sqrt{\phi_s'^2-1}du_s,\ f_s(\bar{u}_s)=u_s,\ \bar{v}_s=v_s,$$

then the line element of the rotational surface is rewritten as

$$ds_s^2=\left(e^{g'(v)}\right)^2\left[d\bar{u}_s^2+f_s^2(\bar{u}_s)d\bar{v}_s^2\right]. \tag{7}$$

Comparing the line element of helicoidal surface in eq. (6) with rotational surface in eq. (7) if we consider that

$$\bar{u}=\bar{u}_s\ ,\ \bar{v}=v_s\ \text{and}\ f(\bar{u})=f_s(\bar{u}_s)$$

then we obtain the function ϕ_s of the rotational surface as

$$\phi_s(u) = -\int \sqrt{\frac{a^2 - u^2\phi'^2}{a^2 - u^2}} du.$$

So, we obtain a homothetic relation between the helicoidal surface and rotational surface. Therefore a generalized helicoid

$$H(u,v) = \begin{bmatrix} \phi(u) + av \\ u\cosh v \\ u\sinh v \end{bmatrix}$$

and the rotational surface

$$R(u,v) = \begin{bmatrix} -e^{g'(v)} \int \sqrt{\frac{a^2-u^2\phi'^2}{a^2-u^2}} du \\ e^{g'(v)}\sqrt{a^2 - u^2}\sinh(v - \int \frac{a\phi'}{u^2-a^2}) \\ e^{g'(v)}\sqrt{a^2 - u^2}\cosh(v - \int \frac{a\phi'}{u^2-a^2}) \end{bmatrix}, \; e^{g'(v)} = \text{constant}$$

have a homothetic relation.

İf $e^{g'(v)} = 1$ then it is clearly seeng that we obtain an isometric relation between a generalized helicoid and the rotational surface ,that is, we obtain the Ikawa's theorem in [11].

Case (iii) $0 < u^2 < a^2/(1+\phi'^2)$. In this case, the line element of the generalized helicoid reduces to

$$ds^2 = d\bar{u}^2 + f(\bar{u})^2 d\bar{v}^2. \tag{8}$$

Similar calculation as above we obtain a (S,S)-type generalized helicoid

$$H(u,v) = \begin{bmatrix} \phi(u) + av \\ u\cosh v \\ u\sinh v \end{bmatrix}$$

and the (S,T)-type rotational surface

$$R(u,v) = \begin{bmatrix} -e^{g'(v)} \int \sqrt{\frac{a^2-u^2\phi'^2}{a^2-u^2}} du \\ e^{g'(v)}\sqrt{a^2 - u^2}\sinh(v - \int \frac{a\phi'}{u^2-a^2}) \\ e^{g'(v)}\sqrt{a^2 - u^2}\cosh(v - \int \frac{a\phi'}{u^2-a^2}) \end{bmatrix}, \; e^{g'(v)} = \text{constant}$$

have a homothetic relation.

If $e^{g'(v)}$ equal to 1 then it is clearly seeng that we obtain an isometric relation between a generalized helicoid and the rotational surface, that is, we obtain the Ikawa's theorem in [11]. □

Remark 4.3. *The other cases; (S,S)-type, (S,T)-type, (T,S)-type, (T,T)-type have similar proofs as above theorems. So, we only give sketch proofs of them.*

Case 4.4. *In the following section, we obtain the* Bour's theorem *under the conformal map with the spacelike axis l and the spacelike profile curve γ which are given in the Case 1. We assume that the profile curve is on the x_1x_3-plane.*

Theorem 4.5. *Let $H(u,v)$ and $S(u,v)$ be (S,S)-type helicoidal surface and (S,S)-type spiral surface which are given in the Case 1, respectively. If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equation*

$$S(u,v) = \begin{bmatrix} e^{g'(v)}\phi_s(u) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\sinh(v) \\ e^{g'(v)}\sqrt{\frac{a^2-u^2-b^2\phi_s(u)^2}{b^2-1}}\cosh(v) \end{bmatrix}, \; b^2 \neq 1.$$

In this theorem

$$\phi_s = \frac{b(b^2-1)a\phi'^2 \pm \sqrt{D}}{b^4 - 3b^2(b^2-1)\phi'^2}$$

Where

$$D = b^2(1-b^2)^2a^2\phi'^4 + (-b^4 + 3b^2(b^2-1)\left(2+\phi'^2\right))((b^2-1)a^2\phi'^2 + b^2u^2 + b^2a^2).$$

Consequently, we can easily say that helices on the (S,S)-type helicoidal surface can be transformed to spirals on the (S,S)-type spiral surface.

If $e^{g'(v)}$ is constant then the spiral surface reduces to the rotational surface so we obtain a generalized helicoid

$$H(u,v) = \begin{bmatrix} \phi(u) + av \\ u\sinh v \\ u\cosh v \end{bmatrix}$$

and the rotational surface

$$R(u,v) = \begin{bmatrix} e^{g'(v)}\int\sqrt{\frac{u^2\phi'^2-a^2}{a^2+u^2}}du \\ e^{g'(v)}\sqrt{a^2+u^2}\sinh(v + \int\frac{a\phi'}{u^2+a^2}) \\ e^{g'(v)}\sqrt{a^2+u^2}\cosh(v + \int\frac{a\phi'}{u^2+a^2}) \end{bmatrix}, \; e^{g'(v)} = \text{constant}$$

have a homothetic relation.

If $e^{g'(v)} = 1$ then we obtain Ikawa's theorem in [11], that is, we obtain an isometric relation between a helicoid and the rotational surface.

In this theorem, if $a^2/(\phi'^2 - 1) > u^2$, then the surfaces are timelike, if $a^2/(\phi'^2 - 1) < u^2$, then the surfaces are spacelike.

Case 4.6. *In the following section, we obtain the* Bour's theorem *under the conformal map with the timelike axis l and the spacelike profile curve γ which are given in the Case 2.*

Theorem 4.7. *Let $H(u,v)$ and $S(u,v)$ be a (T,S)-type helicoidal surface and spiral surface which are given in the Case 1, respectively. If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equation*

$$S(u,v) = \begin{bmatrix} e^{g'(v)}\sqrt{\frac{a^2+u^2-b^2\phi_s(u)^2}{b^2+1}}\cos(v) \\ e^{g'(v)}\sqrt{\frac{a^2+u^2-b^2\phi_s(u)^2}{b^2+1}}\sin(v) \\ e^{g'(v)}\phi_s(u) \end{bmatrix}.$$

In this theorem

$$\phi_s = \frac{-ab(b^2+1)\phi'^2 \pm \sqrt{E}}{-b^4 - b^2(b^2+1)\phi'^2}$$

Where

$$E = b^2(b^2+1)^2a^2\phi'^4 - (b^4 + b^2(b^2+1)\phi'^2)((b^2+1)a\phi'^2 - b^2a^2 - b^2u^2),\ 1 - \phi'^2 > 0.$$

Consequently, we can easily say that helices on the (T,S)-type helicoidal surface can be transformed to spirals on the (T,S)-type spiral surface.

If $e^{g'(v)}$ is constant then the spiral surface reduces to the rotational surface so we obtain a (T,S)-type generalized helicoid

$$H(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ \phi(u) + av \end{bmatrix}$$

and the rotational surface

$$R(u,v) = \begin{bmatrix} e^{g'(v)}\sqrt{u^2-a^2}\cos(v - \int \frac{a\phi'}{u^2-a^2}) \\ e^{g'(v)}\sqrt{u^2-a^2}\sin(v - \int \frac{a\phi'}{u^2-a^2})e^{g'(v)} \\ \int \sqrt{\frac{u^2\phi'^2+a^2}{u^2-a^2}}du \end{bmatrix},\ e^{g'(v)} = \text{constant},\ a^2 < u^2$$

or

$$R(u,v)=\begin{bmatrix} -e^{g'(v)}\int\sqrt{\frac{u^2(1-\phi'^2)-a^2+u^2}{u^2-a^2}}du \\ e^{g'(v)}\sqrt{a^2-u^2}\cosh(v+\int\frac{a\phi'}{a^2-u^2}) \\ e^{g'(v)}\sqrt{a^2-u^2}\sinh(v+\int\frac{a\phi'}{a^2-u^2}) \end{bmatrix},\ e^{g'(v)}=\text{constant},\ u^2<a^2$$

have a homothetic relation. If $e^{g'(v)}=1$ then we obtain Ikawa's theorem in [11]. That is we obtain an isometric relation between a helicoid and the rotational surface. In this theorem, if $a^2>u^2$ then the surfaces are timelike, if $a^2<u^2$ then the surfaces are spacelike.

Case 4.8. *In the following section, we obtain the* Bour's theorem *under the conformal map with the timelike axis l and timelike profile curve γ which are given in the Case 2.*

Theorem 4.9. *Let $H(u,v)$ and $S(u,v)$ be (T,T)-type helicoidal surface and (T,T)-type spiral surface which are given in the Case 1, respectively. If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equation*

$$S(u,v)=\begin{bmatrix} e^{g'(v)}\sqrt{\frac{a^2+u^2-b^2\phi_s(u)^2}{b^2+1}}\cos(v) \\ e^{g'(v)}\sqrt{\frac{a^2+u^2-b^2\phi_s(u)^2}{b^2+1}}\sin(v) \\ e^{g'(v)}\phi_s(u) \end{bmatrix}.$$

In this theorem

$$\phi_s=\frac{-ab(b^2+1)\phi'^2\pm\sqrt{F}}{-b^4-b^2(b^2+1)\phi'^2}$$

Where

$$F=b^2(b^2+1)^2a^2\phi'^4-(b^4+b^2(b^2+1)\phi'^2)((b^2+1)a\phi'^2-b^2a^2-b^2u^2),\ 1-\phi'^2<0.$$

Consequently, we can easily say that helices on the (T,T)-type helicoidal surface can be transformed to spirals on the (T,T)-type spiral surface.

If $e^{g'(v)}$ is constant then the spiral surface reduces to the rotational surface so we obtain a (T,T)-type generalized helicoid

$$H(u,v)=\begin{bmatrix} u\cos v \\ u\sin v \\ \phi(u)+av \end{bmatrix}$$

and the rotational surface

$$R(u,v)=\begin{bmatrix} e^{g'(v)}\sqrt{u^2-a^2}\cos(v-\int\frac{a\phi'}{u^2-a^2}) \\ e^{g'(v)}\sqrt{u^2-a^2}\sin(v-\int\frac{a\phi'}{u^2-a^2})e^{g'(v)} \\ \int\sqrt{\frac{u^2\phi'^2+a^2}{u^2-a^2}}du \end{bmatrix},\ e^{g'(v)}=\text{constant},\ a^2<u^2$$

or

$$R(u,v)=\begin{bmatrix} -e^{g'(v)}\int\sqrt{\frac{u^2(1-\phi'^2)-a^2+u^2}{u^2-a^2}}du \\ e^{g'(v)}\sqrt{a^2-u^2}\cosh(v+\int\frac{a\phi'}{a^2-u^2}) \\ e^{g'(v)}\sqrt{a^2-u^2}\sinh(v+\int\frac{a\phi'}{a^2-u^2}) \end{bmatrix},\ e^{g'(v)}=\text{constant},\ u^2<a^2$$

have a homothetic relation.If $e^{g'(v)}=1$ then we obtain Ikawa's theorem.That is we obtain an isometric relation between a helicoid and the rotational surface. In this theorem the surfaces are timelike.

Case 4.10. *In the following section, we obtain the* Bour's theorem *under the conformal map with the lightlike axis l and the profile curve which are given in the Case 3.*

Theorem 4.11. *Let $H(u,v)$ and $S(u,v)$ which are given in the Case 3, respectively. If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equation*

$$S(u,v)=\begin{bmatrix} 2e^{g'(v)}\frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}v \\ e^{g'(v)}\left(\phi_s+\frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}(1-v^2)\right) \\ e^{g'(v)}\left(\phi_s-\frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}(1+v^2)\right) \end{bmatrix}.$$

Where

$$\phi_s=2ab-2\frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}\phi',\ b\neq 0.$$

Consequently, we can easily say that helices on the type helicoidal surface(with lightlike axis) can be transformed to spirals on the spiral surface(with lightlike axis).

Proof. The first fundamental form of the helicoid as follows

$$E=g(H_u,H_u)=4\phi'$$

$$F=g(H_u,H_v)=2a$$

$$G=g(H_v,H_v)=4u^2.$$

So, we obtain the line element of the $H(u,v)$

$$ds^2 = 4u^2du^2 + 4adudv + 4u^2dv^2. \tag{9}$$

On the other hand, the first fundamental form of the spiral surface as follows

$$E_s = g\left(S_{u_s}, S_{u_s}\right) = 4\phi_s'$$
$$F_s = g\left(S_{u_s}, S_{v_s}\right) = 2bu_s\phi_s' + b\phi_s$$
$$G_s = g\left(S_{v_s}, S_{v_s}\right) = 4u_s^2 + 4bu_s^2 + 4b^2u_s\phi_s.$$

So, we obtain the line element of the spiral surface $S(u_s, v_s)$

(10)

$$ds_s^2 = \left(e^{g'(v)}\right)^2\left[4\phi_s'du_s^2 + 2\left(2bu_s\phi_s' + b\phi_s\right)du_sdv_s + 4u_s^2 + \left(4bu_s^2 + 4b^2u_s\phi_s\right)dv_s^2\right].$$

If we compare the line element of the hicoid in eq. (9) with the line element of the spiral surface in eq. (10) then we obtain the function ϕ_s as follows

$$\phi_s = 2ab - 2\frac{-ab \pm \sqrt{a^2b^2 + u^2 + bu^2 - 2b^2\phi'}}{1 + b - 2b^2\phi'}\phi',\ \ b \neq 0.$$

Then we obtain

$$E_s = \left(e^{g'(v)}\right)^2 E, F_s = \left(e^{g'(v)}\right)^2 F, G_s = \left(e^{g'(v)}\right)^2 G.$$

Therefore a generalized helicoid

$$H(u,v) = \begin{bmatrix} 2uv \\ \phi(u) + u - uv^2 + av \\ \phi(u) - u - uv^2 + av \end{bmatrix}$$

and the spiral surface

$$S(u,v) = \begin{bmatrix} 2e^{g'(v)}\frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}v \\ e^{g'(v)}\left(\phi_s + \frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}(1-v^2)\right) \\ e^{g'(v)}\left(\phi_s - \frac{-ab\pm\sqrt{a^2b^2+u^2+bu^2-2b^2\phi'}}{1+b-2b^2\phi'}(1+v^2)\right) \end{bmatrix}$$

have a conformal relation.

If $e^{g'(v)}$ is constant, i.e, $b = 0$ then the spiral surface reduce to rotational surface has the line element

$$ds_s^2 = \left(e^{g'(v)}\right)^2\left[4\phi_s'du_s^2 + 4u_s^2dv_s^2\right]. \tag{11}$$

On the other hand, helices in $H(u,v)$ are defined by u =constant. From the orthogonal condition , it follows that

$$2u^2du + adv = 0$$

hence

$$v = \frac{a}{2u} + c,\ \ c = \text{constant}.$$

Thus, if we consider that

$$\bar{v} = v - \frac{a}{2u}.$$

then the curve that are orthogonal to helices are given by $\bar{v}$ =constant.

if we consider that the equation

$$d\bar{v} = dv + \frac{a}{2u^2}du.$$

into eq. (9) then we obtain

$$ds^2 = (4\phi' - \frac{a^2}{u^2})du^2 + 4u^2d\bar{v}^2. \tag{12}$$

Comparing eq. (11) with eq. (12), we obtain

$$u = u_s,\ \ v_s = \bar{v}_s,\ \ \phi_s = \phi + \frac{a^2}{4u}$$

Thus a generalized helicoid

$$H(u,v) = \begin{bmatrix} 2uv \\ \phi(u) + u - uv^2 + av \\ \phi(u) - u - uv^2 + av \end{bmatrix}$$

and the rotational surface

$$R(u,v) = \begin{bmatrix} e^{g'(v)}2u\left(v - \frac{a}{2u}\right) \\ e^{g'(v)}\left(\phi(u) + \frac{a^2}{4u} + u - u\left(v - \frac{a}{2u}\right)^2\right) \\ e^{g'(v)}\left(\phi(u) + \frac{a^2}{4u} + u - u\left(v - \frac{a}{2u}\right)^2\right) \end{bmatrix}$$

have a homothetic relation. If $e^{g'(v)} = 1$ then we obtain the Ikawa's theorem, that is, we obtain an isometric relation between a generalized helicoid and the rotational surface. $\square$

Corollary 4.12. *A spiral surface and a rotational surface have a conformal relation. So, spirals on the spiral surface can be transformed to circles on the rotational surface. Consequently, helicoidal, rotational and spiral surface have following diagram which is comutative.*

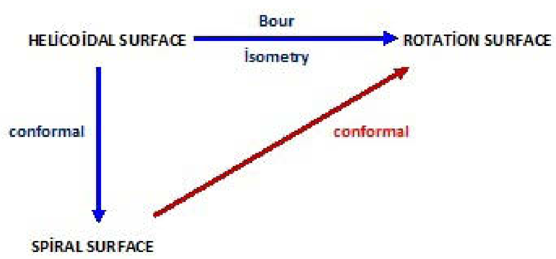

Figure 1: *Relation among the helicoidal, rotational and spiral surface.*

In this section, we give some figures which related to helicoidal surfaces and their image spiral surfaces under the conformal map.

5. Examples

Example 5.1. *If we consider any (S, S)-type helicoidal surface with the following equation*

$$H(u, v) = \left(1/(u-1)^2 + v, u \cosh v, u \sinh v\right)$$

then we can easily obtain that the (S, S)-type spiral surface $S(u, v)$ which is image of the helicoidal surface $H(u, v)$ under the conformal map is given in Figure 2.

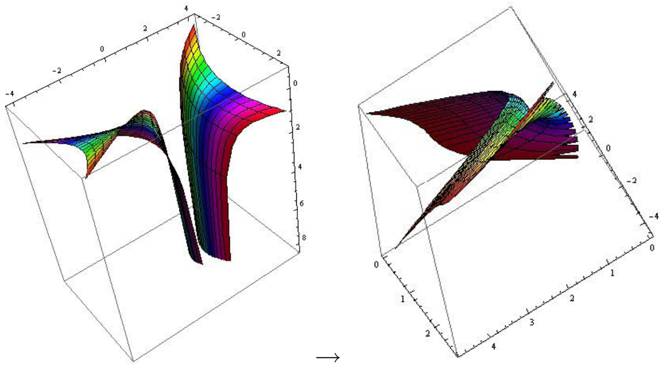

Figure 2: *A (S,S)-type helicoidal surface and its image (S,S)-type spiral surface under the conformal map.*

Example 5.2. *If we consider any (L,S)-type spacelike helicoidal surface with the following equation*

$$H\,(u,v) = (2uv, u^{-2} - 2u - 1 + u - uv^2 + v, u^{-2} - 2u - 1 - u - uv^2 + v)$$

then we can easily obtain that the (L,S)-type spacelike spiral surface $S(u,v)$ which is image of the helicoidal surface $H\,(u,v)$ under the conformal map is given in Figure 3.

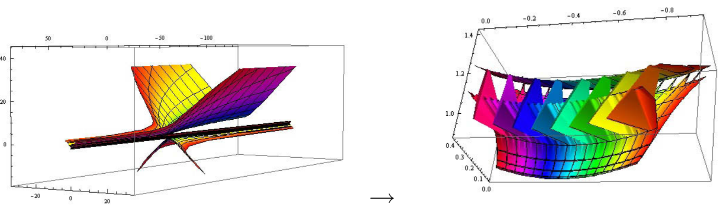

Figure 3: *A (L,S)-type helicoidal surface and its image (L,S)-type spiral surface under the conformal map.*

Example 5.3. *If we consider any (L,S)-type timelike helicoidal surface with the following equation*

$$H\,(u,v) = (2uv, u^2 - 2u + u - uv^2 + v, u^2 - 2u - u - uv^2 + v)$$

then we can easily obtain that the (L,S)-type timelike spiral surface $S(u,v)$ which is image of the helicoidal surface $H(u,v)$ under the conformal map is given in Figure 4.

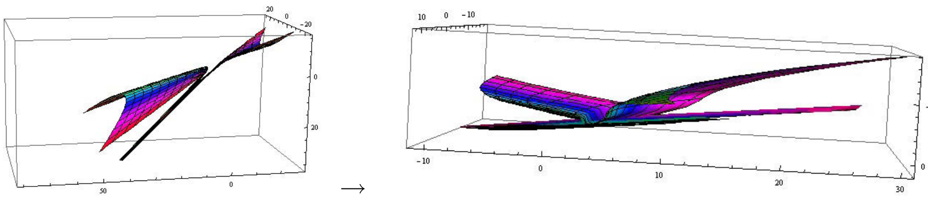

Figure 4: *A (L,S)-type helicoidal surface and its image (L,S)-type spiral surface under the conformal map.*

References

[1] Bour E., *Memoire sur le deformation de surfaces*, Journal de I'E cole Polytechnique XXXIX Cahier, (1862), 1-148.

[2] Bozkurt, Z., Gök. İ., Ekmekci, F. N. and Yaylı, Y., *A conformal approach to Bour's theorem*, Mathematica Aeterna Int. J. Appl. Math. , 8(2), (2012), 701 - 713.

[3] Boyadzhiev, K., N., *Spirals and Conchospirals in the Flight of Insect*s, College Math. J., 30 (1999) 1, 23–31.

[4] Do Carmo, M.P.and Dajczer, M., *Helicoidal surfaces with constant mean curvature*, Tohoku Math.J., 34, (1982), 425-435.

[5] Güler, E., Yaylı, E., Hacısalihoğlu, H.H., *Bour's theorem on Gauss map in 3-Euclidean space*, Hacettepe Math.J.39(4), 515-525, 2010.

[6] Güler, E., Vanlı, T., *Bour's theorem ın Minkowski 3-space*, Kyoto J. Math., 46(1), (2006), 47-63.

[7] Güler, E., Vanlı, T., *On the Mean, Gaussian, the Second Gaussian and the Second mean curvature of the helicoidal surfaces with light-like axis in Minkowski-3 Spac*e, Tsukuba Math. J. 32(1), (2008), 49-65.

[8] Güler, E., *Bour's theorem and light-like profile curve,* Yokohama Math. J., 54(1) , (2007), 55-77.

[9] Güler, E., *Bour's theorem on timelike helicoidal surfaces with (L,L)-type in Minkowski-3 Space*, Beykent University J. Sci. Technol., Turkey, 2 (1), (2008), 86-98.

[10] Hacısalihoğlu, H.,H., *Diferensiyel Geometri*, İnönü universty Faculty of Science and Arts Press, Turkey, 2004

[11] Ikawa, T., *Bour's theorem ın Minkowski Geometry*, Tokyo Math.J. 24(2), (2001), 377-394.

[12] Ikawa, T., *Bour's theorem and Gauss map* , Yokohama Math. J. 48(2), (2000), 173-180.

[13] Lopez, R., *Differential Geometry of Curves and Surfaces in Lorentz Minkowski Space*, 2008.

[14] Barrett O'neill, Elementary Differential Geometry, Academic Press, New York, 1966.

[15] Barrett O'neill, *Semi-Riemannian Geometry with Applications to Relativit*y, Academic Press, 1983.

[16] Ikawa, T., *An Introduction To Lorentz Surfaces*, Rutgers Uni., 1996.

[17] Whittemore, J.K, *Spiral mimimal surfaces*, Trans. Amer. Math. Soc., vol.19,