

ON SOME CURVES IN GALILEAN PLANE AND 3-DIMENSIONAL GALILEAN SPACE

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ABSTRACT. In this paper, we investigate some curves of Galilean geometry in plane and in space. We state the position vectors and prove two theorems about such curves in Galilean plane G_2 . Besides, firstly we define the slant helix and obtain the characterization of slant helices in 3-dimensional Galilean space G_3 .

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1. INTRODUCTION

The helix is one of the important notion in the differential geometry. A helix with non-vanishing constant curvature κ and non-vanishing constant torsion τ is a geometric curve for which the tangent makes a constant angle with a fixed direction. The most classic example of helices is a screw. The shape of screw likes stairs in minarets. Also in the nature, we can realize basic examples of helices such as nuts, bolts, horns. In animals helical structures usually appear in both mirror-image forms. In [1], Izumiya and Takeuchi defined slant helices. They have obtained that a curve with $\kappa(s) \neq 0$ is a slant helix if its normal line makes a constant angle with

a fixed direction. A necessary and sufficient condition that a curve be a slant helix the function

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' \right) (s)$$

is constant. The study of slant helices in other spaces is given but it is a new concept in 3- dimensional Galilean space G_3 .

The curvatures of any curve help us to determine of its shape. The Frenet formulas of any curve provide a system and the unique solution provides the curve. These concepts often use in the geometry. Moreover, the investigating position vectors of the curves in the differential geometry aim to determine behaviour of the curves, [2]. In this paper, we study the position vectors of some curves in Galilean plane G_2 .

The basic concepts of Euclidean plane geometry are points and straight lines, also the best known theorem is Pisagor theorem. But in nature, every surface is not a plane and every line is not a straight line like Euclidean geometry. The parallel postulate which is the fifth postulate in Euclid's Elements, is a distinctive axiom. The non-Euclidean geometry is a geometry which does not provide the parallel postulate. Galilean geometry is a non-Euclidean geometry and associated with Galilei principle of relativity. This principle can be explained briefly as " in all inertial frames, all law of physics are the same."

There are some reasons why Galilean geometry is interesting. Firstly, Galilean relativity today usually refers to Newton's laws. Later, in special relativity Einstein considers Galilean relativity and invariance of the speed of light. This leads to Lorentz transformation, [3]. Galilean geometry is physically the simplest model of the nonrelativistic spacetime where one of the coordinates is taken into account as a time and the others as space coordinates, [4].

Galilean geometry is widely developed by Yaglom and Roschel, [5,6]. Ergut, Bektas and Ogrenmis find the characterization for the curves and the helices in Galilean space, [7].

2. Preliminaries

The Galilean plane is defined as a projective plane with an absolute which consists of a line f (the absolute line) and a point F (the absolute point) lying on f . We can choose a basic coordinate system oxy such that the axis oy coincides with the absolute line f and the infinite point of oy coincides with the absolute point F . Then the direction, determined by the axis oy , is said to be the isotropic (or special) direction, [8].

A line in the Galilean (isotropic) plane is called isotropic if it is parallel to the isotropic direction, and two points P_1 and P_2 are said to be parallel if the line P_1P_2 is isotropic, [4]. A distance d between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \begin{cases} |x_1 - x_2|, & x_1 \neq x_2; \\ |y_1 - y_2|, & x_1 = x_2. \end{cases}$$

A scalar product between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined by

$$\langle P_1, P_2 \rangle = \begin{cases} x_1 x_2, & x_1 \neq 0 \vee x_2 \neq 0; \\ y_1 y_2, & x_1 = 0 \wedge x_2 = 0. \end{cases}$$

The Galilean motions is given by the formulas

$$\begin{aligned} x' &= \pm x + a, \\ y' &= vx \pm y + b \end{aligned}$$

where v is the velocity of the origin. We can find three dimensional Galilean space to write these motions in the form

$$\begin{aligned} x' &= (\cos \alpha) x + (\sin \alpha) y + \mu z + a \\ y' &= -(\sin \alpha) x + (\cos \alpha) y + \nu z + b \\ z' &= z + c \end{aligned}$$

where μ, ν, a, b, c and α are real numbers, [5].

Let $\alpha : I \rightarrow G_3, I \subseteq \mathbb{R}$ be a curve in 3-dimensional Galilean space G_3 given by

$$\alpha = (x, y(x), z(x)),$$

where x is a Galilean invariant the arclength on α .

The curvature and the torsion are defined by

$$\kappa(x) = \sqrt{y''^2(x) + z''^2(x)},$$

$$\tau(x) = \frac{\det(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)}.$$

respectively. The orthonormal trihedron of the curve α is defined by

$$T = \alpha'(x) = (1, y'(x), z'(x)),$$

$$N = \frac{1}{\kappa(x)} c''(x) = \frac{1}{\kappa(x)} (0, y''(x), z''(x)),$$

$$B = \frac{1}{\kappa(x)} (0, -z''(x), y''(x)).$$

The vectors T, N and B are called the vectors of the tangent, principal normal and binormal line, respectively. For their derivatives the following Frenet's formulas hold:

$$T'(x) = \kappa(x)N(x),$$

$$N'(x) = \tau(x)B(x),$$

$$B'(x) = -\tau(x)N(x),$$

[7, 9].

Definition 2.1. Let α be a curve in 3-dimensional Galilean space G_3 , and $\{T, N, B\}$ be the Frenet frame in 3-dimensional Galilean space G_3 along α . A curve α such that

$$\frac{\kappa}{\tau} = \text{const.}$$

is called a general helix with respect to Frenet frame, [7].

3.The Position Vectors of Curves in Galilean Plane

Lemma 3.1. For a curve $\varphi : I \rightarrow G_2, I \subseteq \mathbb{R}$ the Frenet derivatives are given by

$$(3.1) \quad \begin{aligned} T' &= \kappa N, \\ N' &= 0 \end{aligned}$$

where the vectors T, N are called the vectors of the tangent, principal normal in Galilean plane G_2 , respectively.

Theorem 3.1. Let $\varphi = \varphi(s)$ be a unit speed curve in Galilean plane G_2 . Then position vector of the curve φ is

$$\varphi(s) = (s + c_1)T + \left(-\frac{1}{2}\kappa s^2 - \kappa c_1 s + c_2\right)N$$

where c_1 and c_2 are integration constants.

Proof. Let $\varphi = \varphi(s)$ be a unit speed curve in G_2 . If we consider the Frenet frame of G_2 , we can write the position vector of the curve φ by

$$(3.2) \quad \varphi(s) = \delta T + \lambda N$$

If we derivate (3.2), we have

$$(\delta' - 1)T + (\delta\kappa + \lambda')N = 0$$

and so

$$(3.3) \quad \begin{cases} \delta' = 1 \\ \delta\kappa + \lambda' = 0 \end{cases}$$

From the first equation of (3.3), we have

$$\delta = s + c_1$$

and from the second equation, we have

$$\lambda = -\frac{1}{2}\kappa s^2 - \kappa c_1 s + c_2$$

where c_1 and c_2 are integration constants. Then, we can write

$$\varphi(s) = (s + c_1)T + \left(-\frac{1}{2}\kappa s^2 - \kappa c_1 s + c_2\right)N$$

This completes the proof.

Theorem 3.2. Let $\varphi = \varphi(s)$ with $\kappa(s) \neq 0$, be a unit speed curve in Galilean plane G_2 . Then position vector of the curve φ satisfies a differential equation of second order.

Proof. Let $\varphi = \varphi(s)$ be a unit speed curve in G_2 . Then from the equations (1), we can write

$$(3.4) \quad N = \frac{1}{\kappa}T' = \frac{1}{\kappa}\frac{dT}{ds}$$

From derivating of the equations (3.4), we have

$$N' = \frac{d}{ds} \left[\frac{1}{\kappa} \frac{dT}{ds} \right] = 0$$

In the last equation, if we do change of variable $\phi = \int \kappa ds$, we have

$$(3.5) \quad \frac{d^2 T}{d\phi^2} = 0$$

and from the solution of the equations (3.5) such that a and b are arbitrary constants, we have

$$T = a \int \kappa ds + b.$$

3. The Slant Helices in 3-dimensional Galilean Space G_3

Definition 4.1. A unit speed curve $\varphi : I \rightarrow G_3$ is called a slant helix in G_3 if its second Frenet vector V_2 makes a constant angle with the fixed direction U .

Theorem 4.1. Let $\varphi = \varphi(s)$ be a unit speed curve in 3-dimensional Galilean plane G_3 . Then, φ is a slant helix if and only if

$$(4.1) \quad \frac{\kappa^2}{\tau^3} \left(\frac{\tau}{\kappa} \right)' \text{ or } -\frac{1}{\tau} \left(\frac{\kappa}{\tau} \right)'$$

are constants.

Proof. Let $\varphi = \varphi(s)$ be a unit speed curve in G_3 . Assume that, φ is a slant helix. Let U be fixed direction which makes a constant angle θ with N . We can write U by

$$(4.2) \quad U = a_1 T + a_2 N + a_3 B$$

where a_1, a_2, a_3 are differentiable functions. From the definition of slant helix, we know that

$$\langle N, U \rangle = \cos \theta = \text{const} = c$$

If we consider this in the last equation (4.2), we have

$$U = a_1 T + cN + a_3 B$$

$$(4.3) \quad \begin{cases} a_1' = 0, \\ a_1\kappa - a_3\tau = 0, \\ c\tau + a_3' = 0 \end{cases}$$

From derivating of the last equation, we have

Here, from the second equation, we get

$$(4.4) \quad \frac{\tau}{\kappa} a_3 = a_1$$

from the equations (4.3) and (4.4),

$$\left(\frac{\tau}{\kappa} a_3\right)' = 0$$

and

$$(4.5) \quad a_3 = \frac{c \frac{\tau^2}{\kappa}}{\left(\frac{\tau}{\kappa}\right)'}$$

and from the equation (4.5), we have

$$(4.6) \quad -c\tau = \left[\frac{c \frac{\tau^2}{\kappa}}{\left(\frac{\tau}{\kappa}\right)'} \right]'$$

After some calculations in the equation (4.6), we have

$$\text{Log} \left[\frac{\tau' \kappa - \tau \kappa'}{\tau^3} \right] = 0$$

This gives that the equation (4.1) are constants. We do the converse of the proof.

If we assume that the equations (4.1) are constants, it can be shown easily that

$\langle N, U \rangle = \text{const}$ and φ is a slant helix.

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