

INTRINSIC EQUATIONS OF SOME SPECIAL CURVES IN EUCLIDEAN 3-SPACE

AHMAD TAWFIK ALI

PRESENT ADDRESS: KING ABDUL AZIZ UNIVERSITY
FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS
PO BOX 80203, JEDDAH, 21589, SAUDI ARABIA
PERMANENT ADDRESS: MATHEMATICS DEPARTMENT
FACULTY OF SCIENCE, AL-AZHAR UNIVERSITY
NASR CITY, 11884, CAIRO, EGYPT
E-MAIL: ATALI71@YAHOO.COM

(Received: 19 May 2012, Accepted: 20 July 2012)

ABSTRACT. In this paper, we obtain the intrinsic equations of some special curves in Euclidean 3-space characterized by $\det(\psi^{(r_1)}(s), \psi^{(r_2)}(s), \psi^{(r_3)}(s)) = 0$, where $r_i < r_{i+1}$, $r_1 \in \{1, 2, 3\}$, $r_2 \in \{2, 3, 4\}$ and $r_3 \in \{3, 4, 5\}$.

AMS Classification: 53A04, 53C40, 53C50

Keywords: General helix; Slant helix; Slant-slant helix; k -slant helix.

1. INTRODUCTION

From the view of differential geometry, a *straight line* is a geometric curve with the curvature $\kappa(s) = 0$. A *plane curve* is a family of geometric curves with torsion $\tau(s) = 0$. *Helix* (*circular helix*) is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ which is the simplest example of three-dimensional spirals [1, 2, 6].

General helix (a curve of constant slope) is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result was stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [15] for details) says that: *A necessary and sufficient condition that a curve to be a general helix is that the function*

$$(1) \quad \sigma_0 = \frac{\tau(s)}{\kappa(s)}$$

is constant along the curve, where κ and τ denote the curvature and the torsion, respectively.

In 2004, Izumiya and Takeuchi [8] introduced the concept of *slant helix* by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the *geodesic curvature* of the principal image of the principal normal indicatrix

$$(2) \quad \sigma_1 = \frac{\sigma'_0(s)}{\kappa(s) \left(1 + \sigma_0^2(s)\right)^{3/2}}$$

is a constant function.

In 2011, Ali [4] solved the above ordinary differential equation and obtained the explicit relation between the torsion and curvature (intrinsic equations) for a slant helix as follows:

$$(3) \quad \kappa = \kappa(s), \quad \tau = \pm \frac{m \kappa(s) \int \kappa(s) ds}{\sqrt{1 - m^2 \left(\int \kappa(s) ds \right)^2}},$$

where $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line (axis of a slant helix) and the principal normal vector of the slant helix.

A family of curves with constant curvature but non-constant torsion is called *Salkowski curves*; a family of curves with constant torsion but non-constant curvature is called *anti-Salkowski curves* which introduced in [13, 14] and a family of curves of *constant precession* [10, 11] which characterized by having

$$\kappa = \frac{\mu}{m} \sin[\mu s], \quad \tau = \frac{\mu}{m} \cos[\mu s],$$

where μ and m are constants are the important examples of a slant helices.

In this paper, we obtain the intrinsic equations of some special curves in Euclidean 3-space characterized by $\det(\psi^{(r_1)}(s), \psi^{(r_2)}(s), \psi^{(r_3)}(s)) = 0$, where $r_i < r_{i+1}$, $r_1 \in \{1, 2, 3\}$, $r_2 \in \{2, 3, 4\}$ and $r_3 \in \{3, 4, 5\}$.

2. PRELIMINARIES

In Euclidean space $\mathbf{E}^3$, it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ which are respectively called, the tangent, the principal normal and the binormal vector fields. We consider the usual metric in Euclidean 3-space $\mathbf{E}^3$, that is,

$$\langle, \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\psi : I \subset \mathbf{R} \rightarrow \mathbf{E}^3$, $\psi = \psi(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve ψ is said to be of unit speed (or parameterized by the arc-length) if $\langle \psi'(s), \psi'(s) \rangle = 1$ for any $s \in I$. In particular, if $\psi'(s) \neq 0$ for any s , then it is possible to re-parameterize ψ , that is, $\alpha = \psi(\phi(s))$ so that α is parameterized by the arc-length. A unit speed curve ψ is called a *Frenet curve* if $\psi''(s) \neq 0$, that is, it has non-zero curvature. Let $\psi : I \rightarrow \mathbf{e}^3$ be a Frenet curve and $\{\psi'(s) = \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the Frenet frame along ψ , where the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal vectors satisfying $\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1$. The Frenet equations for ψ are given by [7]

$$(4) \quad \begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where κ and τ are smooth functions on I called the curvature and the torsion of ψ , respectively. If $\tau(s) = 0$ for any $s \in I$, then $\mathbf{B}(s)$ is a constant vector V and the curve ψ lies in a 2-dimensional affine subspace orthogonal to V , which is isometric to the Euclidean 2-space $\mathbf{e}^2$.

3. INTRINSIC EQUATIONS OF SOME WELL KNOWN SPECIAL CURVES

From recent some papers [3, 4, 5, 9, 12], we can introduce the conclusion of the intrinsic equations for some important special curves in Euclidean 3-space as the following:

Proposition 3.1. [12] *A curve $\psi : I \rightarrow \mathbf{E}^3$ is spherical, i.e., it is contained in a sphere of radius r , if and only if*

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)^2 = r^2.$$

The curvature and the torsion of a spherical curve with radius r are given by:

$$\kappa = \kappa(s), \quad \tau = \frac{\dot{\kappa}}{\kappa\sqrt{r^2\kappa^2 - 1}}.$$

The curvature and the torsion of a spherical generalized helix are given by:

$$\kappa = \frac{1}{\sqrt{1 - c^2s^2}}, \quad \tau = \frac{c}{\sqrt{1 - c^2s^2}}.$$

Table 1. Intrinsic equations of some special curves

The curve	Conditions	Intrinsic Equations
Plane curve	$\tau = 0$	$\kappa = \kappa(s), \tau = 0$
Circular helix	$\kappa = a, \tau = b$	$\kappa = a, \tau = m a$
General helix	$\frac{\tau}{\kappa} = m$	$\kappa = \kappa(s), \tau = m \kappa(s)$
Slant helix	$\frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa} \right)' = m$	$\kappa = \kappa(s), \tau = \frac{m \kappa(s) \int \kappa(s) ds}{\sqrt{1 - m^2 (\int \kappa(s) ds)^2}}$
Spherical curve	$\left(\frac{1}{\kappa} \right)^2 + \left(\frac{\kappa'}{\kappa^2 \tau} \right)^2 = r^2$	$\kappa = \kappa(s), \tau = \frac{\kappa'(s)}{\kappa(s) \sqrt{r^2 \kappa^2(s) - 1}}$
Spherical plane curve (Circle)	$\kappa' = 0, \tau = 0$	$\kappa = \frac{1}{r}, \tau = 0$
Spherical general helix	$\frac{\kappa'}{\kappa^2 \sqrt{\kappa^2 - 1}} = \frac{\tau}{\kappa} = m$	$\kappa = \frac{1}{\sqrt{1 - m^2 s^2}}, \tau = \frac{m}{\sqrt{1 - m^2 s^2}}$
Slant helix with constant curvature (Salkowski curve)	$\kappa = 1, \frac{\tau'}{(1 + \tau^2)^{3/2}} = m$	$\kappa = 1, \tau = \frac{ms}{\sqrt{1 - m^2 s^2}}$
Slant helix with constant torsion (Anti-Salkowski curve)	$\tau = 1, \frac{\kappa^2}{(\kappa^2 + 1)^{3/2}} \left(\frac{1}{\kappa} \right)' = m$	$\kappa = \frac{ms}{\sqrt{1 - m^2 s^2}}, \tau = 1$
Spherical curves with constant torsion	$\tau = \mu, \left(\frac{1}{\kappa} \right)^2 + \left(\frac{\kappa'}{\kappa^2 \mu} \right)^2 = r^2$	$\kappa = \frac{\csc[\mu s + \theta]}{r}, \tau = \mu$
Bertrand curves	$A \kappa + B \tau = C$	$\kappa = \kappa(s), \tau = \alpha + \beta \kappa(s)$

4. INTRINSIC EQUATIONS OF SOME SPECIAL CURVES

In this section, we study the characterizations of some special curves in Euclidean 3-space characterized by $\det(\psi^{(r_1)}(s), \psi^{(r_2)}(s), \psi^{(r_3)}(s)) = 0$, where $r_i < r_{i+1}$, $r_1 \in \{1, 2, 3\}$, $r_2 \in \{2, 3, 4\}$ and $r_3 \in \{3, 4, 5\}$. Let $\psi : I \rightarrow \mathbf{E}^3 : s \mapsto \psi(s)$ be a space curve that is parameterized arclength. Using *Mathematica Programm*, we have the

following:

$$(5) \quad \begin{cases} \psi^{(1)}(s) = \mathbf{T} = \{1, 0, 0\}, \\ \psi^{(2)}(s) = \kappa \mathbf{N} = \{0, \kappa, 0\}, \\ \psi^{(3)}(s) = \{-\kappa^2, \kappa', \kappa\tau\}, \\ \psi^{(4)}(s) = \{-3\kappa\kappa', \kappa'' - \kappa\tau^2 - \kappa^3, 2\tau\kappa' + \kappa\tau'\}, \\ \psi^{(5)}(s) = \{\psi_1^{(5)}, \psi_2^{(5)}, \psi_3^{(5)}\}, \end{cases}$$

where

$$(6) \quad \begin{cases} \psi_1^{(5)} = \kappa^2(\kappa^2 + \tau^2) - 3\kappa'^2 - 4\kappa\kappa'', \\ \psi_2^{(5)} = \kappa''' - 3\tau(\kappa\tau' + \tau\kappa') - 6\kappa^2\kappa', \\ \psi_3^{(5)} = \kappa(\tau'' - \tau^3) + 3(\kappa'\tau' + \tau\kappa'') - \kappa^3\tau. \end{cases}$$

4.1. Special curves characterized by $\det(\psi^{(1)}(s), \psi^{(2)}(s), \psi^{(3)}(s)) = 0$. The condition $\det(\psi^{(1)}(s), \psi^{(2)}(s), \psi^{(3)}(s)) = \kappa^2\tau = 0$ if and only if the curve is a plane curve. The intrinsic equations of this case are:

$$(7) \quad \kappa = \kappa(s) \quad \text{and} \quad \tau = 0.$$

4.2. Special curves characterized by $\det(\psi^{(1)}(s), \psi^{(2)}(s), \psi^{(4)}(s)) = 0$. The condition $\det(\psi^{(1)}(s), \psi^{(2)}(s), \psi^{(4)}(s)) = \kappa(2\tau\kappa' + \kappa\tau') = 0$ if and only if the intrinsic equations of this case are:

$$(8) \quad \kappa = \kappa(s) \quad \text{and} \quad \tau = \frac{c_1}{\kappa^2(s)},$$

where c_1 is an arbitrary constant.

4.3. Special curves characterized by $\det(\psi^{(1)}(s), \psi^{(2)}(s), \psi^{(5)}(s)) = 0$. The condition $\det(\psi^{(1)}(s), \psi^{(2)}(s), \psi^{(5)}(s)) = 0$ leads to the following condition:

$$(9) \quad \kappa(\tau'' - \tau^3) + 3(\kappa'\tau' + \tau\kappa'') - \tau\kappa^3 = 0.$$

4.4. Special curves characterized by $\det(\psi^{(1)}(s), \psi^{(3)}(s), \psi^{(4)}(s)) = 0$. The condition $\det(\psi^{(1)}(s), \psi^{(3)}(s), \psi^{(4)}(s)) = 0$ leads to the following condition:

$$(10) \quad \tau\kappa^2(\kappa^2 + \tau^2) + 2\tau\kappa'^2 + \kappa(\kappa'\tau' - \tau\kappa'') = 0.$$

The above condition satisfied if and only if the intrinsic equations are:

$$(11) \quad \kappa = \kappa(s) \quad \text{and} \quad \tau = \frac{\exp(f[s])}{\sqrt{c_1 - 2 \int \left[\frac{\kappa(s)}{\kappa'(s)} \exp(2f[s]) \right] ds}},$$

where c_1 is an arbitrary constant and $f[s] = \int \left[\frac{\kappa(s)\kappa''(s) - 2\kappa'(s) - \kappa^4(s)}{\kappa(s)\kappa'(s)} \right] ds$.

As a special case, when the curve ψ is a general helix, then the intrinsic equations must be equal:

$$(12) \quad \kappa = \frac{1}{\sqrt{1 + m^2 \sqrt{c_2^2 - s^2}}} \quad \text{and} \quad \tau = \frac{n}{\sqrt{c_2^2 - s^2}},$$

where c_2 is an arbitrary constant and $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a general helix) and the tangent vector of the curve ψ . It is worth noting that equations (12) are the intrinsic equations of a spherical generalized helix.

4.5. Special curves characterized by $\det(\psi^{(1)}(s), \psi^{(3)}(s), \psi^{(5)}(s)) = 0$. The condition $\det(\psi^{(1)}(s), \psi^{(3)}(s), \psi^{(5)}(s)) = 0$ leads to the following condition:

$$(13) \quad \tau\kappa^2(5\kappa\kappa' + 3\tau\tau') + 3\kappa'(\kappa'\tau' + \tau\kappa'') + \kappa(3\tau^3\kappa' + \kappa'\tau'' - \tau\kappa''') = 0.$$

As a special case if the curve is circular helix then the condition above is satisfied.

4.6. Special curves characterized by $\det(\psi^{(1)}(s), \psi^{(4)}(s), \psi^{(5)}(s)) = 0$. The condition $\det(\psi^{(1)}(s), \psi^{(4)}(s), \psi^{(5)}(s)) = 0$ leads to the following condition:

$$(14) \quad (\kappa^3 + \kappa\tau^2 - \kappa'')(\tau\kappa^3 - 3(\kappa'\tau' + \tau\kappa'') + \kappa(\tau^3 - \tau'')) + (2\tau\kappa' + \kappa\tau')(3\kappa'(2\kappa^2 + \tau^2) + 3\kappa\tau\tau' - \kappa''') = 0.$$

4.7. Special curves characterized by $\det(\psi^{(2)}(s), \psi^{(3)}(s), \psi^{(4)}(s)) = 0$. The condition $\det(\psi^{(2)}(s), \psi^{(3)}(s), \psi^{(4)}(s)) = \kappa^5 \left(\frac{\tau}{\kappa} \right)' = 0$ if and only if the curve is a general helix. The intrinsic equations of this case are:

$$(15) \quad \kappa = \kappa(s) \quad \text{and} \quad \tau = m\kappa(s),$$

where $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a general helix) and the tangent vector of the curve ψ .

4.8. Special curves characterized by $\det(\psi^{(2)}(s), \psi^{(3)}(s), \psi^{(5)}(s)) = 0$. The condition $\det(\psi^{(2)}(s), \psi^{(3)}(s), \psi^{(5)}(s)) = 0$ leads to the following condition:

$$(16) \quad \kappa \left(\frac{\tau}{\kappa} \right)'' + 5\kappa' \left(\frac{\tau}{\kappa} \right)' = 0.$$

Solving the above equation, we have the intrinsic equations of such curve as the following:

$$(17) \quad \kappa = \kappa(s) \quad \text{and} \quad \tau = \kappa(s) \left[c_2 + c_1 \int \frac{ds}{\kappa^5(s)} \right],$$

where c_1 and c_2 are arbitrary constants. When $c_1 = 0$, the curve ψ is general helix.

4.9. Special curves characterized by $\det(\psi^{(2)}(s), \psi^{(4)}(s), \psi^{(5)}(s)) = 0$. The condition $\det(\psi^{(2)}(s), \psi^{(4)}(s), \psi^{(5)}(s)) = 0$ leads to the following condition:

$$(18) \quad \left(\kappa^4 \left[1 + \left(\frac{\tau}{\kappa} \right)^2 \right] + 12\kappa'^2 \right) \left(\frac{\tau}{\kappa} \right)' + \kappa \left[3\kappa' \left(\frac{\tau}{\kappa} \right)'' - 4\kappa'' \left(\frac{\tau}{\kappa} \right)' \right] = 0.$$

If the curve ψ is a general helix then the condition above is satisfied.

4.10. Special curves characterized by $\det(\psi^{(3)}(s), \psi^{(4)}(s), \psi^{(5)}(s)) = 0$. The condition $\det(\psi^{(3)}(s), \psi^{(4)}(s), \psi^{(5)}(s)) = 0$, leads the following condition:

$$(19) \quad \begin{aligned} & (12\kappa'^3 - 9\kappa\kappa'\kappa'' + \kappa^2\kappa''') \left(\frac{\tau}{\kappa} \right)' + \kappa(\kappa^4 + 3\kappa'^2 - \kappa\kappa'') \left(\frac{\tau}{\kappa} \right)'' \\ & + \kappa^5 \left[\left(\frac{\tau}{\kappa} \right) \left(\frac{\tau}{\kappa} \right)'' - 3 \left(\frac{\tau}{\kappa} \right)'{}^2 \right] \left(\frac{\tau}{\kappa} \right) = 0. \end{aligned}$$

We can solve the condition above to find intrinsic equations for some special curves satisfied the above condition.

(1): If the curve ψ is a general helix then the condition (19) is satisfied.

(2): If we take

$$(20) \quad \kappa(s) = \frac{1}{s^{r/2}} \quad \text{and} \quad \tau(s) = \frac{d(a+s)}{s^{r/2}\sqrt{b-(a+s)^2}},$$

the condition (19) becomes:

$$(21) \quad r(r-1)[(r-2)(a^2-b)s^r + a(2r-1)s^{r+1} + (r+1)s^{r+2}] = 6(d^2-1)(as^3 + s^4).$$

where a, b, d and r are arbitrary constants. From equation above, the value of r must be a positive integer number. When we take $r \geq 3$, the condition above not satisfied, so that the value of must be $r \in \{0, 1, 2\}$. Then we have the following three cases:

(2.1): When $r = 0$, then $d = \pm 1$ and so the intrinsic equations of such curve are

$$(22) \quad \kappa(s) = 1 \quad \text{and} \quad \tau(s) = \pm \frac{a+s}{\sqrt{b-(a+s)^2}} = \pm \frac{m(a+s)}{\sqrt{1-m^2(a+s)^2}}.$$

which are the intrinsic equations of Salkowski curves where $m = \frac{1}{\sqrt{b}} = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ and ϕ is the angle between the fixed straight line $\mathbf{e}_3$ (axis of a slant helix) and the normal vector of the curve ψ .

(2.2): When $r = 1$, then $d = \pm 1$ and so the intrinsic equations of such curve are

$$(23) \quad \kappa(s) = \frac{1}{\sqrt{s}} \quad \text{and} \quad \tau(s) = \pm \frac{a+s}{\sqrt{s}\sqrt{b-(a+s)^2}}.$$

(2.3): When $r = 2$, then $d = \pm\sqrt{2}$ and so the intrinsic equations of such curve are

$$(24) \quad \kappa(s) = \frac{1}{s} \quad \text{and} \quad \tau(s) = \pm \frac{\sqrt{2}(a+s)}{s\sqrt{b-(a+s)^2}}.$$

REFERENCES

- [1] Ali A.T. Position vectors of spacelike general helices in Minkowski 3-space, Nonl. Anal. Theory Meth. Appl. Vol 73, 1118–1126, (2010).
- [2] Ali A.T., and Turgut M. Position vector of a time-like slant helix in Minkowski 3-space, J. Math. Anal. Appl. Vol 365, 559–569, (2010).
- [3] Ali A.T. Position vectors of general helices in Euclidean 3-space, Bull. Math. Anal. Appl. Vol 3(2), 198–205, (2010).
- [4] Ali A.T. Position vectors of slant helices in Euclidean 3-space, J. Egyptian Math. Soc. Vol 20, 1–6, (2012).
- [5] Ali A.T. New special curves and their spherical indicatrices, Global J. Adv. Res. Class. Mod. Geom. Vol 1(2), 28–38, (2012).

- [6] Barros M. General helices and a theorem of Lancret, *Proc. Amer. Math. Soc.* Vol 125, 1503–1509, (1997).
- [7] Do Carmo M.P. *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, NJ, 1976.
- [8] Izumiya S. and Takeuchi N. New special curves and developable surfaces, *Turk. J. Math.* Vol 28, 531–537, (2004).
- [9] Izumiya S. and Takeuchi N. Generic properties of helices and Bertrand geometry, *J. Geom.* Vol 74, 97–109, (2002).
- [10] Kula L. and Yayli Y. On slant helix and its spherical indicatrix, *Appl. Math. Comp.* Vol 169, 600–607, (2005).
- [11] Kula L., Ekmekci N., Yayli Y. and Ilarslan K. Characterizations of slant helices in Euclidean 3-space, *Turk. J. Math.* Vol 34, 261–274, (2010).
- [12] Monterde J. Curves with constant curvature ratios, *Bull. Mexican Math. Soc. 3a.* Vol 13, 177–186, (2007).
- [13] Monterde J. Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion, *Comput. Aided Geomet. Design* Vol 26, 271–278, (2009).
- [14] Salkowski E. Zur transformation von raumkurven, *Mathematische Annalen*, Vol 66, 517–557, (1909).
- [15] Struik D.J. *Lectures in Classical Differential Geometry*, Addison-Wesley, Reading, MA, 1961.