HOMOTOPY PERTURBATION METHOD TO APPROXIMATE THE STABLE MANIFOLD

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Abstract. In this paper, the homotopy perturbation method is applied to approximate the stable manifold of a nonlinear autonomous system of ordinary differential equations. The results reveal that the proposed method is very effective and simple.

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1. Introduction

To find the explicit solutions of nonlinear differential equations, many powerful methods have been used. The homotopy perturbation method (HPM) [1]–[3] is a general analytic approach to get series solutions of various types of nonlinear equations. The HPM is based on homotopy, a fundamental concept in topology and differential geometry [7]. This method proposed first by He 1998 [2]. In recent years, the applications of the HPM have appeared in many works, which show the method is a powerful technique to study numerical solutions. In this paper, we

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would like to apply the HPM to approximate the stable manifold of the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

near a hyperbolic equilibrium point $\mathbf{x}_0$.
By the stable manifold Theorem [6], the nonlinear (1) has stable and unstable manifolds S and U tangent at $\mathbf{x}_0$ to the stable and unstable subspaces E^S and E^U of the linearized system

$$\dot{\mathbf{x}} = A(\mathbf{x}), \tag{2}$$

where $A = D\mathbf{f}(\mathbf{x}_0)$.
Remark 1.1 By hypothesis of the stable manifold theorem, E is an open subset of $\mathbf{R}^n$ containing the origin, $\mathbf{f} \in C^1(E)$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, hence the system (1) can be written as

$$\dot{\mathbf{x}} = A(\mathbf{x}) + \mathbf{F}(\mathbf{x}), \tag{3}$$

where $A = D\mathbf{f}(\mathbf{0}),\ \mathbf{F}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - A\mathbf{x},\ \mathbf{F} \in C^1(E),\ \mathbf{F}(\mathbf{0}) = \mathbf{0}$ and $D\mathbf{F}(\mathbf{0}) = \mathbf{0}$.

Furthermore, by the Jordan canonical form Theorem [4], there is an $n \times n$ invertible matrix C such that

$$B = C^{-1}AC = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

where the eigenvalues $\lambda_1, ..., \lambda_k$ of the $k \times k$ matrix P have negative real part and the eigenvalues $\lambda_{k+1}, ..., \lambda_n$ of the $(n-k) \times (n-k)$ matrix Q have positive real part. Letting $\mathbf{y} = C^{-1}\mathbf{x}$, the system (3) then has the form

$$\dot{\mathbf{y}} = B(\mathbf{y}) + \mathbf{G}(\mathbf{y}), \tag{4}$$

where $\mathbf{G}(\mathbf{y}) = C^{-1}\mathbf{F}(C\mathbf{y}) \in C^{-1}(\tilde{E}),$ and $\tilde{E} = C^{-1}(E)$.
Also, by the proof of stable manifold Theorem, there are $n-k$ differentiable functions $\psi_j(y_1, ..., y_k)$ such that the equations

$$y_j = \psi_j(y_1, ..., y_k),\ j = k+1, ..., n$$

define a $k-$dimensional differentiable manifold $\tilde{S}$ in $\mathbf{y}-$space. The differentiable manifold S in $\mathbf{x}-$space is then obtained from $\tilde{S}$ under the linear transformation of coordinates $\mathbf{x} = C\mathbf{y}$. Let

$$U(t) = \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix}.$$

Then $\dot{U} = BU$, $\dot{V} = BV$ and $e^{Bt} = U(t) + V(t)$.

Next consider the integral equation

$$\mathbf{u}(t,\mathbf{a}) = U(t)\mathbf{a} + \int_0^t U(t-s)\mathbf{G}(\mathbf{u}(s,\mathbf{a}))ds - \int_t^\infty V(t-s)\mathbf{G}(\mathbf{u}(s,\mathbf{a}))ds. \tag{5}$$

The continuous function $\mathbf{u}(t,\mathbf{a})$ satisfies the integral equation (5) and hence the differential equation (4)[6].

It is clear from (5) that the last $n-k$ components of the vector $\mathbf{a}$ do not enter the computation and hence they may be taken as zero. Thus, the components $u_j(t,\mathbf{a})$ of the solution $\mathbf{u}(t,\mathbf{a})$ satisfy the initial conditions

$$u_j(t,\mathbf{a}) = a_j, \text{ for } j = 1, ..., k,$$

and

$$u_j(0,\mathbf{a}) = -\int_t^\infty V(-s)\mathbf{G}(\mathbf{u}(s, a_1, ..., a_k, \mathbf{0}))ds, \text{ for } j = k+1, ..., n.$$

For $j = k+1, ..., n$ we define the functions

$$\psi_j(a_1, ..., a_k) = u_j(0, a_1, ..., a_k, 0, ..., 0). \tag{6}$$

Then the initial values $y_j = u_j(0, a_1, ..., a_k, 0, ..., 0)$ satisfy

$$y_j = \psi_j(y_1, ..., y_k), \text{ for } j = k+1, ..., n$$

according to the definition (6). These equation then define a differentiable manifold $\tilde{S}$ for $\mathbf{y}$ sufficiently near the origin. For more information see section 2.7 of [6].

The aim of this article is to directly applied the HPM to approximate the stable manifold from (5).

2. Applications

We construct the following simple homotopy:

$$\begin{aligned} H(\mathbf{u}, q) &= (\mathbf{u}(t,\mathbf{a}) - U(t)\mathbf{a}) \\ &\quad - q\left(\int_0^t U(t-s)\mathbf{G}(\mathbf{u}(s,\mathbf{a}))ds + \int_t^\infty V(t-s)\mathbf{G}(\mathbf{u}(s,\mathbf{a}))ds\right) = 0, \end{aligned} \tag{7}$$

where $q \in [0,1]$ is an embedding parameter. The HPM uses the homotopy parameter q as an expanding parameter [5] to obtain

$$\mathbf{v} = \mathbf{v}_0 + q\mathbf{v}_1 + q^2\mathbf{v}_2 + q^3\mathbf{v}_3 + \cdots \tag{8}$$

and

$$\mathbf{u} = \lim_{q\to 1} \mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \cdots . \tag{9}$$

The series (9) is convergent for most cases [3].

In order to illustrate the advantages and the accuracy of the HPM for approximation the stable manifold, we have applied the method to three examples.

Example 1. Consider the nonlinear system of ordinary differential equations

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2^2,\\ \dot{x}_2 &= x_1^2 + x_2.\end{aligned}$$

We have

$$Df(\mathbf{0}) = A = B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{F}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) = \begin{pmatrix} -x_2^2 \\ x_1^2 \end{pmatrix},$$

$$U(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix},\ V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{t} \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}.$$

Substituting (8) together with the above quantities into (7), and equating the terms with the identical power of q, we have

$$\mathbf{v}(t,\mathbf{a}) = U(t)\mathbf{a} + q\left(\int_0^t U(t-s)\mathbf{G}(\mathbf{v}(s,\mathbf{a}))ds - \int_t^\infty V(t-s)\mathbf{G}(\mathbf{v}(s,\mathbf{a}))ds\right), \tag{10}$$

and hence,

$$q^0 :\ \mathbf{v}_0(t,\mathbf{a}) = U(t)\mathbf{a} = \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix},$$

$$q^1 :\ \mathbf{v}_1(t,\mathbf{a}) = -\int_t^\infty \begin{pmatrix} 0 \\ e^{t-3s}a_1^2 \end{pmatrix} ds = \begin{pmatrix} 0 \\ -\frac{1}{3}e^{-2t}a_1^2 \end{pmatrix},$$

$$q^2 :\ \mathbf{v}_2(t,\mathbf{a}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$q^3 :\ \mathbf{v}_3(t,\mathbf{a}) = -\frac{1}{9}\int_0^t \begin{pmatrix} e^{-t-3s}a_1^4 \\ 0 \end{pmatrix} ds = \begin{pmatrix} \frac{1}{27}a_1^4(e^{-4t} - e^{-t}) \\ 0 \end{pmatrix},$$

so,

$$\begin{aligned}\mathbf{u}(t,\mathbf{a}) &\simeq \mathbf{v}_0+\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3\\ &= \begin{pmatrix} e^{-t}a_1+\frac{1}{27}a_1^4(e^{-4t}-e^{-t})\\ -\frac{1}{3}e^{-2t}a_1^2\end{pmatrix}.\end{aligned}$$

Therefore the function $\psi_2(a_1)=u_2(0,a_1,0)$ is approximated by

$$\psi_2(a_1)\simeq -\frac{1}{3}a_1^2.$$

Hence, the stable manifold S is approximated by

$$S: x_2 \simeq -\frac{x_1^2}{3}.$$

Example 2. Consider

$$\begin{aligned}\dot{x}_1 &= -x_1,\\ \dot{x}_2 &= x_1^2-2x_2,\\ \dot{x}_3 &= x_1^3+x_3.\end{aligned}$$

We have

$$D\mathbf{f}(\mathbf{0})=A=B=\begin{pmatrix}-1&0&0\\0&-2&0\\0&0&1\end{pmatrix},\ \mathbf{F}(\mathbf{x})=\mathbf{G}(\mathbf{x})=\begin{pmatrix}0\\x_1^2\\x_1^3\end{pmatrix},$$

$$U(t)=\begin{pmatrix}e^{-t}&0&0\\0&e^{-2t}&0\\0&0&0\end{pmatrix},\ V(t)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&e^{t}\end{pmatrix}\text{ and }\mathbf{a}=\begin{pmatrix}a_1\\a_2\\0\end{pmatrix}.$$

Substituting (8) together with the above quantities into (7), and equating the terms with the identical power of q, we have

$$q^0:\ \mathbf{v}_0(t,\mathbf{a})=U(t)\mathbf{a}=\begin{pmatrix}e^{-t}a_1\\e^{-2t}a_2\\0\end{pmatrix},$$

$$q^1:\ \mathbf{v}_1(t,\mathbf{a})=\begin{pmatrix}0\\te^{-2t}a_1^2\\-\frac{1}{4}e^{-3t}a_1^3\end{pmatrix},$$

and $\mathbf{v}_j(t,\mathbf{a})=\mathbf{0},$ for $j=2,3,\cdots$. Hence,

$$\mathbf{u}(t,\mathbf{a}) = \mathbf{v}_0 + \mathbf{v}_1 = \begin{pmatrix} e^{-t}a_1 \\ e^{-2t}a_1^2 \\ -\frac{1}{4}e^{-3t}a_1^3 \end{pmatrix}.$$

Therefore the function $\psi_3(a_1, a_2) = u_3(0, a_1, a_2, 0)$ is given by

$$\psi_3(a_1, a_2) = -\frac{1}{4}a_1^3,$$

and the global stable manifold S is given by

$$S : x_3 = -\frac{x_1^3}{4}.$$

Example 3. Consider

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= x_1^2 - x_2, \\ \dot{x}_3 &= x_2^2 + x_3. \end{aligned}$$

We have

$$Df(\mathbf{0}) = A = B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{F}(\mathbf{x}) = \mathbf{G}(\mathbf{x}) = \begin{pmatrix} 0 \\ x_1^2 \\ x_2^2 \end{pmatrix},$$

$$U(t) = \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ V(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{t} \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}.$$

Substituting (8) together with the above quantities into (7), and equating the terms with the identical power of q, we have

$$q^0 : \ \mathbf{v}_0(t,\mathbf{a}) = U(t)\mathbf{a} = \begin{pmatrix} e^{-t}a_1 \\ e^{-t}a_2 \\ 0 \end{pmatrix},$$

$$q^1 : \ \mathbf{v}_1(t,\mathbf{a}) = \begin{pmatrix} 0 \\ (e^{-t} - e^{-2t})a_1^2 \\ -\frac{1}{3}e^{-2t}a_2^2 \end{pmatrix},$$

$$q^2 : \ \mathbf{v}_2(t,\mathbf{a}) = \begin{pmatrix} 0 \\ 0 \\ (-\frac{2}{3}e^{-2t} + \frac{1}{2}e^{-3t})a_1^2 a_2 \end{pmatrix},$$

$$q^3: \ \mathbf{v}_3(t,\mathbf{a}) = \begin{pmatrix} 0 \\ 0 \\ (-\frac{1}{3}e^{-2t} + \frac{1}{2}e^{-3t} - \frac{1}{5}e^{-4t})a_1^4 \end{pmatrix},$$

and $\mathbf{v}_j(t,\mathbf{a}) = \mathbf{0}$, for $j = 4, 5, \cdots$. Hence,

$\mathbf{u}(t,\mathbf{a}) = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

Therefore the function $\psi_3(a_1, a_2) = u_3(0, a_1, a_2, 0)$ is given by

$$\psi_3(a_1, a_2) = -\frac{1}{3}a_2^2 + \frac{1}{2}a_1^2 a_2 - \frac{2}{3}a_1^2 a_2 - \frac{1}{30}a_1^4,$$

and the global stable manifold S is given by

$$S : x_3 = -\frac{1}{3}x_2^2 - \frac{1}{6}x_1^2 x_2 - \frac{1}{30}x_1^4.$$

The approximations for the stable manifolds S in a neighborhood of the origin are shown in Figs. 1, 2 and 3.

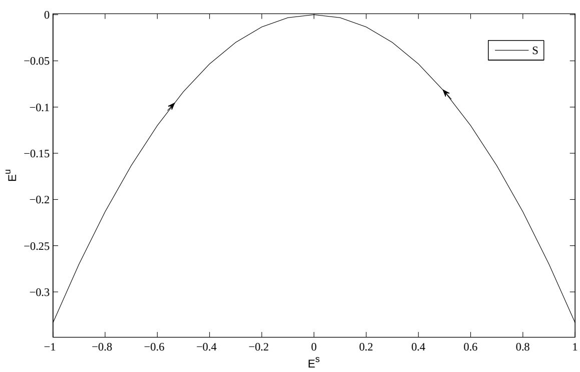

FIGURE 1. The approximation for stable manifold of Example 1.

We can approximate the unstable manifold $\tilde{U}$ in the same way by considering the system (4) with $t \longrightarrow -t$ i.e.

$$\dot{\mathbf{y}} = -B(\mathbf{y}) - \mathbf{G}(\mathbf{y}).$$

The stable manifold for this system will then be the unstable manifold $\tilde{U}$ for (4). Note that it is necessary to replace the vector $\mathbf{y}$ by the vector $(y_{k+1}, ..., y_n, y_1, ..., y_k)$ in order to determine the $n - k$ dimensional manifold $\tilde{U}$ by the above process.

3. Conclusion

In this paper, the homotopy perturbation method has been applied for approximate the stable manifold of a nonlinear system of differential equations. The results obtained in this work confirm the notion that the HPM is a powerful and efficient technique for finding approximate and exact solutions for the stable manifold of these systems.

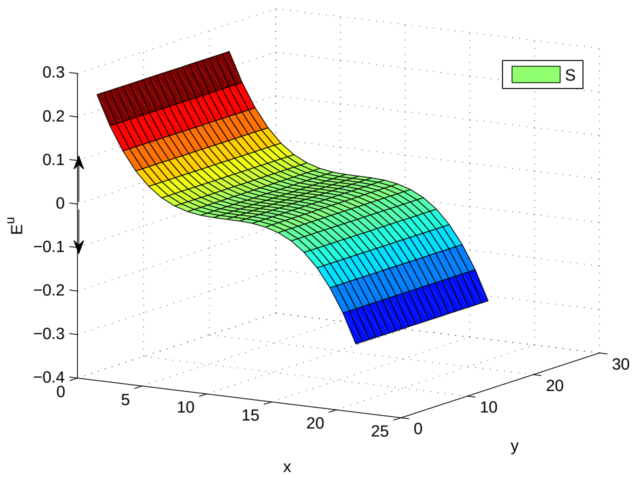

Figure 2. The approximation for stable manifold of Example 2.

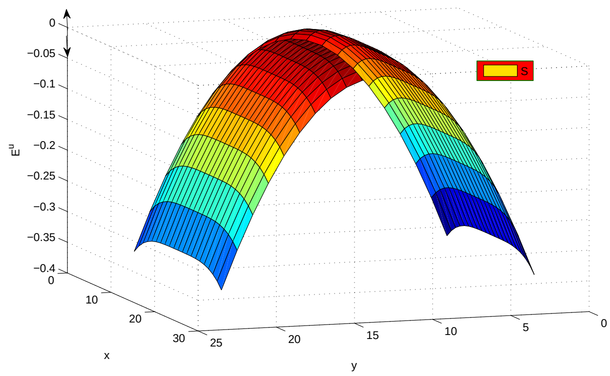

Figure 3. The approximation for stable manifold of Example 3.

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