ASYMPTOTIC AVERAGE SHADOWING PROPERTY FOR FLOWS

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(Received: 19 July 2011, Accepted: 21 March 2012)

ABSTRACT. In this paper we show that if X is a compact metric space and $\varphi : \mathbb{R} \times X \to X$ is a flow and φ is weak uniformly almost periodic with positively asymptotic average shadowing property, then φ is strongly ergodic. As well as corollary we show that if $\varphi : \mathbb{R} \times X \to X$ is a Lyapunov stable flow with the asymptotic average shadowing property then φ is not distal.

AMS Classification: 54H20, 37D30
Keywords: Asymptotic average shadowing property δ-average-pseudo-orbit; Lyapunov stable; strongly ergodic.

1. Introduction

The shadowing property is an important notion in dynamical systems (see [2]). In [1] Blank introduced the notion of the average shadowing property (Abbrev.ASP) and Gu in [3] introduced a new shadowing property,the asymptotic average shadowing property (Abbrev.AASP),which is weaker than the asymptotic shadowing property ,and studied the relation between the AASP and transitivity. In [4] authors studied the relation between AASP and attractors, and C^1 interior AASP in a manifold with dimension 2. Gu and Guo in [6] showed that there is a close

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 10 , NUMBER 2 (2012) 115-123.

relationship among the average-shadowing property, Lyapunov stability and topological ergodicity on flows. In a recent work Gu in [5] introduced the notion of the asymptotic average-shadowing property and showed that a Lyapunov stable flow with the asymptotic average-shadowing property is a minimal flow. In this paper we introduce strong ergidicity, stronger than ergodicity, and weak uniformly almost periodic flow, weakerthan uniformly almost periodic flow, and show that a weak uniformly almost periodic flow with asymptotic average-shadowing property is strong ergodic.

Suppose (X, d) is a compact metric space with metric d. Let $\varphi : \mathbb{R} \times X \to X$ be a flow, that is, φ is a continuous map and satisfies the following conditions:

(1) $\varphi(0, x) = x$ for all $x \in X$.

(2) $\varphi(s, \varphi(t, x)) = \varphi(t + s, x)$ for all $x \in X$ and all $s, t \in \mathbb{R}$.

For $x \in X$, the set $O(x, \varphi) = \{\varphi(t, x) : t \in \mathbb{R}\}$ is said to be the orbit of φ thorough the point x.

A flow φ is called to be topologically ergodic if

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \chi(\varphi(t, U) \cap V) dt > 0$$

for any two non-empty open subsets U and V of X, where $\chi(U) = 1$ when $U \neq 0$ and $\chi(U) = 0$ when $U = \phi$. We say that a flow φ is strongly ergodic if for any two non-empty open subsets U and V of X, there is $T > 0$ such that for any $l \in \mathbb{R}$, there is $t \in [l, l + T]$ such that $\varphi(t, U) \cap V \neq \phi$.

It is clear that

Strong ergodicity$\Rightarrow$topological ergodicity$\Rightarrow$topological transivity.

A flow φ is called to be uniformly almost periodic if for any $\epsilon > 0$ there exist $L(\epsilon) > 0$ and a countable real number set $\{\tau_i : 0 \leq i < \infty\}$ such that $\{\tau_i\} \cap (t, t + L(\epsilon)) \neq \phi$ for any $t \in \mathbb{R}$ and

$$d(\varphi(t, x), \varphi(t + \tau_i, x)) < \epsilon$$

for any $x \in X$, any $t \in \mathbb{R}$ and any $0 \leq i < \infty$. We say that a flow φ is weak uniformly almost periodic if there exist a dense subset A of X such that for any $\epsilon > 0$ there exist $L(\epsilon) > 0$ and a countable real number set $\{\tau_i : 0 \leq i < \infty\}$ such that $\{\tau_i\} \cap (t, t + L(\epsilon)) \neq \phi$ for any $t \in \mathbb{R}$ and

$$d(\varphi(t, x), \varphi(t + \tau_i, x)) < \epsilon$$

2. Some basic terminology for flows

Given $\delta > 0$ and $T > 0$, a bi-sequence $(\{x_i\} - \infty < i < \infty, \{t_i\} - \infty < i < \infty)$ is said to be (δ, T)-pseudo-orbit of φ if $t_i \geq T$ and $d(\varphi(t_i, x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$.
A bi-sequence $(\{x_i\} - \infty < i < \infty, \{t_i\} - \infty < i < \infty)$ is said to be ϵ-shadowed by the orbit of φ through x, if there is an orientation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(\varphi(\alpha(t), x), \varphi(t - s_i, x_i)) < \epsilon, \quad \text{for } s_i \leq t < s_{i+1}, i \geq 0$$

and

$$d(\varphi(\alpha(t), x), \varphi(t + s_{-i}, x_{-i})) < \epsilon, \quad \text{for } -s_{-i} \leq t < -s_{-i+1}, i \geq 1$$

where $s_0 = 0$, $s_n = \sum_{i=0}^{n-1} t_i, \sum_{i=-n}^{-1} t_i, n = 1, 2, \ldots$
A flow φ is said to have the shadowing property if for any $\epsilon > 0$ there is a $\delta > 0$ such that every $(\delta, 1)$-pseudo orbit of φ can be ϵ-shadowed by some orbit of φ.
A bi-sequence $(\{x_i\} - \infty < i < \infty, \{t_i\} - \infty < i < \infty)$ is said to be an asymptotic T-average-pseudo-orbit of φ if $t_i \geq T$ for all $i \in \mathbb{Z}$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{i=n} d(\varphi(t_i, x_i), x_{i+1}) = 0$$

For convenience, we say asymptotic average-pseudo-orbit as asymptotic T-average-pseudo-orbit with $T = 1$.
A bi-sequence $(\{x_i\} - \infty < i < \infty, \{t_i\} - \infty < i < \infty)$ is said to be positively (resp. negatively) asymptotically shadowed in average by the orbit of φ through x, if there is an orientation preserving homomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{i=n} \int_{s_i}^{s_{i+1}} d(\varphi(\alpha(t), x), \varphi(t - s_i, x_i))dt = 0$$

$$(\text{resp.} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{i=n} \int_{-s_{-i}}^{-s_{-i+1}} d(\varphi(\alpha(t), x), \varphi(t + s_{-i}, x_{-i}))dt = 0)$$

where $s_0 = 0, s_n = \sum_{i=0}^{n-1} t_i, \sum_{i=-n}^{-1} t_i, n = 1, 2, \ldots$
A flow φ is said to have the positively (resp. negatively) asymptotic average shadowing property if every asymptotic average-pseudo-orbit of can be positively (resp. negatively) asymptotically shadowed in average by some orbit of φ.
A flow φ is said to have the asymptotic average shadowing property if φ has both positively and negetively asymptotic average shadowing property.

Theorem 1. *Let X be a compact metric space and $\varphi : \mathbb{R} \times X \to X$ be a flow. If φ is weak uniformly almost periodic with positively asymptotic average shadowing property, then φ is strongly ergodic.*

Proof. Suppose that U and V are two nonempty open subsets of X. We can choose $x \in U \cap A$ and $y \in V \cap A$ and $\epsilon > 0$ such that $B(x, \epsilon) \subset U$ and $B(y, \epsilon) \subset V$, where $B(a, \epsilon) = \{b \in X : d(a, b) < \epsilon\}$. There exists $L(\frac{\epsilon}{2}) > 0$ and a countable real number set $\{\tau_i : 0 \leq i < \infty\}$ such that $\{\tau_i\} \cap (t, t + L(\epsilon)) \neq \phi$ for any $t \in \mathbb{R}$ and

$$d(x, \varphi(\tau_i, x)) < \frac{\epsilon}{2}$$

and

$$d(x, \varphi(\tau_i, y)) < \frac{\epsilon}{2}$$

for any $0 \leq i < \infty$.

By uniform continuity of φ on $[0, L(\frac{\epsilon}{2}] \times X$, take $\delta > 0$ such that $d(u, v) < \delta$ implies $d(\phi(t, u), \phi(t, v)) < \frac{\epsilon}{2}$ for all $t \in [0, L(\frac{\epsilon}{2})]$.

We use the method of Gu in [5] to construct a bi-sequence $(\{x_i\} - \infty < i < \infty, \{t_i\} - \infty < i < \infty)$ with $t_i = 1$ for all $i \in \mathbb{Z}$ as follows. Let

$$\begin{aligned} w_{-i} &= \varphi(-i, x), i = 1, 2, \ldots \\ w_0 &= x, \quad y = w_1, \\ w_2 &= x, \quad y = w_3, \\ w_4 &= x, \varphi(1, x), y, \quad \varphi(1, y) = w_7 \\ &\ldots \\ w_{2^k} &= x, \varphi(1, x), \ldots, \varphi(2^{k-1} - 1, x), y, \varphi(1, y), \ldots, \varphi(2^{k-1} - 2, y), \\ &\varphi(2^{k-1} - 1, y) = w_{2^{k+1} - 1}, \\ &\ldots \end{aligned}$$

It is to see that for $2^k \leq n < 2^{k+1}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{i=n} d(\varphi(t_i, w_i), w_{i+1}) < \frac{2(k+1) \times D}{n},$$

Where D is diameter of X. Hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-n}^{i=n} d(\varphi(t_i, w_i), w_{i+1}) = 0$$

Thus, the bi-sequence $(\{x_i\} - \infty < i < \infty, \{t_i\} - \infty < i < \infty)$ is an asymptotic average-pseudo-orbit of φ, hence it can be positively asymptotically shadowed in average by the orbit of φ through some point z in X, that is, there is an oriantation preserving homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{i=n} \int_i^{i+1} d(\varphi(\alpha(t), z), \varphi(t - i, w_i))dt = 0$$

We have the following claim,

(1) There are infinitymany positive integers j and $t_j \in [j, j + 1]$ such that

$$w_{i_j} \in \{x, \varphi(1, x), \ldots \varphi(2^i - 1, x)\}$$

and

$$d(\varphi(\alpha(t_j), z), \varphi(t_j - j, w_{i_j})) < \delta$$

(2) There are infinity many positive integers l and $t_l \in [l, l + 1]$ such that

$$w_{i_l} \in \{y, \varphi(1, y), \ldots \varphi(2^i - 1, y)\}$$

$$d(\varphi(\alpha(t_l), z), \varphi(t_l - l, w_{i_l})) < \delta$$

Proof of claim: Without lost of generality we prove only (1). Suppose on the contrary that there is a positive integer N such that for all integers $j > N$, whenever

$$w_j \in \{x, \varphi(1, x), \ldots \varphi(2^i - 1, x)\},$$

then

$$d(\varphi(\alpha(t), z), \varphi(t - j, w_j)) \geq \delta$$

for all $i \geq N$.
Thus, it would be obtained that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{i=n} \int_i^{i+1} d(\varphi(\alpha(t), z), \varphi(t - j, w_i))dt \geq \frac{\delta}{2}.$$

This is a contradiction.
So (1) and (2) are true. Hence we can choose positive integers $j_0, l_0, t_{j_0}, t_{l_0}$ such that

$$d(\varphi(\alpha(t_{j_0}), z), \varphi(t_{j_0} - j_0, w_{j_0})) < \delta,$$

and

$$d(\varphi(\alpha(t_{l_0}), z), \varphi(t_{l_0} - l_0, w_{l_0})) < \delta,$$

Where $w_{j_0} = \varphi(j_1, x)$ for some $j_1 \in \{0, 1, \ldots, 2^{i_{j_0}} - 1\}$ and $w_{l_0} = \varphi(l_1, y)$ for some $l_1 \in \{0, 1, \ldots, 2^{i_{j_0}} - 1\}$.
So

$$d(\varphi(\alpha(t_{j_0}), z), \varphi(t_{j_0} - j_0 + j_1, x)) < \delta,$$

$$d(\varphi(\alpha(t_{l_0}), z), \varphi(t_{l_0} - l_0 + l_1, x)) < \delta,$$

There are $j_2, l_2 \in [0, L(\frac{\epsilon}{2})]$ such that $t_{j_0} - j_0 + j_1 + j_2 = \tau_j$ and $t_{l_0} - l_0 + l_1 + l_2 = \tau_l$ where

$$d(x, \varphi(\tau_j, x)) < \frac{\epsilon}{2}$$

and

$$d(y, \varphi(\tau_l, y)) < \frac{\epsilon}{2}.$$

By uniform continuity of φ we have

$$d(\varphi(\alpha(t_{j_0} + j_2, z), \varphi(\tau_j, x)) < \frac{\epsilon}{2},$$

and

$$d(\varphi(\alpha(t_{j=l_0} + l_2, z), \varphi(\tau_l, y)) < \frac{\epsilon}{2}.$$

So $d(\varphi(\alpha(t_{j_0} + j_2, z), \varphi(\tau_j, x)) < \frac{\epsilon}{2}$, and $d(\varphi(\alpha(t_{j=l_0} + l_2, z), \varphi(\tau_l, y)) < \frac{\epsilon}{2}$.
If $t_0 = \alpha(t_{l_0}) - \alpha(t_{j_0}) + l_2 - j_2$ then $\varphi(t_0, U) \cap V \neq \phi$. We Consider $p \in A \cap W$ and choose $\epsilon_0 > 0$ such that $B(p, \epsilon_0) \subset W$. There exist a countable real number set $\{\tau_i : 0 \leq i < \infty\}$ such that $\{\tau_i\} \cap (t, t + L(\epsilon_0)) \neq \phi$ for any $t \in \mathbb{R}$ and $d(p, \varphi(\tau_i, p)) < \epsilon_0$ for any $0 \leq i < \infty$. So $\varphi(\tau_i, p) \in W$ for any $0 \leq i < \infty$. Hence $\varphi(\tau_i, W) \cap W \neq \phi$ and so $\varphi(t_0, U) \cap V \cap \varphi(\tau_i, (\varphi(t_0, U) \cap V)) \neq \phi$. In other word

$$\varphi(t_0, U) \cap V \cap \varphi(\tau_i, (\varphi(t_0, U) \cap V)) \subset \varphi(t_0 + \tau_i, U) \cap V.$$

Hence $\varphi(t_0 + \tau_i, U) \cap V \neq \phi$ for any $0 \leq i < \infty$. Now we consider $T = t_0 + L(\epsilon_0)$ so for any $l \in \mathbb{R}$, there is $t \in [l, l + T]$ such that $\varphi(t, U) \cap V \neq \phi$.
Since U, V are arbitary hence φ is strongly ergodic. □

Let $\varphi_1 : \mathbb{R} \times X \to X$ and $\varphi_2 : \mathbb{R} \times Y \to Y$ be two flows on compact metric space X, Y with metric d_1, d_2, respectively. We define

$$\begin{aligned} \varphi_1 \times \varphi_2 : \mathbb{R} \times X \times Y &\to X \times Y \\ (\varphi_1 \times \varphi_2)(t, (x, y)) &= (\varphi_1(t, x), \varphi_2(t, y)) \end{aligned}$$

For $x = (x_1, y_1), y = (x_2, y_2) \in X \times Y$,

$$\rho(x, y) = max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

We say that $\varphi_1 \times \varphi_2$ has asymptotic average shadowing property if for every asymptotic average pseudo orbit of $\varphi_1 \times \varphi_2$, $(\{(x_i, y_i)\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$, there are $x \in X$ and $y \in Y$ and orientation preserving homeomorphisms $\alpha, \alpha' : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = \alpha'(0) = 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \rho(\varphi_1(\alpha(t), x), \varphi_2(\alpha'(t), y), (\varphi_1(t - s, x_i), \varphi_2(t - s_i, y_i)))dt = 0$$

Theorem 2. *Let $\varphi_1 : \mathbb{R} \times X \to X$ and $\varphi_2 : \mathbb{R} \times Y \to Y$ be flows with asymptotic average shadowing property. Then $\varphi_1 \times \varphi_2 : \mathbb{R} \times X \times Y \to X \times Y$ has asymptotic average shadowing property.*

Proof. Let $(\{x_i, y_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ be an asymptotic average pseudo orbit of $\varphi_1 \times \varphi_2$.

So we have $(\{x_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ and $(\{y_i\}_{-\infty<i<\infty}, \{t_i\}_{-\infty<i<\infty})$ are asymptotic pseudo orbit for φ_1 and φ_2, respectively. Hence there are $x \in X$ nad $y \in Y$ and orientation preserving homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$ and $\alpha' : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = \alpha'(0) = 0$ such that

$$\lim_{n\to\infty} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} d_1(\varphi_1(\alpha(t), x), \varphi_1(t - s, x_i))dt = 0$$

and

$$\lim_{n\to\infty} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} d_2(\varphi_2(\alpha'(t), y), \varphi_2(t - s, y_i))dt = 0.$$

We claim that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \rho((\varphi_1(\alpha(t), x), \varphi_2(\alpha', y)), (\varphi_1(t - s, x_i), \varphi_2(t - s, y_i))dt = 0$$

suppose not. So there is a sequence $\{n_k\}$ such that $n_k \to +\infty$ and $\varepsilon > 0$ such that

$$\frac{1}{n_k} \sum_{i=0}^{n_k} \int_{s_i}^{s_{i+1}} \rho((\varphi_1(\alpha(t), x), \varphi_2(\alpha'(t), y)), (\varphi_1(t - s_i, x_i), \varphi_2(t - s_i, y_i)))dt > \varepsilon \quad (*)$$

put

$$c_k = d_1(\varphi_1(\alpha(t), x), \varphi_1(t - s_k, x_k))$$

and

$$d_k = d_2(\varphi_2(\alpha'(t), y), \varphi_2(t - s_k, y_k)),$$

$$m_k = max\{c_k, d_k\}, ,$$

$$D_{t,n} = Card(\{k < n : \int_{s_k}^{s_{k+1}} d_k \geq t\}),$$

$$C_{t,n} = Card(\{k < n : \int_{s_k}^{s_{k+1}} c_k \geq t\}),$$

$$M_{t,n} = Card(\{k < n : \int_{s_k}^{s_{k+1}} m_k \geq t\}).$$

We can see $M_{t,n} \leq C_{t,n} + D_{t,n}$. Consider $D = max\{\rho(x, y) : x, y \in X \times Y\}$. Choose $\sigma > 0$ such that $2(D + 1)\sqrt{\sigma} \leq \varepsilon$. Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} c_k dt = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} d_k dt = 0$$

by Lemma 3.4 (a) in [7], $\limsup_{n\to+\infty} \frac{C_{\sqrt{\sigma},n}}{n} \leq \sqrt{\sigma}$ and $\limsup_{n\to+\infty} \frac{D_{\sqrt{\sigma},n}}{n} \leq \sqrt{\sigma}$, So $\limsup_{n\to+\infty} \frac{M_{\sqrt{\sigma},n}}{n} \leq 2\sqrt{\sigma}$. By Lemma 3.4 (b) in [7].
we have $\limsup \frac{1}{n} \sum_{k=0}^{n-1} \int_{s_i}^{s_{i+1}} m_k dt \leq 2(D + 1)\sigma \leq \varepsilon$. This is a contradiction with $(*)$.

□

$x \in X$ is said to be stable point of flow φ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $d(\varphi(t, x), \varphi(t, y)) < \varepsilon$ for every $y \in X$ with $d(x, y) < \delta$ and all $t \geq 0$. φ is called Lyapunov stable if every point of X is a stable point of φ.
$x \in X$ is said to be distal if

$$\liminf_{t\to\infty} d(\varphi(t, x), \varphi(t, y)) > 0$$

for all $y \neq x$. We say φ is distal if every $x \in X$ is distal. Gu in [5] showed that a Lyapunov stable flow with the asymptotic average shadowing property is a minimal flow. We can deduce the following corollary.

Corollary. *Let $\varphi : \mathbb{R} \times X \to X$ be a Lyapunov stable flow with the asymptotic average shadowing property. Then φ is not distal.*

Proof. By the above theorem $\varphi \times \varphi : \mathbb{R} \times X \times X \to X \times X$ has asymptotic average shadowing property and since φ is a Lyapinov flow we can see $\varphi \times \varphi$ is also Lyapunov stable. Hence by argument in [2] $\varphi \times \varphi$ is a minimal flow. So for $x \neq y$, $\varphi \times \varphi(x, y)$ is dense in $X \times X$. Therefore

$$\liminf_{t \to +\infty} d(\varphi(t, x), \varphi(t, y)) = 0$$

then φ is not distal. $\square$

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