

# ARITHMETIC DIFFERENCE OF MIDDLE CANTOR SETS: SELF-SIMILARITY AND MEASURE

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**ABSTRACT.** Let  $\mathcal{C}$  be the space of all middle Cantor sets, it is shown that for a dense subset  $\mathcal{L}$  of  $\mathcal{C} \times \mathcal{C} \times \mathbb{R}$ , set  $C_\alpha - \lambda C_\beta$  forms a uniformly contracting self-similar set when  $(C_\alpha, C_\beta, \lambda) \in \mathcal{L}$ . By selecting an appropriate element  $(C_\alpha, C_\beta, \lambda) \in \mathcal{L}$ , we see that  $C_\alpha - \lambda C_\beta$  does not satisfy open set condition and then we calculate its Lebesgue measure that is zero.

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## 1. INTRODUCTION

Regular Cantor sets as an important family of iterated function systems appear in some mathematics branches like dynamical systems and number theory. Arithmetic difference of regular Cantor sets comes up in various studies of homoclinic phenomena, such as infinitely sinks, antimonotonicity and  $\Omega$ -explosions in dynamical systems (see [7], [4] and [9]) also the classical Markov and Lagrange spectra related to diophantine approximations in number theory (see [1] and [2]). In this direction, about 1987, J. Palis made the following

**Conjecture.** The arithmetic difference of two affine Cantor sets contains an interval or has zero Lebesgue measure.

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Many papers have been written on this conjecture (see [5], [6], [10], [11] and [12]). This conjecture remains open, also for middle Cantor sets and the aim of this paper is a step forward to this problem. Moreover, a new result has been obtained in Section 3 is the existence of a pair of affine Cantor sets with the sum of above Hausdorff dimension which their arithmetic difference has zero Lebesgue measure while it is not an affine Cantor set. Here we remind the meaning of a middle Cantor set.

A Cantor set  $K$  is regular or dynamically defined if:

- i) There are disjoint compact intervals  $K_1, K_2, \dots, K_r$  such that  $K \subseteq K_1 \cup \dots \cup K_r$  and the boundary of each  $K_i$  is contained in  $K$ ,
- ii) There is a  $C^{1+\epsilon}$  expanding map  $\psi$  defined in a neighborhood of set  $K_1 \cup K_2 \cup \dots \cup K_r$  such that  $\psi(K_i)$  is the convex hull of a finite union of some intervals  $K_j$  satisfying:
  - ii.1 For each  $i, 1 \leq i \leq r$  and  $n$  big,  $\psi^n(K \cap K_i) = K$ ,
  - ii.2  $K = \bigcap_{n=0}^{\infty} \psi^{-n}(K_1 \cup \dots \cup K_r)$ .

The set  $\{K_1, K_2, \dots, K_r\}$  is, by definition, a Markov partition for  $K$ , and the set  $D := \bigcup_{i=1}^r K_i$  is the corresponding Markov domain of  $K$ .

A regular Cantor set  $K$  is affine if  $D\psi$  be constant on every interval  $K_i$ . Cantor set  $C_\alpha$  is a middle- $(1 - 2\alpha)$  or, in simple words, middle Cantor set, if the convex hull of  $C_\alpha$  is  $[0, 1]$  and the Markov partition of  $C_\alpha$  has exactly two members with  $D\psi = \frac{1}{\alpha}$  on their intervals. We call all middle Cantor sets to  $\mathcal{C}$ .

## 2. SET $\mathcal{L}$ AND IT'S RENORMALIZATION OPERATORS

At first, we define the set  $\mathcal{L}$ . An element  $(C_\alpha, C_\beta, \lambda)$  belongs to  $\mathcal{L}$  if, and only if  $\frac{\log \alpha}{\log \beta} \in \mathbb{Q}$ . It is obvious that  $\mathcal{L}$  is a dense subset in space  $\mathcal{C} \times \mathcal{C} \times \mathbb{R}$ . Let  $p, q > 2$ , it has been shown in [3] that transferred renormalization operators corresponding to pair  $(C_{\frac{1}{p}}, C_{\frac{1}{q}})$  are

$$\begin{aligned} \mathbb{R}^* \times \mathbb{R} &\rightarrow \mathbb{R}^* \times \mathbb{R} \\ (s, t) &\xrightarrow{T_0} (ps, pt) \quad * \quad (s, t) \xrightarrow{T_1} (ps, pt - p + 1) \\ (s, t) &\xrightarrow{T'_0} \left(\frac{s}{q}, t\right) \quad (s, t) \xrightarrow{T'_1} \left(\frac{s}{q}, t + \frac{q-1}{q}s\right) \end{aligned}$$

If  $\frac{\log p}{\log q} =: \frac{n_0}{m_0} \in \mathbb{Q}$ , then every vertical lines  $s = \lambda$  pass over itself with suitable compositions of the operators  $(*)$ . Therefore we can transfer the operators on these lines by

**Theorem 1.** Let  $\lambda \in \mathbb{R}^*$  and  $\{a_k\}_{k=0}^{m_0-1}$ ,  $\{b_k\}_{k=0}^{n_0-1}$  be two finite sequences of numbers 0 and 1, then the maps

$$T_\lambda(t) := p^{m_0}t + a_\lambda, \quad **$$

$$a_\lambda := -(p-1)p^{m_0-1} \left( \sum_{k=0}^{m_0-1} \frac{a_k}{p^k} - \frac{p(q-1)}{q(p-1)} \lambda \sum_{k=0}^{n_0-1} \frac{b_k}{q^k} \right)$$

are return maps to the vertical line  $\lambda$  that its attractor is  $C_\alpha - \lambda C_\beta$ .

Above theorem that has been proven in [8] give us an iterated function system which its attractor forms  $C_{\frac{1}{p}} - \lambda C_{\frac{1}{q}}$ . Now we consider  $\alpha := \frac{1}{p} := \frac{1}{\gamma^3}$  and  $\beta := \frac{1}{q} := \frac{1}{\gamma^2}$ , where  $\gamma$  is the golden number  $\frac{\sqrt{5}+1}{2}$ . Also, let  $C_\alpha$  and  $C_\beta$  be two middle Cantor sets with expanding maps  $\phi_\alpha$  and  $\phi_\beta$ , respectively, as follows:

$$C_\alpha : \quad \underline{\frac{1}{p}} \quad \underline{\frac{1}{p}} \quad C_\beta : \quad \underline{\frac{1}{q}} \quad \underline{\frac{1}{q}}$$

$$\phi_\alpha(x) := \begin{cases} px & x \in [0, \frac{1}{p}] \\ px - p + 1 & x \in [1 - \frac{1}{p}, 1] \end{cases}$$

$$\phi_\beta(x) := \begin{cases} qx & x \in [0, \frac{1}{q}] \\ qx - q + 1 & x \in [1 - \frac{1}{q}, 1] \end{cases}$$

On our selection of numbers  $\alpha$  and  $\beta$ , we see that  $m_0 = 2$  and  $n_0 = 3$ .

### 3. LEBESGUE MEASURE

In this section we observe the complication of calculating  $C_\alpha - \lambda C_\beta$  with the simplest possible choice of the middle Cantor sets  $C_\alpha$  and  $C_\beta$ . Note that  $\frac{\log \alpha}{\log \beta} = \frac{3}{2} \in \mathbb{Q}$ , together with the special number  $\lambda = \frac{2}{\gamma}$ .

In this direction, we obtain below result

**Proposition 1.** Lebesgue measure of set  $C_\alpha - \frac{2}{\gamma} C_\beta$  is zero, while  $HD(C_\alpha) + HD(C_\beta) > 1.2$ .

**Proof.** At first, we calculate relations

$$i) \quad HD(C_\alpha) + HD(C_\beta) = \frac{5}{6} \log_\gamma 2 = 1.200350\dots,$$

$$ii) \quad \tau(C_\alpha) \cdot \tau(C_\beta) = \frac{1}{3-\gamma} = 0.723606\dots$$

Thus, pair  $(C_\alpha, C_\beta) \in \Omega$ . Also, we obtain

$$\begin{cases} \gamma^2 = \gamma + 1, & \gamma^3 = 2\gamma + 1, & \gamma^4 = 3\gamma + 2, & \gamma^5 = 5\gamma + 3, & \gamma^6 = 8\gamma + 5 \dots \\ \frac{1}{\gamma} = \gamma - 1, & \frac{1}{\gamma^2} = 2 - \gamma, & \frac{1}{\gamma^3} = 2\gamma - 3, & \frac{1}{\gamma^4} = 5 - 3\gamma \dots \end{cases}$$

and  $\cot \theta := \frac{q(p-1)}{p(q-1)} = \frac{2}{\gamma} = \sqrt{5} - 1$ , that  $\theta$  is projection angle of  $\mathbb{R}^2$  over  $\mathbb{R} \times \{0\}$ .

For simplicity, we consider  $\Pi_\theta := \text{Proj}_\theta$ .

Regarding Theorem 1, we obtain exactly 21 operators  $T = \{T_i \mid T_i(t) = \gamma^6 t + a_i\}_{i=1}^{21}$  on  $\mathbb{R}$  that they are return maps to the vertical line  $\lambda$ . Like the proposition 2 in [8], we can find  $a_i$  easier, which determines by its corresponding point as follows:

$$\begin{aligned} \left(0, \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q} + \frac{1}{q^2}\right)\right) &\implies a_1 = -(6\gamma + 4)(4\gamma - 8) = 8\gamma + 8, \\ \left(0, \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)\right) &\implies a_2 = -(6\gamma + 4)(\gamma - 3) = 8\gamma + 6, \\ \left(\left(1 - \frac{1}{p}\right)\frac{1}{p}, \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q} + \frac{1}{q^2}\right)\right) &\implies a_3 = -(6\gamma + 4)(6\gamma - 11) = 6\gamma + 8, \\ \left(\left(1 - \frac{1}{p}\right)\frac{1}{p}, \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q}\right)\right) &\implies a_4 = -(6\gamma + 4)(3\gamma - 6) = 6\gamma + 6, \\ \left(0, \left(1 - \frac{1}{q}\right)\right) &\implies a_5 = -(6\gamma + 4)(0 - 1) = 6\gamma + 4, \\ \left(\left(1 - \frac{1}{p}\right)\frac{1}{p}, \left(1 - \frac{1}{q}\right)\left(1 + \frac{1}{q^2}\right)\right) &\implies a_6 = -(6\gamma + 4)(5\gamma - 9) = 4\gamma + 6, \\ \left(\left(1 - \frac{1}{p}\right)\frac{1}{p}, \left(1 - \frac{1}{q}\right)\right) &\implies a_7 = -(6\gamma + 4)(2\gamma - 4) = 4\gamma + 4. \end{aligned}$$

For every  $7 < i \leq 14$ , we use relation

$$a_i = a_{i-7} - p(p-1) = a_{i-7} - (6\gamma + 4)$$

and we have:

$$\begin{aligned} a_8 &= 2\gamma + 4, & a_9 &= 2\gamma + 2, & a_{10} &= 4, & a_{11} &= 2, \\ a_{12} &= 0, & a_{13} &= -2\gamma + 2, & a_{14} &= -2\gamma. \end{aligned}$$

Also, for every  $14 < i \leq 21$ , we use relation

$$a_i = a_{i-7} - p(p-1) = a_{i-7} - (6\gamma + 4)$$

and we have:

$$\begin{aligned} a_{15} &= -4\gamma, & a_{16} &= -4\gamma - 2, & a_{17} &= -6\gamma, & a_{18} &= -6\gamma - 2, \\ a_{19} &= -6\gamma - 4, & a_{20} &= -8\gamma - 2, & a_{21} &= -8\gamma - 4. \end{aligned}$$

We are going to show that Lebesgue measure of uniformly contracting self-similar set

$$C_\alpha - \frac{2}{\gamma} C_\beta = \bigcap_{i \in \mathbb{N}} T^{-i} \left( \left[ \frac{-2}{\gamma}, 1 \right] \right)$$

is zero.

Let

$$\begin{aligned} G_1 &:= [T_2^{-1}(\frac{-2}{\gamma}), T_1^{-1}(1)] = [\frac{-10\gamma-4}{p^2}, \frac{-8\gamma-7}{p^2}], \\ G_2 &:= [T_4^{-1}(\frac{-2}{\gamma}), T_3^{-1}(1)] = [\frac{-8\gamma-4}{p^2}, \frac{-6\gamma-7}{p^2}], \\ G_3 &:= [T_5^{-1}(\frac{-2}{\gamma}), T_4^{-1}(1)] = [\frac{-8\gamma-2}{p^2}, \frac{-6\gamma-5}{p^2}], \end{aligned}$$

$$\begin{aligned}
G_4 &:= [T_7^{-1}(\frac{-2}{\gamma}), T_6^{-1}(1)] = [\frac{-6\gamma-2}{p^2}, \frac{-4\gamma-5}{p^2}], \\
R_1 &:= [T_3^{-1}(\frac{-2}{\gamma}), T_2^{-1}(1)] = [\frac{-8\gamma-6}{p^2}, \frac{-8\gamma-5}{p^2}], \\
R_2 &:= [T_6^{-1}(\frac{-2}{\gamma}), T_5^{-1}(1)] = [\frac{-6\gamma-4}{p^2}, \frac{-6\gamma-3}{p^2}], \\
G_i &:= G_{i-4} + \frac{p-1}{p} \quad 4 < i \leq 8, \quad G_i := G_{i-4} + \frac{p-1}{p} \quad 8 < i \leq 12, \\
R_i &:= R_{i-2} + \frac{p-1}{p} \quad 2 < i \leq 4, \quad R_i := R_{i-2} + \frac{p-1}{p} \quad 4 < i \leq 6, \\
H_1 &:= (T_7^{-1}(1), T_8^{-1}(\frac{-2}{\gamma})) = (\frac{-4\gamma-3}{p^2}, \frac{-4\gamma-2}{p^2}), \\
H_2 &:= (T_{14}^{-1}(1), T_{15}^{-1}(\frac{-2}{\gamma})) = (\frac{2\gamma+1}{p^2}, \frac{2\gamma+2}{p^2}), \\
Z_1 &:= ch(G_2 \cup G_3) - G_2 \cup G_3, \\
Z_2 &:= ch(G_6 \cup G_7) - G_6 \cup G_7, \\
Z_3 &:= ch(G_{10} \cup G_{11}) - G_{10} \cup G_{11},
\end{aligned}$$

where  $ch(A)$  is the convex hull of set  $A \subseteq \mathbb{R}$ . Let  $\{Y_i\}_{i=1}^{12}$  be the longest interval set between two subsequent  $G_i, R_j$ , respectively, and  $\{X_i\}_{i=1}^6$  are connected components of complement of above sets on interval  $[\frac{-2}{\gamma}, 1]$ , respectively, see Figure 1.

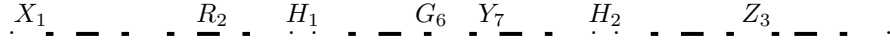


Figure 1:

Now we obtain the length of gaps created by intervals  $H_1$  and  $H_2$  on the  $n$ -step of the construction of  $C_\alpha - \frac{2}{\gamma}C_\beta$  by using contractions  $T_i$ . It is easy to see that

$$\begin{aligned}
X_1 &= T_1^{-1}((ch\{R_6, 1\})^c), \\
X_2 &= T_7^{-1}((ch\{\frac{-2}{\gamma}, R_1\})^c), \\
Y_1 &= T_2^{-1}((ch\{\frac{-2}{\gamma}, R_1\} \cup ch\{R_4, 1\})^c), \\
Y_2 &= T_3^{-1}((ch\{\frac{-2}{\gamma}, R_3\} \cup ch\{R_6, 1\})^c), \\
Y_3 &= T_5^{-1}((ch\{\frac{-2}{\gamma}, R_1\} \cup ch\{R_4, 1\})^c), \\
Y_4 &= T_6^{-1}((ch\{\frac{-2}{\gamma}, R_3\} \cup ch\{R_6, 1\})^c), \\
Z_1 &= T_4^{-1}((ch\{\frac{-2}{\gamma}, R_1\} \cup ch\{R_6, 1\})^c),
\end{aligned}$$

where  $c$  is the complement on interval  $[\frac{-2}{\gamma}, 1]$ . Other sets are similar.

Regardless of indexes, we have:

$$|H| = \frac{|X|}{2} = |Y| = |R| = \frac{1}{\gamma^6}, \quad |G| = \frac{2\gamma-3}{\gamma^6}, \quad |Z| = |X| - |G|.$$

Let  $x_1 := 2$ ,  $y_1 := 1$ ,  $z_1 := 2$ ,  $g_1 := 0$ ,  $r_1 := 0$  be the number of gaps created in intervals  $X_i$ ,  $Y_i$ ,  $Z_i$ ,  $G_i$ ,  $R_i$ , respectively, then their length is

$$\begin{aligned}\ell_1 &:= \frac{1}{(\gamma^6)^2} m_1, \\ m_1 &:= 6x_1 + 12y_1 + 3z_1 + 12g_1 + 6r_1 \\ &= \langle (x_1, y_1, z_1, g_1, r_1), (6, 12, 3, 12, 6) \rangle = 30.\end{aligned}$$

Suppose  $x_{n-1}, y_{n-1}, z_{n-1}, g_{n-1}, r_{n-1}$  are recursively constructed and let  $x_n, y_n, z_n, g_n, r_n$  be the number of gaps created in intervals  $X_i, Y_i, Z_i, G_i, R_i$  on the  $n$ -step with length  $\frac{|H_i|}{p^{2n}}$ , respectively, then for  $n \in \mathbb{N}$ , we obtain

$$\begin{bmatrix} x_n \\ y_n \\ z_n \\ g_n \\ r_n \end{bmatrix} = \begin{bmatrix} 5 & 11 & 3 & 11 & 5 \\ 2 & 6 & 2 & 6 & 2 \\ 4 & 10 & 3 & 10 & 4 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 8 & 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \\ g_{n-1} \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} 5 & 11 & 3 & 11 & 5 \\ 2 & 6 & 2 & 6 & 2 \\ 4 & 10 & 3 & 10 & 4 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 8 & 2 & 6 & 5 \end{bmatrix}^{n-1} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} =: A^{n-1} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

It is important to know that in each  $G_i$  or  $R_i$ , the inverse of  $G_s$  under the operators  $T$  are the same or there is not any intersection between them and the inverse of  $R_s$  are symmetrically, see Figure 2.

$$G : \quad \text{---} \quad \text{---} \quad \text{---} \quad \quad R : \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---}$$

Figure 2:

Therefore, the length of gaps created on  $n$ -step is

$$\begin{aligned}\ell_n &:= \frac{1}{(\gamma^6)^{n+1}} m_n, \\ m_n &:= \langle (x_n, y_n, z_n, g_n, r_n), (6, 12, 3, 12, 6) \rangle \\ &= \langle A^{n-1} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, (6, 12, 3, 12, 6) \rangle.\end{aligned}$$

By adding the lengths of  $\ell_n$ , we have:

$$\ell := 2|H| + \sum_{n=1}^{\infty} \ell_n = \frac{2}{\gamma^6} + \frac{1}{\gamma^{12}} < \sum_{n=1}^{\infty} \frac{A^{n-1}}{(\gamma^6)^{n-1}} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, (6, 12, 3, 12, 6) \rangle.$$

On the other hand, roots of characteristic polynomial

$$x^5 - 20x^4 + 50x^3 - 28x^2 - 3x = x(x-1)(x^3 - 19x^2 + 31x + 3)$$

corresponding to the matrix  $A$  are

$$0, \quad 1, \quad a := -0.09433..., \quad b := 1.84345..., \quad c := 17.25087....$$

Suppose that

$$Q := \begin{bmatrix} 1 & 1 & \frac{13}{2}a - 2a^2 + \frac{3}{2}a^3 - 2 & \frac{13}{2}b - 2b^2 + \frac{3}{2}b^3 - 2 & \frac{13}{2}c - 2c^2 + \frac{3}{2}c^3 - 2 \\ 1 & 2 & -2 + 15a - 6a^2 + a^3 & -2 + 15b - 6b^2 + b^3 & -2 + 15c - 6c^2 + c^3 \\ 2 & 2 & -\frac{5}{2} + \frac{29}{2}a - \frac{11}{2}a^2 + \frac{3}{2}a^3 & -\frac{5}{2} + \frac{29}{2}b - \frac{11}{2}b^2 + \frac{3}{2}b^3 & -\frac{5}{2} + \frac{29}{2}c - \frac{11}{2}c^2 + \frac{3}{2}c^3 \\ -2 & -2 & 4 - 16a + 4a^2 & 4 - 16b + 4b^2 & 4 - 16c + 4c^2 \\ 0 & -2 & -5a - 4a^2 + a^3 & -5b - 4b^2 + b^3 & -5c - 4c^2 + c^3 \end{bmatrix}$$

be the matrix of eigenvectors of  $A$ . Then

$$\begin{aligned}
 A &= Q \cdot \text{diag}[0, 1, a, b, c] \cdot Q^{-1} \implies A^n = Q \cdot \text{diag}[0, 1, a^n, b^n, c^n] \cdot Q^{-1} \\
 &\implies \Sigma_{n=1}^{\infty} \frac{A^{n-1}}{\gamma^{6(n-1)}} = Q \cdot \text{diag}\left[1, \frac{1}{1-\frac{1}{\gamma^6}}, \frac{1}{1-\frac{a}{\gamma^6}}, \frac{1}{1-\frac{b}{\gamma^6}}, \frac{1}{1-\frac{c}{\gamma^6}}\right] \cdot Q^{-1} \\
 &\implies \left\langle \Sigma_{n=1}^{\infty} \frac{A^{n-1}}{\gamma^{6(n-1)}} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, (6, 12, 3, 12, 6) \right\rangle \\
 &= \frac{3}{4} \frac{\gamma^6}{(\gamma^6-a)(\gamma^6-b)(\gamma^6-c)} \left( 40\gamma^{12} - (7(ab+bc+ac) - 9(a+b+c) \right. \\
 &\quad \left. + 7abc + 71)\gamma^6 + (ab+bc+ac) + (a+b+c) + 17abc - 7 \right) \\
 &= \gamma^6 \cdot \frac{30\gamma^{12}-72\gamma^6-6}{\gamma^{18}-19\gamma^{12}+31\gamma^6+3} = \gamma^6 \cdot \frac{3744\gamma+2304}{96\gamma+64} = \gamma^6(18\gamma+9),
 \end{aligned}$$

the above relations are obtained by using Maple program and the following relations:

- i)  $a + b + c = 19$ ,  $ab + bc + ac = 31$ ,  $abc = -3$ ,
- ii)  $(\gamma^6 - a)(\gamma^6 - b)(\gamma^6 - c) = (\gamma^6)^3 - 19(\gamma^6)^2 + 31\gamma^6 + 3$ ,
- iii)  $\gamma^{12} = 144\gamma + 89$ ,  $\gamma^{18} = 2584\gamma + 1597$ .

Therefore, we have:

$$\ell = \frac{2}{\gamma^6} + \frac{18\gamma+9}{\gamma^6} = 1 + \frac{2}{\gamma} = \left| \left[ \frac{-2}{\gamma}, 1 \right] \right|.$$

Here the proof is completed.  $\square$

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