

A NOTE ON ASYMPTOTIC DISTRIBUTION OF PRODUCTS OF WEIGHTED SUMS

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ABSTRACT. In this paper we establish the asymptotic distribution of products of weighted sums of independent, identically distributed, square integrable, positive random variable, which extends the results in Rempała and Wesolowski (2002).

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1. INTRODUCTION

Let $(X_n)_{n \geq 1}$ be a sequence of independent identically distributed (i.i.d.) positive random variables. Recently there are several studies to the products of partial sums. It is well known that the products of i.i.d. positive, square integrable random variables are asymptotically log-normal. This fact is an immediate consequence of the classical central limit theorem (CLT). This point, up to the knowledge of the author, was first argued by Arnold and Villaseñr [1], who considered the limiting properties of the sums of records. In their paper Arnold and Villaseñr obtained the following version of the CLT for a sequence of i.i.d. exponential r.v.'s $(X_n)_{n \geq 1}$ with

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the mean equal to one:

$$\frac{\sum_{k=1}^n \log S_k - n \log n + n}{\sqrt{2n}} \xrightarrow{d} \Phi, \text{ as } n \rightarrow \infty,$$

where $S_k = \sum_{j=1}^k X_j$, $1 \leq k \leq n$, and Φ is a standard normal r.v. Rempała and Wośowski [13] have noted that this limit behavior of a product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following.

Theorem RW Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu$, $\text{Var}X_1 = \sigma^2 > 0$ and the coefficient of variation $\gamma = \sigma/\mu$. Then

$$\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/(\gamma \sqrt{n})} \xrightarrow{d} e^{\sqrt{2}\Phi}.$$

Recently, Gonchigdanzan and Rempała [3] discussed an almost sure limit theorem for the product of the partial sums of i.i.d. positive random variables as follows.

Theorem GR Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu > 0$, $\text{Var}X_1 = \sigma^2$. Denote $\gamma = \sigma/\mu$ the coefficient of variation. Then for any real x ,

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} I \left(\left(\frac{\prod_{k=1}^n S_k}{n! \mu^n} \right)^{1/(\gamma \sqrt{n})} \leq x \right) = F(x), \text{ a.s.}$$

where $F(\cdot)$ is the distribution function of the r.v. $e^{\sqrt{2}\Phi}$.

For further discussions of the CLT, the author refers to [6, 12]. Huang and Zhang [5] obtained the invariance principle of the product of sums of random variables. It is perhaps worth to notice that by the strong law of large numbers and the property of the geometric mean it follows directly that

$$(1) \quad \left(\frac{\prod_{k=1}^n S_k}{n!} \right)^{1/n} \xrightarrow{\text{a.s.}} \mu$$

if only existence of the first moment is assumed. Recently, Miao [7, 8] obtained CLT and ASCLT for the product of some general partial sums. Miao and Li [9] studied the asymptotic distribution of products of weighted sums of dependent random variables. Miao and Qian [11] established an exponential convergence theorem for the product of sums of i.i.d. positive random variables. Miao and Mu [10] proved

the moderate deviation principle for the product of sums of i.i.d. positive random variables.

In the present paper we are interested in the asymptotic distribution of products of general weighted sums, which extends the above Theorem RW.

2. MAIN RESULTS

Theorem 2.1. *Let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. positive random variables. Denote $\mu = \mathbb{E}(X_1) > 0$, the coefficient of variation $\gamma = \sigma/\mu$, where $\sigma^2 = \text{Var}(X_1)$, and $S_{kk} = a_{1k}X_1 + \cdots + a_{kk}X_k$, $k = 1, 2, \dots$ where $\{a_{ik}, 1 \leq i \leq k, k \geq 1\}$ is a triangular array of positive real numbers with $\sum_{i=1}^k a_{ik} = 1$ for all $k \geq 1$. In addition let $b_{in} = \sum_{k=i}^n a_{ik}$, $i = 1, \dots, n$ and assume that the following conditions hold as $n \rightarrow \infty$,*

(1)

$$\frac{\sum_{i=1}^n b_{in}^2}{b_{1n}^2} \rightarrow \infty, \quad \frac{1}{\sqrt{\sum_{i=1}^n b_{in}^2}} \sum_{k=1}^n \sum_{i=1}^k a_{ik}^2 \rightarrow 0;$$

(2)

$$\frac{S_{nn}}{\mu} - 1 \rightarrow 0 \quad a.e.$$

Then for any A_n satisfying $\lim_{n \rightarrow \infty} \frac{A_n}{\sum_{i=1}^n b_{in}^2} = 1$, we have

$$\left(\prod_{k=1}^n (S_{kk}/\mu) \right)^{\frac{1}{\gamma \sqrt{A_n}}} \xrightarrow{d} e^{\Phi}$$

where Φ is a standard normal random variable.

Remark 2.2. *From the condition (1) it is easy to see that $\sum_{i=1}^n b_{in}^2 \rightarrow \infty$, as $n \rightarrow \infty$.*

Lemma 2.3. *Under the assumptions of Theorem 2.1,*

$$(2) \quad \frac{1}{\gamma \sqrt{A_n}} \sum_{k=1}^n \left(\frac{S_{kk}}{\mu} - 1 \right) \xrightarrow{d} \Phi.$$

Proof. Let $Y_i = (X_i - \mu)/\sigma$, $i = 1, 2, \dots$, then (2) becomes

$$\frac{1}{\sqrt{A_n}} \sum_{k=1}^n \sum_{i=1}^k a_{ik} Y_i \xrightarrow{d} \Phi.$$

Observe that

$$\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n \left(\frac{S_{kk}}{\mu} - 1 \right) = \frac{1}{\sqrt{A_n}} \sum_{k=1}^n \sum_{i=1}^k a_{ik} Y_i = \frac{1}{\sqrt{A_n}} \sum_{i=1}^n \left(\sum_{k=i}^n a_{ik} \right) Y_i,$$

and define now

$$b_{in} = \sum_{k=i}^n a_{ik} \quad \text{and} \quad Z_{in} = \frac{1}{\sqrt{A_n}} b_{in} Y_i,$$

then

$$\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n \left(\frac{S_{kk}}{\mu} - 1 \right) = \sum_{i=1}^n Z_{in}.$$

It is easy to check that

$$\mathbb{E}Z_{in} = 0, \quad \text{Var}(Z_{in}) = \frac{b_{in}^2}{A_n}, \quad i = 1, \dots, n$$

and

$$\text{Var} \left(\sum_{i=1}^n Z_{in} \right) = \frac{1}{A_n} \sum_{i=1}^n b_{in}^2 \rightarrow 1.$$

In order to complete the proof we need to check that the Lindeberg condition (see p.530 in [4]) is satisfied for the triangular array $\{Z_{in}\}$. For any $r > 0$, by the condition (1) we have

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}(Z_{in}^2 I(|Z_{in}| > r)) &= \sum_{i=1}^n \frac{b_{in}^2}{A_n} \mathbb{E} \left(Y_i^2 I \left(|Y_i| > \frac{r\sqrt{A_n}}{b_{in}} \right) \right) \\ &\leq \sum_{i=1}^n \frac{b_{in}^2}{A_n} \mathbb{E} \left(Y_1^2 I \left(|Y_1| > \frac{r\sqrt{A_n}}{b_{1n}} \right) \right) \rightarrow 0, \end{aligned}$$

which implies the Lindeberg condition holds. $\square$

Proof of Theorem 2.1. Here we will use the delta-method expansion to prove our results as Rempała and Wesolowski [13]. Denote $C_k = S_{kk}/\mu$. By the condition (2), for any $\delta > 0$, there exists a number R such that for any $r > R$

$$\mathbb{P} \left(\sup_{k \geq r} |C_k - 1| > \delta \right) \leq \delta.$$

Consequently, there exist two sequences $\{\delta_m\} \downarrow 0$ ($\delta_1 = 1/2$) and $(R_m) \uparrow \infty$ such that

$$\mathbb{P} \left(\sup_{k \geq R_m} |C_k - 1| > \delta_m \right) \leq \delta_m.$$

Taking now any real x and any m , we have

$$\begin{aligned} \mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n \log C_k \leq x\right) &= \mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n \log C_k \leq x, \sup_{k \geq R_m} |C_k - 1| \geq \delta_m\right) \\ &\quad + \mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n \log C_k \leq x, \sup_{k \geq R_m} |C_k - 1| < \delta_m\right) \\ &:= A_{mn} + B_{mn} \end{aligned}$$

where $A_{mn} \leq \delta_m$. Next we will control the term B_{mn} . By the following logarithm:

$$\log(1+x) = x - \frac{x^2}{2(1+\theta x)^2},$$

where $\theta \in (0, 1)$ depends on $x \in (-1, 1)$, we have

$$\begin{aligned} B_{mn} &= \mathbb{P}\left\{\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n \log(1+(C_k-1)) \leq x, \right. \\ &\quad \left. \sup_{k \geq R_m} |C_k - 1| < \delta_m\right\} \\ &= \mathbb{P}\left\{\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n (C_k - 1) \right. \\ &\quad \left. - \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n \frac{(C_k - 1)^2}{2(1+\theta_k(C_k - 1))^2} \leq x, \sup_{k \geq R_m} |C_k - 1| < \delta_m\right\} \\ &= \mathbb{P}\left\{\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n (C_k - 1) \right. \\ &\quad \left. - \left[\frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n \frac{(C_k - 1)^2}{2(1+\theta_k(C_k - 1))^2}\right] I\left(\sup_{k \geq R_m} |C_k - 1| < \delta_m\right) \leq x\right\} \\ &\quad - \mathbb{P}\left\{\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} \log C_k + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n (C_k - 1) \leq x, \sup_{k \geq R_m} |C_k - 1| \geq \delta_m\right\} \\ &:= D_{mn} + F_{mn}, \end{aligned}$$

where $\theta_k, k = 1, \dots, n$ are $(0, 1)$ -valued random variables and $F_{mn} \leq \delta_m$. To estimate the term D_{mn} , we rewrite it as

$$\begin{aligned} D_{mn} &= \mathbb{P}\left\{\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} [\log C_k - (C_k - 1)] + \frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n (C_k - 1) \right. \\ &\quad \left. - \left[\frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n \frac{(C_k - 1)^2}{2(1+\theta_k(C_k - 1))^2}\right] I\left(\sup_{k \geq R_m} |C_k - 1| < \delta_m\right) \leq x\right\}. \end{aligned}$$

For any fixed m ,

$$\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^{R_m} [\log C_k - (C_k - 1)] \xrightarrow{\mathbb{P}} 0$$

as $n \rightarrow \infty$. By the following elementary inequality: for $|x| < 1/2$ and any $\theta \in (0, 1)$ it follows that $x^2/(1 + \theta x)^2 \leq 4x^2$. Then for any m

$$\left[\frac{1}{\gamma\sqrt{A_n}} \sum_{k=R_m+1}^n \frac{(C_k - 1)^2}{(1 + \theta_k(C_k - 1))^2} \right] I\left(\sup_{k \geq R_m} |C_k - 1|\right) \leq \frac{4}{\gamma\sqrt{A_n}} \sum_{k=1}^n (C_k - 1)^2 \xrightarrow{\mathbb{P}} 0,$$

as $n \rightarrow \infty$. The above convergence is a consequence of the Markov's inequality, since for any $r > 0$,

$$\mathbb{P}\left(\frac{4}{\gamma\sqrt{A_n}} \sum_{k=1}^n (C_k - 1)^2 > r\right) \leq \frac{4}{r\gamma\sqrt{A_n}} \sum_{k=1}^n \mathbb{E}(C_k - 1)^2 = \frac{4}{r\gamma\sqrt{A_n}} \sum_{k=1}^n \sum_{i=1}^k a_{ik}^2 \rightarrow 0,$$

as $n \rightarrow \infty$ for any fixed m . Since

$$\mathbb{P}\left(\log\left(\prod_{k=1}^n (S_{kk}/\mu)\right)^{\frac{1}{\gamma\sqrt{A_n}}} \leq x\right) = \mathbb{P}\left(\frac{1}{\gamma\sqrt{A_n}} \sum_{k=1}^n \log C_k \leq x\right) = A_{mn} + D_{mn} + F_{mn}$$

and from Lemma 2.3, we have $A_{mn} + F_{mn} \leq 2\delta_m \rightarrow 0$ as $m \rightarrow \infty$ and $\lim_{n \rightarrow \infty} D_{mn} \rightarrow \Phi(x)$ as $m \rightarrow \infty$, which imply the desired results. $\square$

Remark 2.4. From the proof of Theorem 2.1, we know that condition (2) can be replaced by the following: for any $\delta > 0$, there exists a constants R such that

$$(3) \quad \mathbb{P}\left(\max_{n \geq R} |S_{nn}/\mu - 1| > \delta\right) \leq \delta.$$

On the other hand, if we take $a_{ik} = 1/k$ for all $1 \leq i \leq k$, $k \geq 1$, then the conditions (1) and (2) hold, which induces Theorem RW.

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