

METRIZABLE AND 2-METRIZABLE TOPOLOGICAL SPACES

M.R. FARHANGDOOST

DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCES, SHIRAZ UNIVERSITY
P.O. BOX 71457- 44776, SHIRAZ, IRAN
E-MAIL: FARHANG@SHIRAZU.AC.IR

(Received: 14 March 2011, Accepted: 17 July 2011)

ABSTRACT. In this paper we introduce $(\epsilon - 2)$ -ball centered at each point in 2-metric topological space (X, d) . Theorems on the normal, regular and Hausdorff topological spaces in 2-metrizable topological space are presented. We show that every metrizable topological spaces are coarser than 2-metrizable topological space, and then we conclude that each manifold is a 2-metric topological space.

AMS Classification: 54E35, 11J83, 22E15

Keywords: Metric topological space; 2-Metric space; Manifold.

1. INTRODUCTION

The notion of 2-metric space was initially introduced by *Gähler*(1963/64) in papers [3] and [4]. In recent years, development of this branch, is due to *Diminie* and *White*(1969), *Ehret*(1969), *Iseki*(1975/76), *Siddiqi, Gupta* and *Gähler* (1974).

Some authors have studied the aspects of fixed point theory in 2-metric space and some important fixed point results are obtained by *Khan*(1978), *Iseki*(1975), *Rohoades*(1979), *Miczko* and *Palezwski*(1985), *Saha et. al.* (1993/2009) and *Sehgal*(1969).

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 10, NUMBER 1 (2012) 61-69.
©TARU PUBLICATIONS

In this paper we construct a 2-metric topological space and we find the relation between a 2-metric topological space and a metric topological space. We show that 2-metric topological spaces are normal topological spaces, then they are regular and Hausdorff topological spaces.

2. 2-METRIC SPACES

In this section we introduce 2-metric spaces. We begin with definition of 2-metric space.

Let X be a non-empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if:

- (i) given distinct elements x, y of X , there is an elements z of X such that $d(x, y, z) \neq 0$;
- (ii) $d(x, y, z) = 0$, when at least two of x, y, z are equal;
- (iii) $d(x, y, z) = d(z, y, x) = d(z, x, y)$, for every $x, y, z \in X$;
- (iv) $d(x, y, z) \leq d(w, y, z) + d(x, w, z) + d(x, y, w)$, for every $x, y, z, w \in X$.

The ordered pair (X, d) is called a 2-metric space.

The convergence in a 2-metric space (X, d) introduced by *Hadžić*(1984) by the following way:

A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a 2-metric space (X, d) is said to be convergent with the limited point x in X , if for every $a \in X$, $\lim_{n \rightarrow \infty} d(x_n, x, a) = 0$.

A sequence is called a Cauchy sequence if:

for all $a \in X$, $\lim_{m, n \rightarrow \infty} d(x_m, x_n, a) = 0$.

If every Cauchy sequence in X converges to a point in X , then (X, d) is said to be a complete 2-metric space.

A 2-metric which is continuous in all its arguments is called continuous [5].

Clearly, every 2-metric is non-negative.

Example 2.1. Take $X = \mathbb{R}^3$ and

$$d(x, y, z) = |x_1(y_2z_3 - z_2y_3) - x_2(y_1z_3 - z_1y_3) + x_3(y_1z_2 - z_1y_2)|,$$

where $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ and $Z = (z_1, z_2, z_3)$.

Then $(\mathbb{R}^3, d)$ is 2-metric space.

Example 2.2. Let $d : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by:

$$d(x, y, z) = |(x - y)(y - z)(z - x)|.$$

Then (R, d) is a 2-metric space, because:

(i) If $x \neq y$ are disjoint elements in R and if $z = (x + y)/2$, then $d(x, y, z) = \frac{1}{4}|x - y|^3 \neq 0$.

Conditions (ii) and (iii) are clear. For condition (iv), if $x, y, z, w \in R$ then:

$$\begin{aligned} & d(x, y, w) + d(x, w, z) + d(w, y, z) \\ &= |(x - y)(y - w)(w - x)| + |(x - w)(w - z)(z - x)| + |(w - y)(y - z)(z - w)| \\ &\geq |(x - y)(y - w)(w - x) + (x - w)(w - z)(z - x) + (w - y)(y - z)(z - w)| \\ &= |xy^2 - x^2y - z^2x + x^2z - y^2z + z^2y| \\ &= |(x - y)(y - z)(z - x)| = d(x, y, z). \end{aligned}$$

Example 2.3. Let X be an arbitrary set, $d : X \times X \times X \longrightarrow R$ be defined by:

$d(x, y, z) = 0$, when at least two of x, y, z are equal;

and otherwise $d(x, y, z) = 1$.

Then "d" is a 2-metric which is called discrete 2-metric.

Example 2.4. Let $d : R^k \times R^k \times R^k \longrightarrow R$ defined by:

$$d(x, y, z) = \text{Max}\{|x_i - y_i||x_i - z_i||y_i - z_i|\}_{1 \leq i \leq k}$$

where $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k)$ and $z = (z_1, \dots, z_k)$. It is easy to show that (R^k, d) is a 2-metric space.

3. 2-METRIC TOPOLOGICAL SPACES

In this section we introduce 2-metric topological spaces. We start by definition of $(\epsilon - 2)$ -ball centered at x .

Definition 3.1. Let (X, d) be a 2-metric space. $B_d(x, \epsilon)$ which is defined by $B_d(x, \epsilon) = \{y \in X | \text{Sup}_{a \in X} d(x, y, a) < \epsilon\}$, is called $(\epsilon - 2)$ -ball centered at x .

We show that the collection of all $(\epsilon - 2)$ -balls centered at x , for all $x \in X$, is a basis for a topology on X , which is called the 2-metric topology induced by d :

It is clear that $a \in B_d(a, \epsilon)$, for all $\epsilon > 0$ and $a \in X$.

For the second condition of the basis for a topological space definition, let $c \in B_d(a, \epsilon_1) \cap B_d(b, \epsilon_2)$, for $a, b, c \in X$. Letting positive number ϵ be the smaller of positive number $\frac{1}{2} \min\{\epsilon_1 - \text{Sup}_{x \in X} d(a, c, x), \epsilon_2 - \text{Sup}_{y \in X} d(b, c, y)\}$, we show that $B_d(c, \epsilon)$ is subset $B_d(a, \epsilon_1) \cap B_d(b, \epsilon_2)$:

If $e \in B_d(c, \epsilon)$, then $\text{Sup}_{x \in X} d(c, e, x) < \epsilon$. (*)

Hence

$$d(c, e, x_0) < \epsilon \text{ and } d(c, e, a) < \epsilon, \text{ for all } x_0 \in X.$$

$$\begin{aligned} d(a, e, x_0) &\leq d(c, e, x_0) + d(a, c, x_0) + d(a, e, c) \\ &< \epsilon + d(a, c, x_0) + \epsilon, \text{ for all } x_0 \in X, \end{aligned} \quad (\text{by } (*))$$

so

$$\text{Sup}_{x \in X} d(a, e, x) \leq 2\epsilon + \text{Sup}_{x \in X} d(a, c, x).$$

Since

$$0 < \epsilon < \frac{1}{2} \min\{\epsilon_1 - \text{Sup}_{x \in X} d(a, c, x), \epsilon_2 - \text{Sup}_{y \in X} d(b, c, y)\},$$

then

$$2\epsilon + \text{Sup}_{x \in X} d(a, c, x_0) < (\epsilon_1 - \text{Sup}_{x \in X} d(a, c, x)) + \text{Sup}_{x \in X} d(a, c, x) = \epsilon_1.$$

Hence

$$\text{Sup}_{x \in X} d(a, e, x) < \epsilon_1.$$

So

$$e \in B_d(a, \epsilon_1).$$

By the same process we can show that $e \in B_d(b, \epsilon_2)$ and the proof is complete.

Example 3.2. If d is a discrete metric on X , then (X, d) is a discrete 2-metric topological space.

Example 3.3. In the Example 2.3, each $\{x\}$ is an open set, for all $x \in R$.

Note: It is easy to show that in 2-metric topological space (X, d) a sequence $\{a_n\}_{n \in N}$ is convergent to "a" if and only if for every $(\epsilon - 2)$ -ball centered at "a", there is a natural number M such that $a_m \in B_d(a, \epsilon)$, for every $m \geq M$.

Also, it is easy to show that: "Let X be a topological space and $A \subseteq X$. If there is a sequence of points of A converging to b , then $b \in \overline{A}$; the converse holds if X is 2-metric topological space."

Lemma 3.4. Let (X, d) be a 2-metric topological space and $a_n \in B_d(b, \epsilon_n)$, where $n \in N$, $a_n, b \in X$, $\epsilon_n > 0$ and $\epsilon_n \rightarrow 0$. Then the sequence $\{a_n\}_{n \in N}$ is convergent to b .

Proof: Since $a_n \in B_d(b, \epsilon_n)$, for all $n \in N$, then $\text{Sup}_{x \in X} d(a_n, b, x) < \epsilon_n$,

so

$$d(a_n, b, x) < \epsilon_n, \text{ for all } x \in X \text{ and } n \in N.$$

Therefore

$$\lim_{n \rightarrow \infty} d(a_n, b, x) = 0,$$

finally

$$\{a_n\}_{n \in N}$$

convergent with limit $b \in X$.

Corollary 3.5. Let (X, d) be a 2-metric topological space and $a \in X$. If $\{a_n\}_{n \in N}$ is a sequence in X such that every open set U in X includes " a " which intersects $\{a_n\}_{n \in N}$, then $\{a_n\}_{n \in N}$ converges to " a ".

Proof: The proof obtains by Lemma 3.4.

Theorem 3.6. Let d be a metric on X and $|X| \geq 3$. Then d_1 is a 2-metric, where $d_1 : X \times X \times X \rightarrow R$ is defined by:

$$d_1(x, y, z) = \min\{d(x, y), d(x, z), d(y, z)\},$$

for every $x, y, z \in X$.

Also, the metric topological space (X, d) is coarser than the 2-metric topological space (X, d_1) .

Proof: We show that d_1 is a 2-metric:

(i) Since $|X| \geq 3$, for every $x \neq y$, there is $z \in X$ such that $x \neq z$ and $y \neq z$. then

$$d(x, y) > 0, d(x, z) > 0 \text{ and } d(y, z) > 0.$$

So

$$\min\{d(x, y), d(x, z), d(y, z)\} > 0,$$

then

$$d_1(x, y, z) \neq 0.$$

It is clear that conditions (ii) and (iii) are true.

Now, we show that for every $x, y, z, w \in X$:

$$d_1(x, y, z) \leq d_1(x, y, w) + d_1(x, w, z) + d_1(w, y, z), \quad (**)$$

without loss of generality we can assume that:

$$d_1(x, y, z) = d(x, y).$$

Case(1): Let $d_1(x, y, w) = d(x, y)$, then it is clear that:

$$d_1(x, y, z) = d_1(x, y, w) \leq d_1(x, y, w) + d_1(x, w, z) + d_1(w, y, z).$$

Case(2): Let $d_1(x, y, w) = d(x, w)$ and $d_1(w, y, z) = d(w, y)$, then:

$$\begin{aligned} d_1(x, y, w) &= d(x, y) \leq d(x, w) + d(w, y) = \\ d_1(x, y, w) + d_1(w, y, z) &= d_1(x, y, w) + d_1(x, w, z) + d_1(w, y, z). \end{aligned}$$

Case(3): Let $d_1(x, y, w) = d_1(y, w, z) = d(w, z)$, since $d_1(x, y, z) = d(x, y)$, then $d(x, y) \leq d(x, z)$, so $d(x, y) \leq d(x, w) + d(w, z)$, it is clear that $(^{**})$ holds.

For the other cases with the same process we can complete the proof.

Also, it is clear that $B_{d_1}(x, \epsilon)$ is a subset of $B_d(x, \epsilon)$, so the metric topological space (X, d) is coarser than 2-metric topological space (X, d_1) .

Definition 3.7. A topological space (X, τ) is called 2-metric deformation, if there exists 2-metrizable space (X, d) such that τ is coarser than τ_d , where d is a 2-metric.

Example 3.8. Every metrizable space such as Manifolds, Lie groups and top spaces are 2-metric deformation. [11]

Lemma 3.9. Let $\bar{d} : X \times X \times X \rightarrow R$ be a 2-metric on X and $(X, \bar{d})$ be a 2-metric topological space, then $d : X \times X \times X \rightarrow R$, which is defined by $d(x, y, z) = \min\{\bar{d}(x, y, z), 1\}$ is a bounded 2-metric that induce the same topology as d .

Proof: It is easy to show that the conditions (i), (ii), (iii) and (iv) of the 2-metric definition true. So d is a 2-metric.

It is clear that, for every 2-metric space, the collection of $(\epsilon - 2)$ -balls with $\epsilon < 1$ forms a base for 2-metric topological space X . So 2-metric d and $\bar{d}$ induce the same topology on X , because the collections of $(\epsilon - 2)$ -balls with $\epsilon < 1$ under these two metrics are the same collections.

Example 3.10. By the previous Lemma and Example 2.2

$$\rho(x, y, z) = \min\{|x - y||y - z||x - z|, 1\}, \quad x, y, z \in R,$$

is a bounded 2-metric.

Corollary 3.11. *Let $\bar{\rho}$ be defined as:*

$$\bar{\rho}(x, y, z) = \text{Sup}_{i \in N} \left\{ \frac{\rho(x_i, y_i, z_i)}{i} \right\}.$$

Then $\bar{\rho}$ is 2-metric, where ρ is introduced in the Example 3.10 and $x = (x_i)_{i \in N}$, $y = (y_i)_{i \in N}$ and $z = (z_i)_{i \in N}$ belong to R^w .

Proof: *It is easy to show that the conditions of the definition of 2-metric are true.*

Corollary 3.12. *Each 2-metric topological space is a normal topological space.*

Proof: *Let A, B be two closed subsets of a 2-metric topological space (X, d) , and $U = \bigcup_{a \in A} B_d(a, \frac{\epsilon_a}{4})$ and $V = \bigcup_{b \in B} B_d(b, \frac{\epsilon_b}{4})$, where $\epsilon_a, \epsilon_b > 0$ and $B_d(a, \epsilon_a) \cap B = B_d(b, \epsilon_b) \cap A = \text{empty}$.*

Claiming that U, V are disjoint open subsets containing A, B , respectively:

Let $z \in U \cap V$. Then there are $a, b \in X$ that $z \in B_d(a, \frac{\epsilon_a}{4})$ and $z \in B_d(b, \frac{\epsilon_b}{4})$,

so

$$d(a, b, x) \leq d(z, b, x) + d(a, z, x) + d(a, b, z) \quad , \text{ for all } x \in X.$$

Since

$$d(z, b, x) + d(a, z, x) + d(a, b, z) < \frac{\epsilon_b}{4} + \frac{\epsilon_a}{4} + \frac{\epsilon_a}{4},$$

without loss of generality we assume that $\epsilon_a < \epsilon_b$,

then

$$d(a, b, x) < \frac{3}{4}\epsilon_b \quad , \text{ for all } x \in X$$

therefore

$$\text{Sup}_{x \in X} d(a, b, x) < \epsilon_b,$$

so

$$a \in B_d(b, \epsilon_b),$$

this situation is not possible, so the proof is complete.

A topological space X is regular if for each pair consisting of a point x and a closed set C disjoint from x , there exist disjoint open sets containing x and C , respectively. Note that a topological space X is regular if and only if given a point $x \in X$ and an open set U of x , there is an open set V of x such that $\bar{V} \subseteq U$.

By this corollary we have the following corollary, but we prove it straightforwardly.

Corollary 3.13. *Every 2-metric topological space is a regular topological space.*

Proof: Let (X, d) be a 2-metric topological space, a point $x \in X$ and an open set U of x be given, there is a $(\epsilon - 2)$ -ball of x , where $\epsilon > 0$, we show that the closure $\overline{B_d(x, \frac{\epsilon}{2})}$ is a subset of $B_d(x, \epsilon)$:

Let $a \in \overline{B_d(x, \frac{\epsilon}{2})}$. Then there is a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that converges to a , where $x_n \in B_d(x, \frac{\epsilon}{2})$ for every natural number n ,

so

$$\lim_{n \rightarrow \infty} d(a, x_n, z) = 0, \text{ for all } z \in X \quad (***)$$

since

$$d(x, a, z) \leq d(x_n, a, z) + d(x, x_n, z) + d(x, a, x_n), \text{ for all } n \in \mathbb{N},$$

then by $(***)$ have:

$$d(x, a, z) \leq \epsilon/2, \text{ for all } z \in X$$

therefore " a " belongs to $B_d(x, \epsilon)$, and the proof is complete.

Another corollary is:

Corollary 3.14. *Every 2-metric topological space is a Hausdorff topological space.*

Conclusion: We have the following conditions:

1. For every metric topological space (X, d) , there is a 2-metric topological space (X, d_1) , such that it is coarser than (X, d) .
2. For every 2-metric topological space (X, d) , there is a bounded topological space (X, d_1) which is topological equivalent to (X, d) .
3. Every 2-metric topological space is normal, regular and Hausdorff topological space.

Now, let us to conclude this paper by sketching two problems:

1. What is the relation between 2-metric topological spaces and metric topological spaces?
2. Whether all 2-metric topological spaces are 2-metrizable topological spaces or not?

REFERENCES

- [1] R.L. Bishop, R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York and London, 1964.
- [2] R. Ehret, *Linear 2-normed space*, Dissertation, St. Louis university, 1975.
- [3] S. Gähler, *2-metric Raume and ihre topologische strucktur*, Math. Nachr., 26(1963), 115-148.
- [4] S. Gähler, *Über die unifromisierbarkeit 2-metrischer Raume*, Math. Nachr., 21(1976), 197-200.
- [5] S. Gähler, A.H. Siddiqi, S. C. Gupta, *On certain problems in 2-normed linear space*, Proc. Fifth. National Mathematics Conference, Pahlavi university, Iran, March (1974), 28-31.
- [6] K. Iseki, *A property of orbitally continuous mappings on 2-metric spaces*, Math. Sem. Notes, Kobe university, XVIII, 1975.
- [7] K. Iseki, P.L. Sharma, *Contraction type mapping on 2-metric space*, Math. Japonica, 21(1976), 28-31.
- [8] J. L. Kelly, *Generall Topology*, Spriger-Verlag, New York, 1991.
- [9] M. S. Khan, *On the convergence of sequences of fixed points in 2-metric spaces*, Indian J. Pure Appl. Math., 10(9)(1979), 1062-1067.
- [10] A. Miczko, B. Palezewski, *Common fixed points of contractive type mapping in 2-metric spaces*, Math. Nachr., 124(1985), 341-355.
- [11] M.R. Molaei, G.S. Khadekar, M.R. Farhangdoost, *On Top Spaces.*, Bakan Journal of Geometry and its Application, 12(2)(2005), 95-107.
- [12] S.V.R. Naidu, *Some Fixed Point Theorems in Metric and 2-Metric Space.*, Int. J. Math. Math. Sci., 28(11)(2001), 625636.
- [13] B.E. Rhoades, *Contraction type mappings of 2-metric spaces*, Math. Nachr., 91(1978), 151-165.
- [14] M. Saha, A.P. Baisnab, *Fixed point of mappings with contractive iterate*, Proc. Nat. Acad. Sci. India, 63A(4)(1993), 645-650.
- [15] M. Saha, D. Dey, *On the theory of fixed points of contractive type mappings in 2-metric space*, Int. Journal of Math. Analysis, 3(6)(2009), 283-293.
- [16] V.M. Sehgal, *A fixed point theorem for mappings with contractive iterate*, Proc. Math. Soc., 23(1969), 631-634.