

LIAPUNOV STABILITY VERSUS JACOBI STABILITY

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ABSTRACT. Liapunov stability and Jacobi stability are based on two different concepts and methods. The stability of classical example of a torque-free rigid body motion around a stationary point has been examined from Liapunov stability and Jacobi stability (or KCC theory) viewpoints. Unlike some worked example investigated by other authors, where there are discrepancies between these two methods, our case here shows complete agreement between two methods.

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1. INTRODUCTION

Jacobi stability analysis of dynamical systems has been recently studied by several authors [3, 6, 8, 17, 18, 19] using the Kosambi-Cartan-Chern (KCC) theory. The Jacobi stability is a natural generalization of the stability of the geodesic flow on a differentiable manifold equipped with a Riemannian or Finslerian metrics to the non-metric manifold. The Jacobi (also called KCC) stability studies the robustness of a second-order differential equation. Robustness is a measure of insensitivity and adaptation to change of the system internal parameters and the environment.

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From the mathematical point of view, the KCC theory studies the deviation of nearby trajectories, which allows one to estimate the perturbation permitted around the steady states solutions of the second-order differential equation. This is of interest in physical applications when one needs to identify the "robust arrest" regions, i.e., the regions where one has both Liapunov and Jacobi stability.

The KCC theory initiated from the works of D. D. Kosambi [13], E. Cartan [9] and S. S. Chern [10], and hence the abbreviation KCC (Kosambi-Cartan-Chern). The KCC theory has applications in engineering, physics, chemistry and biology. Recent developments have been made in KCC applications in gravitation and cosmology. Boehmer *et al.* [19] have analyzed Jacobi stability and its relations with the linear Liapunov stability analysis of dynamical systems, and presented a comparative study of these methods in the fields of gravitation and astrophysics. Nevertheless, there are apparent discrepancies observed in their studies in the sense that in some cases investigated, Liapunov stability is not in agreement with Jacobi stability.

The purpose of the present paper is to present a specific example from classical mechanic, i.e., the motion of free rigid body where Liapunov stability is entirely in agreement with Jacobi stability. We review the KCC theory to analyze its relations with the linear Liapunov stability analysis of dynamical systems, and to present a comparative study of these methods for a free rigid body motion. We also give a full dynamical systems treatment of free rigid body motion which is rarely presented in the literature.

The present paper is organized as follows. In Section 2 we review briefly the Liapunov stability analysis. Section 3 is devoted to the KCC theory and the Jacobi stability analysis of the dynamical systems. The motion of a free rigid body is reviewed in Section 4. A full dynamical systems approach of free rigid body motion and one-to-one comparison between Liapunov and Jacobi stability is presented in section 5. A discussion and final remarks appears in section 6.

2. LIAPUNOV STABILITY

2.1. Brief Survey of Techniques in Dynamical Systems. In this section we give precise mathematical expression to the concepts of stability of dynamical systems [12, 15, 23].

Definition: A *singular point* of a system of autonomous ODEs

$$(1) \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

is a point $\bar{\mathbf{x}} \in \mathbb{R}^n$ such that $\mathbf{f}(\bar{\mathbf{x}}) = 0$ where $\mathbf{f}$ is a C^1 vector field in $\mathbb{R}^n$. In the present paper, the singular points are also called "equilibrium point" or "fixed point" interchangeably.

Definition: Let $\bar{\mathbf{x}}$ be a singular point of the DE (1). The point $\bar{\mathbf{x}}$ is called a *hyperbolic* singular point if $\operatorname{Re}(\lambda_i) \neq 0$ for all eigenvalues, λ_i , of the Jacobian of the vector field $\mathbf{f}(\mathbf{x})$ evaluated at $\bar{\mathbf{x}}$. Otherwise the point is called *non-hyperbolic*.

Definition: (Liapunov Stability) $\bar{\mathbf{x}}(t)$ is said to be stable (or Liapunov stable) if, given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any other solution, $\mathbf{y}(t)$, of (1) satisfying $|\bar{\mathbf{x}}(t_0) - \mathbf{y}(t_0)| < \delta$, then $|\bar{\mathbf{x}}(t) - \mathbf{y}(t)| < \epsilon$ for $t > t_0$, $t_0 \in \mathbb{R}$. A solution which is not stable is said to be *unstable*.

Definition: (Asymptotic Stability) $\bar{\mathbf{x}}(t)$ is said to be *asymptotically stable* if it is Liapunov stable and for any other solution, $\mathbf{y}(t)$, of (1), there exists a constant $b > 0$ such that, if $|\bar{\mathbf{x}}(t_0) - \mathbf{y}(t_0)| < b$, then $\lim_{t \rightarrow \infty} |\bar{\mathbf{x}}(t) - \mathbf{y}(t)| = 0$.

Theorem: (Liapunov Stability Theorem) Consider the vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. Let $\bar{\mathbf{x}}$ be an equilibrium point of the above vector field and let $V : U \rightarrow \mathbb{R}$ be a C^1 function, called the Liapunov function, defined on some neighborhood U of $\bar{\mathbf{x}}$ such that

- i) $V(\bar{\mathbf{x}}) = 0$ and $V(\mathbf{x}) > 0$ if $\mathbf{x} \neq \bar{\mathbf{x}}$.
- ii) $\dot{V}(\mathbf{x}) \leq 0$ in $U - \bar{\mathbf{x}}$.

Then $\bar{\mathbf{x}}$ is stable. Moreover, if

$$\text{iii) } \dot{V}(\mathbf{x}) < 0 \text{ in } U - \bar{\mathbf{x}}$$

Then $\bar{\mathbf{x}}$ is asymptotically stable.

One must begin a qualitative analysis of a dynamical system by locating its singular points. Once the singular points are obtained, one should consider the dynamics in a local neighbourhood of each of the points. Assuming that the vector field $\mathbf{f}(\mathbf{x})$ is of class C^1 , then the local behaviour of the orbits can be determined by linear approximation of the vector field in the local neighbourhood of the singular point $\bar{\mathbf{x}}$. In this neighborhood,

$$(2) \quad \mathbf{f}(\mathbf{x}) \approx D\mathbf{f}(\bar{\mathbf{x}})(\mathbf{x} - \bar{\mathbf{x}})$$

where $D\mathbf{f}(\bar{\mathbf{x}})$ is the Jacobian of the vector field at the singular point $\bar{\mathbf{x}}$. The system (2) is referred to as the *linearization of the DE at the singular point*. The classification of the singular points can then be done by studying the eigenvalues of the Jacobian of the linearized vector field at the point.

The classification then follows from the fact that if the singular point is hyperbolic in nature the orbits of the non-linear system and its linear approximation are *topologically equivalent* in a neighborhood of the singular point. The following important theorem supports this fact:

Theorem: (Hartman-Grobman Theorem) Consider a DE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $x \in R^n$ where the vector field $\mathbf{f}$ is of class C^1 . If $\bar{\mathbf{x}}$ is a hyperbolic singular point of the DE then there exists a neighbourhood of $\bar{\mathbf{x}}$ on which the flow is topologically equivalent to the flow of the linearization of the DE at $\bar{\mathbf{x}}$. Given a linear system of ODEs:

$$(3) \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where A is a $n \times n$ square matrix with constant coefficients, it is a straightforward to show when eigenvalues of A have negative real parts, then the singular point is a hyperbolic sink (stable), since all solutions converge to the singular point $\bar{\mathbf{x}} = 0$; a linear system whose eigenvalues have positive real parts is called to have a hyperbolic source (unstable), because the solutions in the neighborhood of $\bar{\mathbf{x}} = 0$ all diverge

from that point; and when one eigenvalue is negative and the other is positive the singular point is said to be a hyperbolic saddle (unstable), where some orbits are attracted to the singular point, and some are repelled away. If the eigenvalues are complex conjugates with non-zero real parts then the fixed point is of type spiral (stable or unstable depends on the sign of $Re(\lambda_i)$) and if $Re(\lambda_i) = 0$, then the singular point is of type center.

In the non-linear case, the topological equivalence of flows allows for a similar classification of the singular points. The equivalence only applies in directions where the eigenvalue has non-zero real parts. If the linearization of the dynamical systems at the singular point has purely imaginary eigenvalues, then the stability type of equilibrium point cannot be deduced from the linear approximation. In such case one must use other methods such as Liapunov stability analysis to analyze the case. This brief introduction concludes our treatment of stability of dynamical systems which is used in this paper.

3. KOSAMBI-CARTAN-CHERN (KCC) THEORY AND JACOBI STABILITY

3.1. Brief Review of General Theory. We now introduce the main ideas of KCC-theory [2, 3, 6, 8]. Let $\mathcal{M}$ be a real, smooth n -dimensional manifold and let $T\mathcal{M}$ be its tangent bundle. Let $\mathbf{u} = (\mathbf{x}, \mathbf{y})$ be a point in $T\mathcal{M}$, where $\mathbf{x} = (x^1, x^2, \dots, x^n)$, and $\mathbf{y} = (y^1, y^2, \dots, y^n)$ which means $y^i = \frac{dx^i}{dt}$, $i = 1, 2, \dots, n$. Consider the second order differential equations in normalized form

$$(4) \quad \frac{d^2 x^i}{dt^2} + 2G^i(\mathbf{x}, \mathbf{y}) = 0, \quad i = 1, 2, \dots, n,$$

where $G^i(\mathbf{x}, \mathbf{y})$ are smooth functions defined in a local system of coordinates on $T\mathcal{M}$. In fact the system of (4) is motivated by Euler-Lagrange equations of classical dynamics

$$(5) \quad \begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} &= F_i \\ y^i &= \frac{dx^i}{dt}, \end{aligned} \quad i = 1, 2, \dots, n,$$

where L is the Lagrangian of $\mathcal{M}$, and F_i are the external forces [5, 11].

In order to find the KCC differential invariants of the system (4) under the non-singular coordinate transformation

$$(6) \quad \begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^1, x^2, \dots, x^n) \\ \tilde{y}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} y^j, \end{aligned} \quad i = 1, 2, \dots, n.$$

We now define the *KCC-covariant differential* of a vector field $\xi = \xi^i(x) \frac{\partial}{\partial x^i}$ in an open subset $\Omega \subseteq R^n \times R^n$ as [2, 3, 6, 19]

$$(7) \quad \frac{D\xi^i}{dt} = \frac{d\xi^i}{dt} + N_j^i \xi^j.$$

where $N_j^i = \frac{\partial G^i}{\partial y^j}$ defines the coefficients of a non-linear connection N on the tangent bundle $T\mathcal{M}$. For $\xi^i = y^i$, one obtains

$$(8) \quad \frac{Dy^i}{dt} = N_j^i y^j - 2G^i = -\epsilon^i.$$

The contravariant vector field ϵ^i on Ω is called the *first KCC invariant* and plays the role of an external force. The terms ϵ^i has geometrical character since with respect to a coordinate transformation (6), we have

$$(9) \quad \tilde{\epsilon}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \epsilon^j.$$

Furthermore, if one varies the paths $x^i(t)$ of the system (4) into nearby ones prescribed by

$$(10) \quad \tilde{x}^i(t) = x^i(t) + \eta \xi^i(t),$$

where $|\eta|$ is a small parameter and $\xi^i(t)$ are the components of some contravariant vector field defined along the path $x^i(t)$. We derive the following variational equations after substituting Eqs.(10) into Eqs.(4) and taking the limit as $\eta \rightarrow 0$ [1, 2, 17]

$$(11) \quad \frac{d^2 \xi^i}{dt^2} + 2N_j^i \frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j} \xi^j = 0.$$

We can write Eqs.(11) in terms of the KCC covariant differential (7) in the covariant form

$$(12) \quad \frac{D^2 \xi^i}{dt^2} = P_j^i \xi^j,$$

where the right hand side (1,1)-tensor

$$(13) \quad P_j^i = -2\frac{\partial G^i}{\partial x^j} - 2G^l G_{jl}^i + y^l \frac{\partial N_j^i}{\partial x^l} + N_l^i N_j^l,$$

and $G_{jl}^i \equiv \partial N_j^i / \partial y^l$ is called the Berwald connection [1, 17]. Eq. (12) is called the *Jacobi equations*, or *the variation equations* associated with the system of second order differential equations (4), and P_j^i is called *the second KCC-invariant*.

The third, fourth and fifth invariants of the system (4) are given by [4]

$$(14) \quad P_{jk}^i \equiv \frac{1}{3} \left(\frac{\partial P_j^i}{\partial y^k} - \frac{\partial P_k^i}{\partial y^j} \right), \quad P_{jkl}^i \equiv \frac{\partial P_{jk}^i}{\partial y^l}, \quad D_{jkl}^i \equiv \frac{\partial G_{jk}^i}{\partial y^l}.$$

The third invariant represents a torsion tensor, while the fourth and fifth invariants are interpreted as the Riemann-Christoffel curvature tensor, and the Douglas tensor, respectively [2, 17]. These tensors which describe the geometrical properties of a system of second-order differential equations always exist [2, 17].

The term Jacobi stability within the KCC theory is justified by the fact that, when (4) represents the second order differential equations for the geodesic equations in Finsler or Riemannian geometry [7, 16], then (12) is the Jacobi field equations for the geodesic deviation. The Jacobi equation (12) of the Finsler manifold $(M; F)$ can be written in the scalar form [7]:

$$\frac{d^2 v}{ds^2} + K \cdot v = 0,$$

where $\xi^i = v(s)\eta^i$ is a Jacobi field along the geodesic $x^i(s)$, η^i is the unit normal vector field, and K is the flag curvature of $(M; F)$. The sign of K affects the geodesic rays: if $K > 0$, then the geodesics bunch together (are Jacobi stable), and if $K < 0$, then they disperse (are Jacobi unstable). Therefore, positive/negative flag curvature is equivalent to negative/positive eigenvalues of P_j^i . The following result is known [19] :

Theorem: The trajectories of (4) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation tensor P_j^i are strictly negative everywhere, and Jacobi unstable, otherwise.

The geometrical meaning of the above theorem in the context of an Euclidean, Riemannian or Finslerian structure is discussed in [19].

3.2. The relation between linear stability and Jacobi stability. The correlation between linear stability and Jacobi stability has been completely discussed in [19] for the 2-dimensional dynamical systems case. The result of this discussion is crucial for the present paper. Here one wants to compare the signs of the eigenvalues of the Jacobian matrix J at a fixed point with the signs of the eigenvalues of the deviation curvature tensor P_i^j evaluated at the same point.

Let us consider the following system of ODE:

$$(15) \quad \frac{du}{dt} = f(u, v) \quad \frac{dv}{dt} = g(u, v)$$

such that the point $(0, 0)$ is an equilibrium, i.e., $f(0, 0) = g(0, 0) = 0$. We denote by J the Jacobian matrix of (15), i.e.

$$(16) \quad J(u, v) = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

where the subscripts are partial derivatives with respect to u and v . The characteristic equation is given by

$$(17) \quad \lambda^2 - (\text{tr}A)\lambda + \det A = 0,$$

where $\text{tr}A$ and $\det A$ are the trace and the determinant of the matrix $A := J|_{(0,0)}$, respectively. The signs of the trace, determinant of A , and of the discriminant $\Delta = (f_u - g_v)^2 + 4f_v g_u = (\text{tr}A)^2 - 4\det A$ give the linear stability of the fixed point $(0, 0)$ as described in the previous section.

In applying the KCC theory to the ODE system (15), by elimination of one of the variables, one can transform (15) into a system of differential equations of the form (4) and compute the deviation curvature. Boehmer *et al.* [19] obtain the following theorem:

Theorem: Let us consider the ODE (15) with the fixed point $P = (0, 0)$ such that $g_u|_{(0,0)} \neq 0$. Then,

$$(18) \quad 4P_1^1|_{(0,0)} = -4g_{,1}^1|_{(0,0)} + (g_{,1}^1)^2|_{(0,0)} = \Delta := (\text{tr}A)^2 - 4\det A.$$

where $g^1(x, y) \equiv -g_u(u(x, y), x) f(u(x, y), x) - g_v(u(x, y), x)y$ and $y \equiv g(u, v)$ and $g_{,1}^1$ is simply $\frac{\partial g^1}{\partial x}$. In this case, the trajectory $v = v(t)$ is Jacobi stable if and only if $\Delta < 0$.

More generally, we can conclude that in the case $g_u|_{(0,0)} \neq 0$ and $f_v|_{(0,0)} \neq 0$ both trajectories $u = u(t)$ and $v = v(t)$ are Jacobi stable if and only if $\Delta < 0$.

Boehmer *et al.* [19] also present the following corollary:

Corollary: Let us consider the ODE (15) with the fixed point $P = (0, 0)$ such that $g_u|_{(0,0)} \neq 0$. Then, the Jacobian J estimated at the point P has complex eigenvalues if and only if P is a Jacobi stable point.

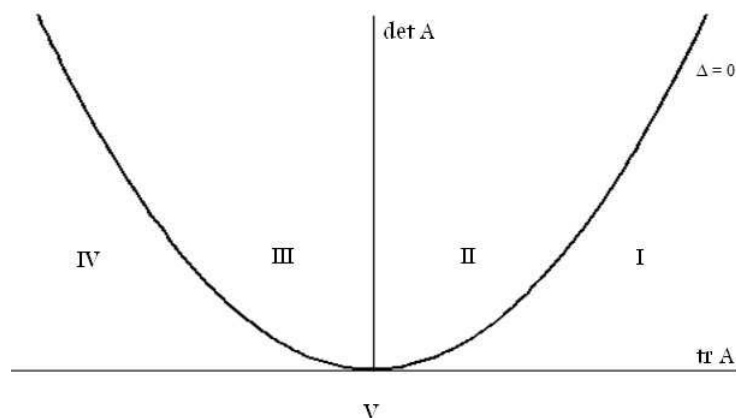


FIGURE 1. Linear stability

A combination of the linear stability [22] as depicted in the Fig. 1, with the definition of Jacobi stability, in the conditions of theorem above, one obtains the following relations:

(1) **Region I**

$\Delta > 0$	Jacobi unstable
$\text{tr} A > 0$	Unstable node
$\det A > 0$	

(2) **Region II**

$\Delta < 0$	Jacobi stable
$\text{tr}A > 0$	Unstable focus
$\det A > 0$	

(3) **Region III**

$\Delta < 0$	Jacobi stable
$\text{tr}A < 0$	Stable focus
$\det A > 0$	

(4) **Region IV**

$\Delta > 0$	Jacobi unstable
$\text{tr}A < 0$	Stable node
$\det A > 0$	

(5) **Region V**

$\Delta > 0$	Jacobi unstable
$\det A < 0$	Saddle point

The above description of all possible cases in 2-dimension suggests that linear stability and Jacobi stability do not have to agree in all cases. In fact, the stability in Jacobi sense refers to a linear stability type of the trajectories in the **curved space** endowed with a nonlinear connection and a curvature tensor. Here the role of usual partial derivative is played by the covariant derivative along the flow. This suggests the difference in the meaning of linear stability and Jacobi stability. Nevertheless, in the present paper, we have found a known physical case, namely, the rotation of a free rigid body, where these two stability concepts match in ALL cases.

4. MOTION OF A TORQUE-FREE RIGID BODY REVISITED

Motion of a rigid body around a stationary point not subject to any torques has been completely solved and is a textbook material [5, 11, 14]. Under such condition, a theorem due to Euler states that the general displacement of rigid body is an instantaneous rotation about some axis passing through the stationary point. In this way, the problem is reduced to the problem with three degrees of freedom. The governing equations of motion are given by Euler's equations:

$$(19) \quad \dot{M}_1 = \frac{I_2 - I_3}{I_2 I_3} M_2 M_3$$

$$(20) \quad \dot{M}_2 = \frac{I_3 - I_1}{I_1 I_3} M_1 M_3$$

$$(21) \quad \dot{M}_3 = \frac{I_1 - I_2}{I_2 I_3} M_1 M_2$$

where "." represents derivative with respect to time, $M_i = I_i \omega_i$, $i = 1, 2, 3$ are components the angular momentum vector along three principal axes and I_i are moments of inertia about the three principal axes. Without loss of generality, one can always assume $I_1 > I_2 > I_3$. The solutions to the above systems of differential equations have been given in details in many classical mechanics textbooks with both analytical and geometrical viewpoints, but rarely a full dynamical systems treatment has been provided in the literature. In this paper, we present such a analysis which is a ground base for applying Liapunov and Jacobi stabilities.

Furthermore, one can show that in the problem of the motion of a rigid body around a stationary point, in the absence of outside forces, there are four first integrals of motion: the three components of the angular momentum vector M_1, M_2, M_3 and the kinetic energy E . In particular the Euler's equations (1), (2) and (3) above have two quadratic first integrals (constant on orbits):

$$(22) \quad M^2 = M_1^2 + M_2^2 + M_3^2$$

$$(23) \quad E = \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3}.$$

One can easily calculate that $\dot{M} \equiv 0$ and $\dot{E} \equiv 0$, using (1), (2) and (3). Further standard analysis of the above problem reveals the following interesting theorem [5, 11, 14].

Rigid Body Stability Theorem: *In the motion of a free rigid body, rotation around the long and short axes is (Liapunov) stable and around the middle axis is unstable.*

In the next section, we will give a full dynamical systems approach to qualitative behavior of the solutions of Euler's equations and prove the Rigid Body Stability Theorem stated above. More importantly, we will show that the Rigid Body Stability Theorem is completely in agreement with stability in the sense of Jacobi.

5. ANALYSIS

5.1. Liapunov Stability Analysis. Linear stability analysis requires finding all fixed points of (19), (20) and (21) of the Euler's equations. One identifies the following six fixed points by putting the right hand side of (19), (20) and (21) equal to zero and solve for ω_1 , ω_2 and ω_3 :

$$(24) \quad A(or A') = \begin{bmatrix} \omega_1 = \pm\omega \\ \omega_2 = 0 \\ \omega_3 = 0 \end{bmatrix}; B(or B') = \begin{bmatrix} \omega_1 = 0 \\ \omega_2 = \pm\omega \\ \omega_3 = 0 \end{bmatrix}; C(or C') = \begin{bmatrix} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_3 = \pm\omega \end{bmatrix}$$

where ω is any arbitrary real number but to be determined by the energy E or the angular momentum M as initial conditions. These six fixed points are mutually located on three principal axes. This simply means that a free rigid body admits a stationary rotation around any of the three principal axes. Since we assumed $I_1 > I_2 > I_3$, then the right-hand side of the Euler equations does not become 0 anywhere else, i.e., there are no other stationary rotations. To investigate the stability (in the sense of Liapunov) of these stationary solutions, one calculates the Jacobian of the non-linear system (19), (20) and (21) given by:

$$J = \begin{bmatrix} 0 & \omega_3 \frac{I_2 - I_3}{I_1} & \omega_2 \frac{I_2 - I_3}{I_1} \\ \omega_3 \frac{I_3 - I_1}{I_2} & 0 & \omega_1 \frac{I_3 - I_1}{I_2} \\ \omega_2 \frac{I_1 - I_2}{I_3} & \omega_1 \frac{I_1 - I_2}{I_3} & 0 \end{bmatrix}$$

Linearization about the fixed point A is given by the following Jacobian matrix:

$$J_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \frac{I_3 - I_1}{I_2} \\ 0 & \omega \frac{I_1 - I_2}{I_3} & 0 \end{bmatrix}$$

On the tangent space at fixed point A to the angular momentum sphere of radius $\|\mathbf{M}\| = \text{constant}$; the linear operator given by this linearized vector has a Jacobian matrix given by the above lower right (2×2) -block whose eigenvalues are the roots of the following quadratic equation:

$$\lambda^2 - \text{tr}(J_A)\lambda + \det J_A = 0.$$

In other words, although the motion of a rigid body appears to be 3-dimensional, but the motion occurs in a 2-dimensional phase space on the sphere $\|M\| = \text{constant}$. The eigenvalues are now given by:

$$(25) \quad \lambda_A = \pm i\omega \sqrt{\frac{(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}}.$$

Since $I_1 > I_2 > I_3$, therefore fixed point A (and A') has pure imaginary conjugate eigenvalues, therefore a center, but the stability type of the fixed points cannot be deduced from linear approximation. The same arguments also holds for fixed point C (and C') where the eigenvalues of the linear system has zero real part and therefore standard linear stability analysis is insufficient to determine the stability of the fixed points.

Unlike the above cases, the linear stability analysis about fixed point B (and B') is straightforward. The Jacobian matrix about B is of form:

$$J_B = \begin{bmatrix} 0 & 0 & \omega \frac{I_2 - I_3}{I_1} \\ 0 & 0 & 0 \\ \omega \frac{I_1 - I_2}{I_3} & 0 & 0 \end{bmatrix}$$

with eigenvalues :

$$(26) \quad \lambda_B = \pm \omega \sqrt{\frac{(I_1 - I_2)(I_2 - I_3)}{I_1 I_3}},$$

which are two real eigenvalues with opposite signs and therefore the equilibrium point B (and B') is a **saddle** point and hence unstable.

In order to investigate the stability of equilibrium points A (and A') and C (and C'), one should construct Liapunov functions for these cases, but since we already have first integrals, employing the following theorem seems to be more appropriate:

Theorem (Dirichlete): *Consider a C^r vector field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in R^n$, which has a fixed point at $\mathbf{x} = \bar{\mathbf{x}}$. Let $V(\mathbf{x})$ be a first integral of this vector field defined in a neighborhood of $\mathbf{x} = \bar{\mathbf{x}}$ such that $\mathbf{x} = \bar{\mathbf{x}}$ is a non-degenerate minimum (maximum) of $V(\mathbf{x})$. Then $\mathbf{x} = \bar{\mathbf{x}}$ is stable (unstable) [20].*

Instability here is guaranteed by testing $DV(\bar{x}) = 0$ and $D^2V(\bar{x}) < 0$. The construction of a Liapunov function V can be done by exploiting the first integrals of F_1 and F_2 given by (22) and (23), and generating a new first integral [21]:

$$F_1(\mathbf{\Omega}) = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2, \quad F_2(\mathbf{\Omega}) = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

where $\mathbf{\Omega} = (\omega_1, \omega_2, \omega_3)$. A suitable ansatz is:

$$V(\mathbf{\Omega}) = (F_2 + \Phi \circ F_1)(\mathbf{\Omega})$$

since F_1 and F_2 are first integrals, then $\dot{F}_1 \equiv 0$ and $\dot{F}_2 \equiv 0$ and therefore $\dot{V} = \dot{F}_2 + \Phi'(F_1(\mathbf{\Omega})) \equiv 0$ which is one of the criteria we need for the Liapunov function. Here $\Phi : R \rightarrow R$ is some smooth function to be found (with Φ' as its derivative) such that V (a first integral) vanishes at the equilibrium point $\mathbf{\Omega}$ and is strictly positive or negative in a punctured neighborhood of $\mathbf{\Omega}$. Now the first and second derivative of V are given by:

$$(27) \quad DV(\mathbf{\Omega}) = (2I_1^2\omega_1 + I_1\omega_1\Phi'(F_1(\mathbf{\Omega})), 2I_2^2\omega_2 + I_2\omega_2\Phi'(F_1(\mathbf{\Omega})), 2I_3^2\omega_3 + I_3\omega_3\Phi'(F_1(\mathbf{\Omega}))),$$

and

$$(28) \quad D^2V(\omega) = \begin{pmatrix} P & I_1 I_2 \omega_1 \omega_2 \Phi''(F_1(\Omega)) & I_1 I_3 \omega_1 \omega_3 \Phi''(F_1(\Omega)) \\ I_1 I_2 \omega_1 \omega_2 \Phi''(F_1(\Omega)) & Q & I_2 I_3 \omega_2 \omega_3 \Phi''(F_1(\Omega)) \\ I_1 I_3 \omega_1 \omega_3 \Phi''(F_1(\Omega)) & I_2 I_3 \omega_2 \omega_3 \Phi''(F_1(\Omega)) & R \end{pmatrix},$$

where

$$P \equiv 2I_1^2 + I_1 \Phi'(F_1(\Omega)) + I_1^2 \omega_1^2 \Phi''(F_1(\Omega)),$$

$$Q \equiv 2I_2^2 + I_2 \Phi'(F_1(\Omega)) + I_2^2 \omega_2^2 \Phi''(F_1(\Omega)),$$

$$R \equiv 2I_3^2 + I_3 \Phi'(F_1(\Omega)) + I_3^2 \omega_3^2 \Phi''(F_1(\Omega)).$$

Now (27) yields

$$DV(A) = \omega \begin{bmatrix} 2I_1^2 + I_1 \Phi'(\frac{1}{2}I_1 \omega^2), & 0, & 0 \end{bmatrix},$$

where $DV(\Omega)$ has been evaluated at the fixed point $A = (\omega, 0, 0)$. Now,

$$DV(A) = 0, \quad \text{iff} \quad 2I_1^2 + I_1 \Phi'(\frac{1}{2}I_1 \omega^2) = 0 \quad \text{which implies} \quad \Phi'(\frac{1}{2}I_1 \omega^2) = -2I_1.$$

Similarly

$$D^2V(A) = \text{Diag} \left[I_1^2 \omega^2 \Phi''(\frac{1}{2}I_1 \omega^2), \quad 2I_2(I_2 - I_1), \quad 2I_3(I_3 - I_1) \right],$$

which yields

$$D^2V(A) < 0 \quad \text{iff} \quad \Phi''(\frac{1}{2}I_1 \omega^2) < 0,$$

since $I_2 - I_1 < 0$ and $I_3 - I_1 < 0$ by previous assumption. Let $\Phi(s) := -2I_1[s + \frac{1}{2}(s - \frac{1}{2}I_1 \omega^2)^2]$, then $\Phi'(\frac{1}{2}I_1 \omega^2) = -2I_1$ and $\Phi''(s) \equiv -2I_1 < 0$ as required for Φ to have a non-degenerate maximum at the equilibrium point A . By the Dirichlete theorem the fixed points A and A' are of type stable **centers**. The method used above to construct a Liapunov function is called the 'Energy-Casimir method' [14].

Although the characteristic of the fixed point A is now determined, but one can complete this procedure by constructing the Liapunov function $V = F_2 + \Phi(F_1)$:

(29)

$$\begin{aligned}
V(\omega_1, \omega_2, \omega_3) &= F_2 + \Phi(F_1) = \\
&= I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 - I_1 \left[I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 + \left(\frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 - \frac{1}{2} I_1 \omega^2 \right)^2 \right] \\
&= I_2 \omega_2^2 (I_2 - I_1) + I_3 \omega_3^2 (I_3 - I_1) - I_1 \left(\frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 - \frac{1}{2} I_1 \omega^2 \right)^2.
\end{aligned}$$

One can easily check that $V(A \text{ or } A') = V(\pm\omega, 0, 0) = 0$, $V < 0$ and $\dot{V} \equiv 0$ in a deleted neighborhood of A which again implies that fixed points of A and A' are stable centers.

One can similarly repeat the same construction for the equilibrium point $C = (0, 0, \omega)$ (or $C' = (0, 0, -\omega)$) and arrive at the following Liapunov function:

$$V(\omega_1, \omega_2, \omega_3) = I_1 \omega_1^2 (I_1 - I_3) + I_2 \omega_2^2 (I_2 - I_3) + I_3 \left(\frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2 - \frac{1}{2} I_3 \omega^2 \right)^2$$

Here again $V(C \text{ or } C') = V(0, 0, \pm\omega) = 0$, $V > 0$ and $\dot{V} \equiv 0$ in a deleted neighborhood of C (or C') which implies that fixed points of C and C' are stable **centers**.

To summarize, we have proven that four of six fixed points (i.e., A, A', C, C') are stable centers and the other two B and B' are unstable saddle. We now arrive at the Rigid Body Stability Theorem stated earlier. Fig. 2¹ depicts the flow of the dynamical system (19), (20) and (21) on the 2-dimensional angular momentum phase space $\|\mathbf{M}\| = \text{constant}$.

5.2. Jacobi Stability Analysis. Recalling the Jacobi stability analysis for a 2-dimensional autonomous system described in section (3.2) and using formula (18),

$$\Delta := \text{tr}(J)^2 - 4\det(J) = \begin{cases} < 0 & \text{Jacobi Stable} \\ \geq 0 & \text{Jacobi Unstable.} \end{cases}$$

¹With kind permission from Springer Science+Business Media: Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems, Chapter 15-The Free Rigid Body, 2nd edition, 1999, Page 491, Jerrold E. Marsden and Tudor S. Ratiu, Figure 15.3.1

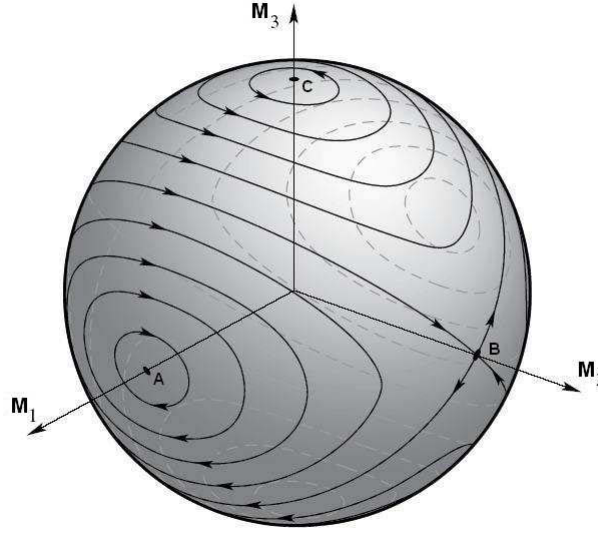


FIGURE 2. Singular points A, B and C and rigid body orbits on the angular momentum spheres for the case $I_1 > I_2 > I_3$. Singular points A' , B' and C' are not shown here.

Calculating Δ for fixed point A (and A') using eigenvalues given by (25) give rises to (knowing that $\text{tr}(J) = 0$ since eigenvalues have the same values but in opposite signs):

$$\Delta \Big|_{\text{fixed point A}} := 4\omega^2 \frac{(I_1 - I_2)(I_3 - I_1)}{I_2 I_3} < 0$$

which implies that fixed point A (and A') is Jacobi stable. Calculating Δ for fixed point B (and B') using (26) yields

$$\Delta \Big|_{\text{fixed point B}} := 4\omega^2 \frac{(I_1 - I_2)(I_3 - I_1)}{I_1 I_3} > 0$$

which reveals that fixed point B (and B') is Jacobi unstable and finally

$$\Delta \Big|_{\text{fixed point C}} := 4\omega^2 \frac{(I_2 - I_3)(I_3 - I_1)}{I_1 I_2} < 0$$

which means that fixed point C (and C') is also Jacobi stable. Our results are summarized in Table 1 below which clearly shows that the nature of singular points are the same in both Liapunov and Jacobi stability analysis methods.

Fixed Point	Liapunov	Jacobi
A & A'	Stable (Center)	Stable
B & B'	Unstable(Saddle)	Unstable
C & C'	Stable (Center)	Stable

Table 1. Stability analysis of all fixed points via Liapunov and Jacobi analysis.

6. DISCUSSIONS AND FINAL REMARKS

Two main stability analysis methods, namely Liapunov stability analysis and the Jacobi stability analysis are reviewed in the present paper. Bohmer *et al.* [19] have considered the stability properties of several dynamical systems that play important roles in gravitation and cosmology where both methods of Liapunov linear stability analysis and Jacobi stability analysis, or the KCC theory, were used. Their study of stability has been done by analyzing the behavior of steady states of the respective dynamical system. Linear stability analysis is performed by the linearization of the dynamical system via the Jacobian matrix of a non-linear system at the fixed points, while the KCC theory involves stability of a whole trajectory in a tubular region [17], where Jacobi stability (or instability) means that the trajectories of the structure equations will bunch together (or disperse) when approaching the fixed point.

Boehmer *et al.* [19] have shown that there is a good correlation (but not perfect) between the linear stability of the critical points, and the robustness of the corresponding trajectory to a small perturbation. In fact there are cases where a singular point is Liapunov stable but Jacobi unstable or vice versa and hence some discrepancies in the results of these two stability methods. In the present paper, we have shown that a known example in classical mechanics, namely the rotational motion of a free rigid body where Liapunov stability and Jacobi stability match exactly for ALL the equilibrium points. To do this task, we have given a full dynamical

systems analysis of the Euler equations in a general case which is rarely discussed in classical mechanics or dynamical systems textbooks.

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