

INTERIOR BLACK-HOLE SOLUTION WITH ANISOTROPIC FLUID

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ABSTRACT. The result of gravitational collapse of a compact body is believed to be a singularity hidden beyond its Schwarzschild radius known as a black hole. In the interior black hole region, a remarkable change occurs in the nature of the spacetime, namely the external spatial radial and temporal coordinates exchange their characters. So, the interior black hole solution is represented by a non-static spacetime. With the assumption that the material content of the spherical body is an anisotropic fluid, we have carried out a brief study of the annular space time metric. The case of isotropic pressures is discussed as a particular case and a regular solution has been obtained.

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1. INTRODUCTION

General relativity predicts the formation of a black hole, whenever gravitation at the surface of a star becomes so large that nothing can escape, even a beam of light. Michell and Laplace had discussed the possibility of such object within the framework of the Newtonian theory at the end of 18th century. In general relativity, the problem arose within a year after Schwarzschild (1916) obtained the first exact solution of Einstein's field equations in vacuum. This solution has a singularity at the center of the symmetry and an additional singularity on the gravitational radius surface. The gravitational collapse of a massive star which produces a black hole was first described by Oppenheimer and Snyder (1939). The term "black hole" was introduced by Wheeler (1967).

For $z < 2m$ the Schwarzschild exterior metric can be written,

$$ds^2 = - \left(\frac{2m}{z} - 1 \right) dt^2 + \left(\frac{2m}{z} - 1 \right)^{-1} dz^2 - z^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

We have chosen the gravitational units such that $G = c = 1$.

Above metric can be rewritten as

$$(1) \quad ds^2 = \left(\frac{2m}{z} - 1 \right)^{-1} dz^2 - \left(\frac{2m}{z} - 1 \right) dt^2 - z^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

which is valid for $z < 2m$.

Above metric represents the the annular region of spacetime for a spherical star inside its Schwarzschild radius $2m$ and outside its physical radius a , that is for a black hole.

2. THE INTERIOR BLACK HOLE SOLUTION

In the interior black hole spacetime, a remarkable change occurs in the nature of the spacetime namely the external spatial radial and temporal co-ordinates exchange their character to temporal and spatial co-ordinates, respectively [3].

The region $a < z < 2m$ is empty and the space time geometry is characterized by the spherically symmetric non static line element

$$(2) \quad ds^2 = B(z, t) dt^2 - A(z, t) dz^2 - t^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where the coefficient functions can be obtained by solving Einstein's field equations for empty space time. That is $R_{ij} = 0$. Here, the Einstein field equation for empty

space are

$$(3) \quad R_{11} = \frac{-A^2}{4AB} + \frac{B'^2}{4B^2} - \frac{\dot{A}\dot{B}}{4B^2} + \frac{A'B'}{4AB} - \frac{B''}{2B} + \frac{\ddot{A}}{2B} + \frac{\dot{A}}{Bt} = 0$$

$$(4) \quad R_{14} = \frac{B'}{Bt} = 0$$

$$(5) \quad R_{22} = \frac{c^2 t \dot{A}}{2AB} - \frac{\dot{B} c^2 t}{2B^2} + \frac{c^2}{B} + 1 = 0$$

$$(6) \quad R_{44} = \frac{\dot{A}^2}{4A^2} - \frac{B'^2}{4AB} + \frac{\dot{A}\dot{B}}{4AB} - \frac{A'B'}{4A^2} + \frac{B''}{2A} - \frac{\ddot{A}}{2A} + \frac{\dot{B}}{Bt} = 0$$

Here and in what follows dot and prime denote the derivatives with respect to t and z , respectively. Solution of these equations give

$$\frac{d}{dt}(AB) = 0 \text{ and } B = B(t).$$

Let $AB = C = C(z)$.

Without a significant loss of generality, we take the integrating constant $C = 1$.

So, $A = B^{-1}$ and $A = A(t)$ As B is related to proper time [3], $A, B > 0$.

With these Einstein's Field equations (3) to (6) become

$$(7) \quad R_{11} = A \left(\frac{\ddot{A}}{2A} + \frac{\dot{A}}{t} \right) = 0$$

$$(8) \quad R_{22} = \dot{A}t + A + 1 = 0$$

$$(9) \quad R_{44} = \frac{\ddot{A}}{2A} + \frac{\dot{A}}{At} = 0$$

Solving and following the notations of Rosa et al [3], we get

$$(10) \quad A = B^{-1} = \left(\frac{2\xi}{t} - 1 \right)$$

so the metric for interior black hole is

$$(11) \quad ds^2 = \left(\frac{2\xi}{t} - 1 \right)^{-1} dt^2 - \left(\frac{2\xi}{t} - 1 \right) dz^2 - t^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

which represents the spacetime inside the Schwarzschild radius and outside physical radius of a black hole.

3. ANISOTROPIC PRESSURE

If the interior of the black hole is filled with an imperfect fluid, Einstein Field equation is

$$(12) \quad R_{ij} = -8\pi \left(T_{ij} - \frac{1}{2} T g_{ij} \right)$$

where g_{ij} and R_{ij} are metric tensor and Ricci tensor, respectively, T_{ij} is the energy momentum tensor and T is its scalar. Following [2], T_{ij} is given by

$$(13) \quad T_{ij} = (\rho + p) u_i u_j - p g_{ij} + \pi_{ij}$$

, where $\pi_{ij} = \sqrt{3}S [c_i c_j - \frac{1}{3}(u_i u_j - g_{ij})]$; S = magnitude of anisotropic stress tensor, $c^i = (-\sqrt{A}, 0, 0, 0)$, $u_i = (0, 0, 0, c\sqrt{B})$, $c^i c_i = -1$, $u^i u_i = 1$,
The pressure along radial direction $p_r = p + \frac{2S}{\sqrt{3}}$ and the pressure along tangential direction $p_\perp = p - \frac{S}{\sqrt{3}}$ will be different and we have

$$(14) \quad S = \frac{p_r - p_\perp}{\sqrt{3}}$$

$$(15) \quad T = \rho - p_r - 2p_\perp = \rho - 3p$$

$$(16) \quad T_{11} = A p_r$$

$$(17) \quad T_{22} = t^2 p_\perp$$

$$(18) \quad T_{33} = T_{22} \sin^2 \theta$$

$$(19) \quad T_{44} = \rho$$

Here, $S \neq 0$ and hence $R_1^1 \neq R_2^2$ and this with (7),(8) and (10) implies $\ddot{\xi} - \frac{2\dot{\xi}}{t} \neq 0$
Assuming

$$(20) \quad \ddot{\xi} - \frac{2\dot{\xi}}{t} = \frac{\alpha}{t^2}$$

This has the complete solution

$$(21) \quad \xi = C_1 + C_2 t^3 - \frac{\alpha}{3} \log t$$

So the solution is

$$(22) \quad ds^2 = \left(2C_2 t^2 + \frac{2C_1}{t} - \frac{2\alpha \log t}{3t} - 1 \right)^{-1} dt^2 - \left(2C_2 t^2 + \frac{2C_1}{t} - \frac{2\alpha \log t}{3t} - 1 \right) dz^2 -$$

$t^2(d\theta^2 + \sin^2 \theta d\phi^2)$ which has singularity at $t = 0$

and

$$(23) \quad p_r = \frac{1}{8\pi} \left[6C_2 - \frac{2\alpha}{3t^3} \right]$$

$$(24) \quad p_\perp = \frac{1}{8\pi} \left[6C_2 + \frac{\alpha}{3t^3} \right]$$

$$(25) \quad \rho = \frac{1}{8\pi} \left[\frac{2\alpha - 18C_2 t^3}{6C_1 t^2 - 3t^3 + 6C_2 t^5 - 2\alpha t^2 \log t} \right]$$

$$(26) \quad S = \frac{-\alpha}{8\sqrt{3}\pi t^3}$$

4. ISOTROPIC PRESSURE

In case of isotropic pressures $\pi_{ij} = 0$ and therefore $S = 0$. Thus, $R_1^1 - R_2^2 = 0$ that gives $\xi = C_1 + C_2 t^3$. If the interior of the black hole is filled with a perfect fluid, the interior black hole solution is

$$(27) \quad ds^2 = \left(2C_2 t^2 + \frac{2C_1}{t} - 1 \right)^{-1} dt^2 - \left(2C_2 t^2 + \frac{2C_1}{t} - 1 \right) dz^2 - t^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

with a removable singularity $t = 0$.

Here,

$$(28) \quad ds^2 = (2C_2 t^2 - 1)^{-1} dt^2 - (2C_2 t^2 - 1) dz^2 - t^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

is a regular solution.

Also one can have

$$(29) \quad ds^2 = \left(\frac{2C_1}{t} - 1 \right)^{-1} dt^2 - \left(\frac{2C_1}{t} - 1 \right) dz^2 - t^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

which is of the form (11).

5. CONCLUSION

In this work, first we have obtained the interior black hole solution by considering a non static space-time. Then, for anisotropic fluid as a material content of it, the solution has been obtained that has a singularity at $t = 0$. This singularity has arisen as a result of our choice $\ddot{\xi} - \frac{2\dot{\xi}}{t} = \frac{\alpha}{t^2}$. For large value of t anisotropy vanishes as $S \propto \frac{1}{t^3}$. The case of isotropic pressures is discussed as its particular case. There

is a regular solution in the case of isotropic pressures.

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