

HILBERT BIMODULES AND PARTIAL-ISOMETRIC CROSSED PRODUCTS BY ONE ENDOMORPHISM

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ABSTRACT. Let α be an extendible endomorphism of a C^* -algebra A , and let X be the Hilbert bimodule $\overline{\alpha}(1)A$. We analyze the Toeplitz algebra of this Hilbert bimodule and we show that the partial-isometric crossed product $A \rtimes_{\alpha} \mathbb{N}$ is isomorphic to the Toeplitz algebra $\mathcal{T}_X$. Then we have two crossed product exact sequences.

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1. INTRODUCTION

Suppose X is a Hilbert bimodule over a C^* -algebra A with a left action on it defined by a homomorphism φ from A to the $\mathcal{L}(X)$ consists of all adjointable operators on X . Our motivating Hilbert bimodule is $X = \overline{\alpha}(1)A$ where α is an extendible endomorphism, in the sense that, α can be extend to the multiplier $M(A)$ such that for an approximate identity (e_{λ}) in A , $\alpha(e_{\lambda})$ converges strictly to the projection $\overline{\alpha}(1)$ in $M(A)$ [1]. As we shall see, the partial-isometric crossed product of the system (A, α) is isomorphic to the Teoplitz algebra $\mathcal{T}_X$. By applying

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results in [5, Corrolary 3.2] on this particular Hilbert bimodule leads us to indicate that the quotient $(A \times_{\alpha} \mathbb{N})/(I \times_{\alpha} \mathbb{N})$ is isomorphic to the quotient crossed product $A/I \times_{\alpha_{A/I}} \mathbb{N}$. To do this, we assume that I is extendibly α -invariant, that is, $\alpha(I) \subseteq I$, and $\alpha|_I$ extends to a strictly continuous endomorphism $\overline{\alpha}|_I$ such that $\overline{\alpha}|_I(1) = \overline{\psi}(\overline{\alpha}(1))$, where $\overline{\psi}$ is the extension of the canonical non degenerate homomorphism $\psi : A \rightarrow M(I)$. Moreover, we can get a partial-isometric crossed product exact sequence.

In section 2, the fundamental definitions and notions of right Hilbert modules, adjointable operators on right Hilbert modules, extendible homomorphisms and multiplier algebras are given. Hilbert bimodules and Partial-isometric crossed product are discussed in section 3. For the Hilbert bimodule $X = \overline{\alpha}(1)A$ we have a theorem that describes the the Toeplitz algebra related to X . In order to prove this theorem we need to extend Toeplitz representations on X so that this extension devotes section 4. Finally in the section 5, invariant ideals for an extendible endomorphism are considered. So we have the quotient Hilbert bimodules and their relationships with quotient partial-isometric crossed products.

2. PRELIMINARIES

Let X be a vector space over the complex numbers $\mathbb{C}$, and A be a C^* -algebra. The vector space X is an inner product A -module if X is a right A -module (the right action is denoted by $x \cdot a$ where $x \in X, a \in A$), and there is an A -valued map $\langle \cdot, \cdot \rangle : X \times X \rightarrow A$ such that

- (1) $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle, \quad x, y \in X, \lambda, \mu \in \mathbb{C},$
- (2) $\langle x, y \cdot a \rangle = \langle x, y \rangle a, \quad x, y \in X, a \in A,$
- (3) $\langle x, y \rangle^* = \langle y, x \rangle, \quad x, y \in X,$
- (4) $\langle x, x \rangle \geq 0; \quad \text{if } \langle x, x \rangle = 0 \text{ then } x = 0 \quad x \in X,$

moreover, for all $x \in X, a \in A$ and $\lambda \in \mathbb{C}, \lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)$. Every inner product A -module X is equipped by a norm defined by

$$\|x\|_X := \|\langle x, x \rangle\|_A^{1/2}.$$

Definition 2.1. A *Hilbert A -module* is an inner product A -module X which is complete in the norm arising from the A -valued inner product. We say that X is a *full Hilbert A -module* if $A = \overline{\text{span}}\{\langle x, y \rangle : x, y \in X\}$.

We denote a Hilbert A -module by X_A . To emphasize that X_A is a right module we put A as an index in the right hand of X . It is denoted by X where there is no ambiguity. In the theory of Hilbert spaces the bounded operators have adjoint automatically. But this is not true in general that every bounded operator between two right Hilbert modules has an adjoint.

A map $T : X \rightarrow Y$ between right Hilbert A -modules X and Y is *adjointable* if there exists a map $T^* : Y \rightarrow X$ such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for all } x \in X \text{ and } y \in Y.$$

If the adjoint T^* exists, it is unique. Every adjointable map $T : X \rightarrow Y$ between Hilbert A -modules is a bounded linear operator. Moreover, it is A -linear, in the sense that for all $x \in X$ and $a \in A$, $T(x \cdot a) = T(x) \cdot a$ (see [8, Lemma 2.18]). The norm of an operator $T : X \rightarrow Y$ of right Hilbert A -modules is defined as usual; that is, $\|T\| = \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$. We denote the set of all adjointable operators from X to Y by $\mathcal{L}(X, Y)$. We write $\mathcal{L}(X)$, or $\mathcal{L}(X_A)$ for $\mathcal{L}(X, X)$. If X is a Hilbert A -module, then by [8, Proposition 2.21] $\mathcal{L}(X)$ is a C^* -algebra with respect to the operator norm.

Let X, Y be right Hilbert A -modules. An operator $u \in \mathcal{L}(X, Y)$ is said to be unitary if $uu^* = Id_Y$ and $u^*u = Id_X$. If there exists a unitary element in $\mathcal{L}(X, Y)$ then we say that X and Y are unitary equivalent Hilbert A -modules, and we write $X \cong Y$.

Theorem 2.1. [12, Theorem 3.5] *Let A be a C^* -algebra, let X, Y be Hilbert A -modules and let u be a linear map from X to Y . Then the following conditions are equivalent.*

- (i) u is isometry, surjective and A -linear;
- (ii) u is unitary element of $\mathcal{L}(X, Y)$.

Suppose X and Y are Hilbert A -modules. For each $x \in X$ and $y \in Y$, setting $\Theta_{y,x}(z) := y \cdot \langle x, z \rangle$ defines an adjointable operator $\Theta_{y,x} : X \rightarrow Y$ such that $\Theta_{y,x}^* = \Theta_{x,y}$. The adjointable operator $\Theta_{y,x}$ is called *compact*. The closed linear subspace

$$\mathcal{K}(X) := \overline{\text{span}}\{\Theta_{x,y} : x, y \in X\}$$

of $\mathcal{L}(X)$ is called the *algebra of compact operators* on X . For any Hilbert A -module X , $\mathcal{K}(X)$ is a closed two-sided ideal in $\mathcal{L}(X)$ (see [12] and [8] for more details).

We can find a unital C^* -algebra denoted by $M(A)$ such that A embeds in it as an *essential* ideal, in the sense that A has nonzero intersection with any other nonzero ideal of $M(A)$. We recall the definition of multiplier algebra from [8, Definition 2.49]. The strict topology on a multiplier algebra is applied for extendible endomorphism. So we need know a bit about this topology. Let $\|\cdot\|_a$ be the seminorm on $M(A)$ defined by $\|b\|_a = \|ab\| + \|ba\|$. The strict topology on $M(A)$ is the topology generated by the seminorms $\{\|\cdot\|_a : a \in A\}$. A net b_i in $M(A)$ converges strictly to b if and only if $b_i a \rightarrow ba$ and $ab_i \rightarrow ab$ for all $a \in A$.

Definition 2.2. A homomorphism $\alpha : A \rightarrow M(B)$ of C^* -algebras A and B , is said to be extendible if there is an approximate identity (e_λ) in A such that $\alpha(e_\lambda)$ converges strictly to a projection p in $M(B)$.

Adji in [3] showed that a homomorphism $\alpha : A \rightarrow M(B)$ is extendible if and only if it has a unique extension $\bar{\alpha} : M(A) \rightarrow M(B)$. In fact, $\alpha(e_\lambda) \rightarrow \bar{\alpha}(1_{M(A)})$.

Let α be an extendible endomorphism on a C^* -algebra A . If I is an ideal of A , then define $\psi : A \rightarrow M(I)$ by $\psi(a)i := ai$ for $a \in A$ and $i \in I$. A routine calculation shows that ψ is nondegenerate and then it has an extension $\bar{\psi} : M(A) \rightarrow M(I)$.

Definition 2.3. An ideal I of A is α -*extendible* if I is α -invariant ($\alpha(I) \subseteq I$) and there is an approximate identity (i_λ) in I such that $\alpha(i_\lambda)$ converges strictly to $\bar{\psi}(\bar{\alpha}(1))$ in $M(I)$, where $\bar{\psi}$ is the extension of the canonical homomorphism $\psi : A \rightarrow M(I)$.

Example 2.1. The forward shift map $\tau : \mathbf{c}_0 \rightarrow \mathbf{c}_0$, defined by

$$\tau(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots)$$

is an extendible endomorphism. Where $\mathbf{c}_0$ is the subspace of $\ell^\infty(\mathbb{N})$ contains all sequences which converge to zero. Define for each $n \in \mathbb{N}$

$$1_n(m) = \begin{cases} 1 & m \geq n, \\ 0 & m < n. \end{cases}$$

So, $(1 - 1_n)$ is an approximate identity in $\mathbf{c}_0$. It is easy to check that $\tau(1 - 1_n)$ converges strictly to $\tau(1, 1, 1, \dots)$. Indeed, $\mathbf{c}_0$ is a τ -extendible ideal of the C^* -algebra $\mathbf{c}$ the set of all convergent sequences in the complex numbers.

3. TOEPLITZ ALGEBRA AND PARTIAL-ISOMETRIC CROSSED PRODUCTS

By a Hilbert bimodule over A , we shall mean right Hilbert A -module X with a left action $a \cdot x := \varphi(a)x$ where $\varphi : A \rightarrow \mathcal{L}(X)$ is a homomorphism.

Definition 3.1. A Toeplitz representation (ψ, π) of a Hilbert bimodule X in a C^* -algebra B consists of a linear map $\psi : X \rightarrow B$, and a homomorphism $\pi : A \rightarrow B$ such that

- a) $\psi(a \cdot x) = \pi(a)\psi(x)$,
- b) $\psi(x \cdot a) = \psi(x)\pi(a)$,
- c) $\pi(\langle x, y \rangle_A) = \psi(x)^*\psi(y)$,

for all $x, y \in X$ and $a \in A$.

Note that $\psi : X \rightarrow B$ is bounded as Hilbert bimodules; for $x \in X$,

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\|^2 = \|\pi(\langle x, x \rangle)\|^2 \leq \|\langle x, x \rangle\|^2 = \|x^*x\|^2 = \|x\|^2$$

which implies that $\|\psi\| \leq 1$.

Definition 3.2. A Toeplitz algebra $\mathcal{T}_X$ of a Hilbert bimodule X is a C^* -algebra together with a Toeplitz representation (i_X, i_A) satisfying the following conditions:

- a) for every Toeplitz representation (ψ, π) of X on a Hilbert space $\mathcal{H}$, there is a homomorphism $\psi \times \pi$ of $\mathcal{T}_X$ on $\mathcal{H}$ such that $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$,
- b) $\mathcal{T}_X$ is generated by $i_X(X) \cup i_A(A)$ as a C^* -algebra.

For every Hilbert bimodule X , the Toeplitz algebra $\mathcal{T}_X$ related to X exist [4, Proposition 1.3]. An element $v \in A$ is called partial isometry if $vv^*v = v$. A partial isometry v is called a power partial isometry if v^n is a partial isometry for every $n \in \mathbb{N}$. Note that in a unital C^* -algebra A , $v \in A$ is said to be an isometry if $v^*v = 1$, so every isometry is a partial isometry. If v is a partial isometry then so is v^* .

A partial isometric representation of $\mathbb{N}$ on a Hilbert space H is a map $V : \mathbb{N} \rightarrow B(H)$ such that V_n is partial isometry and $V_{n+k} = V_n V_k$ for all $n, k \in \mathbb{N}$. Every partial-isometric representation V of $\mathbb{N}$ is always determined by the single partial isometry V because we can write $V^n = V_1^n$. Conversely a single partial isometry W generates a partial isometric representation $V : n \mapsto W^n$ if and only if W is a power partial isometry.

For a bounded operator u on a Hilbert space H the following conditions are equivalent (see [9, Theorem 2.3.3]).

- 1) $u = uu^*u$;
- 2) uu^* is a projection;

- 3) u^*u is a projection;
- 4) u is a partial isometry.

A system (A, α) consists of a C^* -algebra A together with an action α of $\mathbb{N}$ by extendible endomorphisms on A . The following definition of crossed product for a system (A, α) comes from [7]. We suppose α is extendible when A is non-unital.

Definition 3.3. A *covariant partial-isometric representation* of a system (A, α) in a C^* -algebra B is a pair (π, V) consisting of a nondegenerate representation $\pi : A \rightarrow B$ and a homomorphism $V : \mathbb{N} \rightarrow M(B)$ such that V_s is a partial isometry for all $s \in \mathbb{N}$ and for all $a \in A$,

$$\pi(\alpha_s(a)) = V_s \pi(a) V_s^*, \quad V_s^* V_s \pi(a) = \pi(a) V_s^* V_s.$$

By [7, Lemma 4.2 and 4.3], every system (A, α) has a nontrivial covariant representation and every covariant partial-isometric representation (π, V) on a Hilbert space can be extended to a covariant partial-isometric representation $(\bar{\pi}, V)$ of $(M(A), \bar{\alpha})$ such that

$$\pi(\alpha_s(a)) = V_s \pi(a) V_s^* \text{ and } V_s V_s^* = \bar{\pi}(\bar{\alpha}(1)) \text{ for all } a \in A \text{ and } s \in \mathbb{N}.$$

Definition 3.4. A *partial-isometric crossed product* is a C^* -algebra B together with a covariant partial-isometric representation $(i_A, i_{\mathbb{N}})$ in B such that

- (i) for each covariant partial-isometric representation (π, V) of (A, α) on a Hilbert space $\mathcal{H}$, there is a nondegenerate representation $\pi \times V : B \rightarrow B(\mathcal{H})$ with $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_{\mathbb{N}} = V$;
- (ii) B is generated by $i_A(A) \cup i_{\mathbb{N}}(\mathbb{N})$; in fact,

$$B = \overline{\text{span}}\{i_{\mathbb{N}}(x)^* i_A(a) i_{\mathbb{N}}(y) : a \in A, x, y \in \mathbb{N}\}.$$

Every system (A, α) has a partial-isometric crossed product which are denoted by $A \times_{\alpha}^{iso} \mathbb{N}$ [7]. The universal property implies that if there is another triple $(B, j_A, j_{\mathbb{N}})$ satisfying (i) and (ii), then there is an isomorphism ρ of B onto $A \times_{\alpha} \mathbb{N}$ such that $\rho \circ i_A = j$ and $\rho \circ j_{\mathbb{N}} = i_{\mathbb{N}}$ [7].

We set $X := \bar{\alpha}(1)A$. By the equations

$$x \cdot a := xa, \quad \langle x, y \rangle := x^* y, \quad x, y \in X \text{ and } a \in A,$$

X is a right Hilbert A -module, $\varphi(a)x := \alpha(a)x$ defines a left action $\varphi : A \rightarrow \mathcal{L}(X)$ on X . So X is a Hilbert bimodule over A by these operations. Every element x in X has the form $x = \bar{\alpha}(1)a$ for some $a \in A$. Moreover, if (e_λ) is an approximate identity in A , then for each $a \in A$,

$$\lim_\lambda \varphi(e_\lambda)\bar{\alpha}(1)a = \lim_\lambda \alpha(e_\lambda)\bar{\alpha}(1)a = \lim_\lambda \alpha(e_\lambda)a = \bar{\alpha}(1)a.$$

It implies that X is *essential*, that is $X = \overline{\text{span}}\{a \cdot x : a \in A, x \in X\}$. During the rest of this work we keep the notation X just for the above Hilbert bimodule.

Lemma 3.1. *Then for all $a \in A$, $\varphi(a) \in \mathcal{K}(X)$ and $\Theta_{\alpha(a), \alpha(e_\lambda)}$ converges to $\varphi(a)$ in $\mathcal{K}(X)$ where (e_λ) is an approximation identity in A .*

Proof. Since A_A is a right Hilbert A -module, by [6, Proposition 2.31] a can be written as $a = bc^*$ with $b, c \in A$, so for the compact operator $\Theta_{\alpha(b), \alpha(c)}$ and for every $x \in X$,

$$\Theta_{\alpha(b), \alpha(c)}(x) = \alpha(b) \cdot \langle \alpha(c), x \rangle = \alpha(b)\alpha(c)^*x = \alpha(bc^*)x = \phi(a)x,$$

and hence $\phi(a)$ is a compact operator. Suppose (e_λ) is an approximate identity in A and $a \in A$. First, note that

$$\Theta_{\alpha(a), \alpha(e_\lambda)}(x) = \alpha(a) \cdot \langle \alpha(e_\lambda), x \rangle = \alpha(ae_\lambda)x,$$

and $\alpha(ae_\lambda)$ converges to $\alpha(a)x = \varphi(a)x$. Then for all $x \in X$ with $\|x\| \leq 1$,

$$\begin{aligned} \|\Theta_{\alpha(a), \alpha(e_\lambda)}(x) - \varphi(a)x\| &= \|\alpha(a)\alpha(e_\lambda)x - \alpha(a)x\| \\ &= \|(\alpha(ae_\lambda) - \alpha(a))x\| \\ &\leq \|\alpha(ae_\lambda) - \alpha(a)\|\|x\| \\ &\leq \|ae_\lambda - a\|, \end{aligned}$$

which implies that

$$\|\Theta_{\alpha(a), \alpha(e_\lambda)} - \varphi(a)\| = \sup_{\|x\| \leq 1} \|\Theta_{\alpha(a), \alpha(e_\lambda)}(x) - \varphi(a)x\| \leq \|ae_\lambda - a\|.$$

So $\Theta_{\alpha(a), \alpha(e_\lambda)} \rightarrow \varphi(a)$ as ae_λ reaches to a . □

The main focus of interest is the particular Hilbert bimodule $X = \bar{\alpha}(1)A$. It shall be shown that the Toeplitz algebra arisen by this Hilbert bimodule is certainly a crossed product, without using the results in [6] as were used in [7]. We are now ready to state our main theorem.

Theorem 3.1. *Let (A, α) be a system, and let $X = \bar{\alpha}(1)A$ with $a \cdot x = \alpha(a)x$, $x \cdot a = xa$ and $\langle x, y \rangle_A = x^*y$ where α is an extendible endomorphism on a C^* -algebra A . Then $\mathcal{T}_X \cong A \times_{\alpha}^{piso} \mathbb{N}$.*

To prove the Theorem 3.1 we apply [4, Proposition 1.3]. Using [4, Proposition 1.3] leads us to extend any Toeplitz representation (ψ, π) of the Hilbert bimodule X on a Hilbert space H to a Toeplitz representation $(\bar{\psi}, \bar{\pi})$ of the Hilbert bimodule $\tilde{X} := \bar{\alpha}(1)M(A)$ on H . Attempting to prove the Theorem 3.1 needs going through the construction of Toeplitz representations of the Hilbert bimodule $X = \bar{\alpha}(1)A$. So all next section is about the structure of X .

4. EXTENDING A TOEPLITZ REPRESENTATION OF $\bar{\alpha}(1)A$

Lemma 4.1. *Let (ψ, π) be a Toeplitz representation of a Hilbert bimodule X over a C^* -algebra A on a Hilbert space $\mathcal{H}$ where π is not necessary nondegenerate, and $\mathcal{H}_0 = \overline{\text{span}}\{\pi(a)h | a \in A, h \in \mathcal{H}\}$. Then for all $a \in A, x \in X$ and $k \in \mathcal{H}_0^{\perp}$*

- 1) $\pi(a)k$, $\psi(x)k$ and $\psi(x)^*k$ vanish on $\mathcal{H}_0^{\perp}$,
- 2) $\psi(x)\mathcal{H}_0 \subseteq \mathcal{H}_0$,
- 3) $\psi(x)^*\mathcal{H}_0 \subseteq \mathcal{H}_0$.

Proof. For each $a \in A$ and $k \in \mathcal{H}_0^{\perp}$, $\|\pi(a)k\|^2 = (\pi(a)k | \pi(a)k) = (k | \pi(a^*a)k) = 0$, so $\pi(a)k = 0$. Now, for each element e_{λ} of an approximate identity (e_{λ}) in A , $x \in X, h \in \mathcal{H}$ and $k \in \mathcal{H}_0^{\perp}$ we have

$$\begin{aligned} (\psi(x \cdot e_{\lambda})k | h) &= (k | (\psi(x \cdot e_{\lambda}))^*h) = (k | (\psi(x)\pi(e_{\lambda}))^*h) = (k | \pi(e_{\lambda})\psi(x)^*h) = 0 \text{ and} \\ (\psi(e_{\lambda} \cdot x)^*k | h) &= (k | (\psi(e_{\lambda} \cdot x)h) = (k | \pi(e_{\lambda})\psi(x)h) = 0 \end{aligned}$$

because both $\pi(e_{\lambda})\psi(x)h$ and $\pi(e_{\lambda})\psi(x)^*h$ are in $\mathcal{H}_0$. We know that $\psi(x \cdot e_{\lambda})$ and $\psi(e_{\lambda} \cdot x)$ converge to $\psi(x)$ because ψ is continuous, and $x \cdot e_{\lambda}$ and $e_{\lambda} \cdot x$ converge to x in X . Thus $(\psi(x)k | h) = 0$ for all $h \in \mathcal{H}$ and part (1) holds. Furthermore, $\psi(e_{\lambda} \cdot x)h = \pi(e_{\lambda})\psi(x)h$ and $(\psi(x \cdot e_{\lambda}))^*h = (\psi(x)\pi(e_{\lambda}))^*h = \pi(e_{\lambda})\psi(x)^*h$ are in $\mathcal{H}_0$. Since $\psi(e_{\lambda} \cdot x)h$ and $\psi(x \cdot e_{\lambda})^*h$ converge to $\psi(x)h$ and $\psi(x)^*h$ respectively, so (2) and (3) follow. $\square$

Lemma 4.2. *If (ψ, π) is a Toeplitz representation of X on a Hilbert space $\mathcal{H}$, then there is bounded operator $\psi(\bar{\alpha}(1))$ in $B(\mathcal{H})$ such that for all $a_i, b_j \in A, h_i \in \mathcal{H}$ and $n, m \in \mathbb{N}$,*

$$\begin{aligned} 1) \quad & \psi(\bar{\alpha}(1))(\sum_{i=1}^n \pi(a_i)h_i) = \sum_{i=1}^n \psi(\bar{\alpha}(1)a_i)h_i, \\ 2) \quad & \psi(\bar{\alpha}(1))^* \sum_{j=1}^m \pi(b_j)k_j = \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j. \end{aligned}$$

Proof. Suppose $\mathcal{H}_0 := \overline{\text{span}}\{\pi(a)h \mid a \in A, h \in \mathcal{H}\}$ and (e_λ) be an approximate identity in A where λ varies in an index set Λ . If $\sum_{i=1}^n \pi(a_i)h_i \in \mathcal{H}_0$, then for each $\lambda \in \Lambda$, we have

$$\begin{aligned} (4.1) \quad & \left\| \sum_{i=1}^n \pi(e_\lambda) \psi(\bar{\alpha}(1)a_i)h_i \right\| = \left\| \sum_{i=1}^n \psi(e_\lambda \cdot \bar{\alpha}(1)a_i)h_i \right\| = \left\| \sum_{i=1}^n \psi(\alpha(e_\lambda)\bar{\alpha}(1)a_i)h_i \right\| \\ & = \left\| \sum_{i=1}^n \psi(\bar{\alpha}(1)\alpha(e_\lambda) \cdot a_i)h_i \right\| = \left\| \sum_{i=1}^n \psi(\alpha(e_\lambda)\pi(a_i))h_i \right\| \\ & = \left\| \psi(\alpha(e_\lambda)) \sum_{i=1}^n \pi(a_i)h_i \right\| \leq \left\| \sum_{i=1}^n \pi(a_i)h_i \right\|. \end{aligned}$$

The last inequality follows by the fact that $\|\psi(\alpha(e_\lambda))\| \leq \|\alpha(e_\lambda)\| \leq \|e_\lambda\| \leq 1$. The inequality 4.1 implies that

$$\lim_{\lambda} \left\| \sum_{i=1}^n \pi(e_\lambda) \psi(\bar{\alpha}(1)a_i)h_i \right\| = \left\| \sum_{i=1}^n \psi(\bar{\alpha}(1)a_i)h_i \right\| \leq \left\| \sum_{i=1}^n \pi(a_i)h_i \right\|$$

and the map $\psi(\bar{\alpha}(1))$ defined by

$$\psi(\bar{\alpha}(1))\left(\sum_{i=1}^n \pi(a_i)h_i\right) := \sum_{i=1}^n \psi(\bar{\alpha}(1)a_i)h_i,$$

is well-defined, linear and bounded on $\text{span}\{\pi(a)h \mid a \in A, h \in \mathcal{H}\}$. Lemma 4.1 implies that $\psi(\bar{\alpha}(1))\mathcal{H}_0 \subseteq \mathcal{H}_0$ in that if $h_i \in \mathcal{H}_0^\perp$, $\psi(\bar{\alpha}(1)a_i)h_i = 0$, otherwise $\psi(\bar{\alpha}(1)a_i)h_i \in \mathcal{H}_0$. Hence it can be extended to a bounded operator on $\mathcal{H}_0$. Since $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, $\psi(\bar{\alpha}(1))$ can be defined on whole $\mathcal{H}$ by $\psi(\bar{\alpha}(1)) \oplus T$ where $Tk = 0$ for all $k \in \mathcal{H}_0^\perp$. Thus $\psi(\bar{\alpha}(1)) \in B(\mathcal{H})$ and it vanishes on $\mathcal{H}_0^\perp$.

To prove (2) we show that for $\sum_{j=1}^m \pi(b_j)k_j \in \mathcal{H}$ where $b_j \in A, k_j \in \mathcal{H}$,

$$(4.2) \quad \left(h \left| \psi(\bar{\alpha}(1))^* \left(\sum_{j=1}^m \pi(b_j)k_j \right) - \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j \right. \right) = 0$$

for all $h \in \mathcal{H}$. For all $a_i, b_j \in A, h_i, k_j \in \mathcal{H}$ and $m, n \in \mathbb{N}$,

$$\begin{aligned}
& \left(\sum_{i=1}^n \pi(a_i) h_i \left| \psi(\bar{\alpha}(1))^* \sum_{j=1}^m \pi(b_j) k_j \right. \right) = \left(\psi(\bar{\alpha}(1)) \sum_{i=1}^n \pi(a_i) h_i \left| \sum_{j=1}^m \pi(b_j) k_j \right. \right) \\
&= \left(\sum_{i=1}^n \psi(\bar{\alpha}(1) a_i) h_i \left| \sum_{j=1}^m \pi(b_j) k_j \right. \right) = \sum_{i=1}^n \sum_{j=1}^m (\psi(\bar{\alpha}(1) a_i) h_i | \pi(b_j) k_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m (h_i | \psi(\bar{\alpha}(1) a_i)^* \pi(b_j) k_j) = \sum_{i=1}^n \sum_{j=1}^m (h_i | \psi(\bar{\alpha}(1) a_i)^* \pi(b_j) k_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m (h_i | (\pi(b_j^*) \psi(\bar{\alpha}(1) a_i))^* k_j) = \sum_{i=1}^n \sum_{j=1}^m (h_i | (\psi(\alpha(b_j^*) \bar{\alpha}(1) a_i))^* k_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m (h_i | (\psi(\alpha(b_j^*) a_i))^* k_j) = \sum_{i=1}^n \sum_{j=1}^m (h_i | (\psi(\alpha(b_j^*)) \pi(a_i))^* k_j) \\
&= \sum_{i=1}^n \sum_{j=1}^m (h_i | \pi(a_i)^* \psi(\alpha(b_j^*))^* k_j) = \sum_{i=1}^n \sum_{j=1}^m (\pi(a_i) h_i | \psi(\alpha(b_j^*))^* k_j) \\
&= \left(\sum_{i=1}^n \pi(a_i) h_i \left| \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j \right. \right).
\end{aligned}$$

Since $\mathcal{H}_0 = \overline{span}\{\pi(a)h : a \in A, h \in \mathcal{H}\}$, it follows that, for all $h \in \mathcal{H}_0$,

$$\left(h \left| \psi(\bar{\alpha}(1))^* \sum_{j=1}^m \pi(b_j) k_j \right. \right) = \left(h \left| \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j \right. \right).$$

If h in $\mathcal{H}_0^\perp$, $\psi(\bar{\alpha}(1))h = 0$ and by Lemma 4.1 $\psi(\alpha(b_j^*))h = 0$, so

$$\begin{aligned}
& \left(h \left| \psi(\bar{\alpha}(1))^* \sum_{j=1}^m \pi(b_j) k_j \right. \right) = \left(\psi(\bar{\alpha}(1)) h \left| \sum_{j=1}^m \pi(b_j) k_j \right. \right) = 0, \\
& \left(h \left| \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j \right. \right) = \sum_{j=1}^m (\psi(\alpha(b_j^*)) h | \pi(b_j) k_j) = 0,
\end{aligned}$$

and then

$$\left(h \left| \psi(\bar{\alpha}(1))^* \sum_{j=1}^m \pi(b_j) k_j \right. \right) = 0 = \left(h \left| \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j \right. \right).$$

Thus for all $h \in \mathcal{H}$,

$$\left(h \left| \psi(\bar{\alpha}(1))^* \sum_{j=1}^m \pi(b_j) k_j \right. \right) = \left(h \left| \sum_{j=1}^m \psi(\alpha(b_j^*))^* k_j \right. \right),$$

and 4.2 follows. $\square$

Proposition 4.1. Let (ψ, π) be a Toeplitz representation of $X = \bar{\alpha}(1)A$ in which π is non-degenerate, i.e $\mathcal{H} = \overline{\text{span}}\{\pi(a)h \mid a \in A, h \in \mathcal{H}\}$. Then there is a unique bounded operator namely $\psi(\bar{\alpha}(1)) : \mathcal{H} \rightarrow \mathcal{H}$ such that

- (1) $\psi(\bar{\alpha}(1))\pi(a)h = \psi(\bar{\alpha}(1)a)h$,
- (2) $\psi(\bar{\alpha}(1))^*\pi(a)h = \psi(\alpha(a^*))^*h$,
- (3) $\pi(a)\psi(\bar{\alpha}(1))h = \psi(\alpha(a))h$,

for all $a \in A$, $h \in \mathcal{H}$. Moreover, for any approximate identity (e_λ) in A , we have $\psi(\alpha(e_\lambda))$ is convergent $*$ -strongly to $\psi(\bar{\alpha}(1))$, that is $\psi(\alpha(e_\lambda))h \rightarrow \psi(\bar{\alpha}(1))h$ and $\psi(\alpha(e_\lambda))^*h \rightarrow \psi(\bar{\alpha}(1))^*h$ in $\mathcal{H}$ for every $h \in \mathcal{H}$.

Proof. Given any bounded operator such as $\psi(\bar{\alpha}(1))$ satisfying the condition (1) is unique: Suppose T is another operator satisfied (1). Since T and $\psi(\bar{\alpha}(1))$ are linear, for all $a_i \in A$, $h_i \in \mathcal{H}$ and $n \in \mathbb{N}$, we have

$$\psi(\bar{\alpha}(1))\left(\sum_{i=1}^n \pi(a_i)h_i\right) = \sum_{i=1}^n \psi(\bar{\alpha}(1)a_i)h_i = T\left(\sum_{i=1}^n \pi(a_i)h_i\right).$$

So T and $\psi(\bar{\alpha}(1))$ coincide on the dense subspace $\text{span}\{\pi(a)h \mid a \in A, h \in \mathcal{H}\}$ of $\mathcal{H}$ and the continuity of T and $\psi(\bar{\alpha}(1))$ implies that $\psi(\bar{\alpha}(1)) = T$ on $\mathcal{H}$. The rest of the proof is just similar to the Lemma 4.2. $\square$

Proposition 4.2. Let $X = \bar{\alpha}(1)A$ and $\tilde{X} := \bar{\alpha}(1)M(A) = \{\bar{\alpha}(1)m : m \in M(A)\}$ with the operations $x \cdot m := xm$, $\langle x, y \rangle_{M(A)} := x^*y$ and $m \cdot x := \bar{\alpha}(m)x$ for all $x, y \in \tilde{X}$ and $m \in M(A)$. Then for every Toeplitz representation (ψ, π) of X on a Hilbert space $\mathcal{H}$ where π is nondegenerate, there is a Toeplitz representation $(\bar{\psi}, \bar{\pi})$ of $\tilde{X}$ on $\mathcal{H}$ such that $\bar{\psi}|_X = \psi$. In particular, $\bar{\psi}(\bar{\alpha}(1)) = \psi(\bar{\alpha}(1))$.

Proof. Suppose (ψ, π) is a Toeplitz representation of X on a Hilbert space $\mathcal{H}$ and π is nondegenerate. By the Proposition 4.1 there is $\psi(\bar{\alpha}(1)) \in B(\mathcal{H})$ satisfying the related conditions. Take an approximate identity (e_λ) in A . For each λ , $m \in M(A)$

and $\alpha(e_\lambda)m$ in A we have

$$\begin{aligned}
\psi(\alpha(e_\lambda)m) &= \psi(\alpha(e_\lambda)\alpha(e_\lambda)m) \\
&= \psi(\alpha(e_\lambda) \cdot \alpha(e_\lambda)m) \\
&= \psi(\alpha(e_\lambda))\pi(\alpha(e_\lambda)m) \\
&= \psi(\alpha(e_\lambda))\pi(\alpha(e_\lambda))\bar{\pi}(m) \\
&= \psi(\alpha(e_\lambda) \cdot \alpha(e_\lambda))\bar{\pi}(m) \\
&= \psi(\alpha(e_\lambda))\bar{\pi}(m),
\end{aligned}$$

so $\psi(\alpha(e_\lambda)m)h$ converges to $\psi(\bar{\alpha}(1))\bar{\pi}(m)h$ for every $h \in \mathcal{H}$ by part (1) of the Proposition 4.1. Now, define $\bar{\psi} : \tilde{X} \rightarrow B(\mathcal{H})$ by

$$\bar{\psi}(\bar{\alpha}(1)m) := \lim_{\lambda} \psi(\alpha(e_\lambda)m).$$

We claim that $\bar{\psi}$ is well-defined. Assume $\bar{\alpha}(1)m = \bar{\alpha}(1)m'$, then $\bar{\alpha}(1)m - \bar{\alpha}(1)m' = 0$, that is $\alpha(e_\lambda)(m - m')$ converges to zero in $M(A)$, but

$$\|\alpha(e_\lambda)(m - m')\|_A = \|\alpha(e_\lambda)(m - m')\|_{M(A)} = \|\alpha(e_\lambda)m - \alpha(e_\lambda)m'\|_{M(A)}$$

therefore $\alpha(e_\lambda)(m - m')$ converges to zero in A . So, for all $h \in \mathcal{H}$,

$$\psi(\alpha(e_\lambda)m)h - \psi(\alpha(e_\lambda)m'h) = \psi(\alpha(e_\lambda)(m - m'))h$$

converges to zero in $\mathcal{H}$, that is $\psi(\bar{\alpha}(1)\bar{\pi}(m))h = \psi(\bar{\alpha}(1)m')h$ and hence $\bar{\psi}(\bar{\alpha}(1)m) = \bar{\psi}(\bar{\alpha}(1)m')$. In order to show that $(\bar{\psi}, \bar{\pi})$ is a Topelitz representation of $\tilde{X}$ on $\mathcal{H}$, we show that $(\bar{\psi}, \bar{\pi})$ satisfies the following conditions

- (1) $\bar{\psi}(m \cdot \bar{\alpha}(1)n) = \bar{\pi}(m)\bar{\psi}(\bar{\alpha}(1)n)$,
- (2) $\bar{\psi}(\bar{\alpha}(1)n \cdot m) = \bar{\psi}(\bar{\alpha}(1)n)\bar{\pi}(m)$,
- (3) $\bar{\pi}(\langle \bar{\alpha}(1)m, \bar{\alpha}(1)n \rangle_{M(A)}) = \bar{\psi}(\bar{\alpha}(1)m)^* \bar{\psi}(\bar{\alpha}(1)n)$

for all $m, n \in M(A)$. To verify these conditions, first we show that $\psi(\bar{\alpha}(1))$ defined in 4.1 has the following properties:

$$(4.3) \quad \psi(\bar{\alpha}(1))\bar{\pi}(\bar{\alpha}(1)m) = \bar{\pi}(m)\psi(\bar{\alpha}(1)), \quad \psi(\bar{\alpha}(1))^* \psi(\bar{\alpha}(1)) = \bar{\pi}(\bar{\alpha}(1)).$$

For each $h \in \mathcal{H}$ and $m \in M(A)$, $\pi(e_\lambda m)\psi(\bar{\alpha}(1))h \longrightarrow \bar{\pi}(m)\psi(\bar{\alpha}(1))h$, but

$$\begin{aligned}\pi(e_\lambda m)\psi(\bar{\alpha}(1))h &= \psi(\alpha(e_\lambda m))h = \psi(\alpha(e_\lambda)\alpha(e_\lambda m))h \\ &= \psi(\alpha(e_\lambda) \cdot \alpha(e_\lambda m))h = \psi(\alpha(e_\lambda))\pi(\alpha(e_\lambda)\alpha(m))h \\ &= \psi(\alpha(e_\lambda))\pi(\alpha(e_\lambda))\bar{\pi}(\bar{\alpha}(1)m)h = \psi(\alpha(e_\lambda) \cdot \alpha(e_\lambda))\bar{\pi}(\bar{\alpha}(m))h \\ &= \psi(\alpha(e_\lambda))\bar{\pi}(\bar{\alpha}(m))h \\ &\rightarrow \psi(\bar{\alpha}(1))\bar{\pi}(\bar{\alpha}(m)),\end{aligned}$$

so $\psi(\bar{\alpha}(1))\bar{\pi}(\bar{\alpha}(m)) = \bar{\pi}(m)\psi(\bar{\alpha}(1))h$. For the second equation note that

$$\psi(\alpha(e_\lambda))^*\psi(\alpha(e_\lambda))h = \pi(\langle \alpha(e_\lambda), \alpha(e_\lambda) \rangle_A)h = \pi(\alpha(e_\lambda)^*\alpha(e_\lambda))h = \pi(\alpha(e_\lambda))h,$$

so, $\psi(\alpha(e_\lambda))^*\psi(\alpha(e_\lambda))h$ converges to $\bar{\pi}(\bar{\alpha}(1))h$, that is $\psi(\bar{\alpha}(1))^*\psi(\bar{\alpha}(1))h = \bar{\pi}(\bar{\alpha}(1))h$ for all $h \in \mathcal{H}$. Now, for given $m, n \in M(A)$,

$$\begin{aligned}\bar{\psi}(m \cdot \bar{\alpha}(1)n) &= \bar{\psi}(\bar{\alpha}(m)\bar{\alpha}(1)n) = \bar{\psi}(\bar{\alpha}(1)\bar{\alpha}(m)n) \\ &= \bar{\psi}(\bar{\alpha}(1))\bar{\pi}(\bar{\alpha}(m))\bar{\pi}(n) = \bar{\pi}(m)\psi(\bar{\alpha}(1))\bar{\pi}(n) \\ &= \bar{\pi}(m)\bar{\psi}(\bar{\alpha}(1)n),\end{aligned}$$

and part (1) holds. For any $m, n \in M(A)$,

$$\bar{\psi}(\bar{\alpha}(1)n \cdot m) = \bar{\psi}(\bar{\alpha}(1)nm) = \psi(\bar{\alpha}(1))\bar{\pi}(nm) = \psi(\bar{\alpha}(1))\bar{\pi}(n)\bar{\pi}(m) = \bar{\psi}(\bar{\alpha}(1)n)\bar{\pi}(m)$$

and

$$\begin{aligned}\bar{\pi}(\langle \bar{\alpha}(1)m, \bar{\alpha}(1)n \rangle_{M(A)}) &= \bar{\pi}((\bar{\alpha}(1)m)^*\bar{\alpha}(1)n) = \bar{\pi}(m^*\bar{\alpha}(1)\bar{\alpha}(1)n) \\ &= \bar{\pi}(m^*\bar{\alpha}(1)n) = \bar{\pi}(m^*)\bar{\pi}(\bar{\alpha}(1))\bar{\pi}(n) \\ &= \bar{\pi}(m^*)\psi(\bar{\alpha}(1))^*\psi(\bar{\alpha}(1))\bar{\pi}(n) = (\psi(\bar{\alpha}(1))\bar{\pi}(m))^*\psi(\bar{\alpha}(1))\bar{\pi}(n) \\ &= \bar{\psi}(\bar{\alpha}(1)m)^*\bar{\psi}(\bar{\alpha}(1)n).\end{aligned}$$

So, part (2) and (3) follow. $\square$

Lemma 4.3. *If (ψ, π) is a Toeplitz representation of $X = \bar{\alpha}(1)A$ on a Hilbert space $\mathcal{H}$, then (π, V) is a partial-isometric covariant representation of the system (A, α) on $\mathcal{H}$ where $V := \psi(\bar{\alpha}(1))^*$.*

Proof. Showing (π, V) is covariant representation, needs to

- 1) V is a power partial isometry.

$$2) \pi(\alpha(a)) = V\pi(a)V^*, \quad V^*V\pi(a) = \pi(a)V^*V.$$

Suppose $\mathcal{H}_0 := \overline{\text{span}}\{\pi(a)h \mid a \in A, h \in \mathcal{H}\}$ and $h \in \mathcal{H}_0^\perp$. Then $\psi(\bar{\alpha}(1))\psi(\bar{\alpha}(1))^*\psi(\bar{\alpha}(1))h = 0 = \psi(\bar{\alpha})^*h$. For an approximate identity (e_λ) in A , $a_i \in A$, and $h_i \in \mathcal{H}$,

$$\begin{aligned} \psi(\bar{\alpha}(1))\psi(\alpha(e_\lambda))^*\psi(\bar{\alpha}(1))\left(\sum_{i=1}^n \pi(a_i h_i)\right) &= \psi(\bar{\alpha}(1))\psi(\alpha(e_\lambda))^*\left(\sum_{i=1}^n \psi(\bar{\alpha}(1)a_i)h_i\right) \\ &= \psi(\bar{\alpha}(1))\left(\sum_{i=1}^n \psi(\alpha(e_\lambda))^*\psi(\bar{\alpha}(1)a_i)h_i\right) \\ &= \psi(\bar{\alpha}(1))\left(\sum_{i=1}^n \pi(\langle \alpha(e_\lambda), \bar{\alpha}(1)a_i \rangle)\right) \\ &= \psi(\bar{\alpha}(1))\left(\sum_{i=1}^n \pi(\alpha(e_\lambda a_i))h_i\right) \end{aligned}$$

This implies that $\psi(\bar{\alpha}(1))\psi(\alpha(e_\lambda))^*\psi(\bar{\alpha}(1))h$ converges to $\psi(\bar{\alpha}(1))h$, that is, $V^*VV^* = V^*$ and hence V is partial isometry. The Proposition 4.2 together with part (1) of the Proposition 4.1 imply that for all $a \in A$

$$\pi(a)V^* = \pi(a)\bar{\psi}(\bar{\alpha}(1)) = \psi(\alpha(a)) = \psi(\bar{\alpha}(1))\pi(\alpha(a)) = V^*\pi(\alpha(a)),$$

and by taking adjoints, $V\pi(a) = \pi(\alpha(a))V$. This shows that

$$V^*V\pi(a) = V^*\pi(\alpha(a))V = \pi(a)V^*V,$$

and hence, $(V^*)^n V^n \pi(a) = \pi(a)(V^*)^n V^n$.

Now, for all $h \in \mathcal{H}$,

$$\begin{aligned} (4.4) \quad V\pi(a)V^*h &= V\psi(\alpha(a))h = \lim \psi(\alpha(e_\lambda))^*\psi(\alpha(a))h \\ &= \lim \pi(\langle \alpha(e_\lambda), \alpha(a) \rangle)h = \lim \pi(\alpha(e_\lambda a))h \\ &= \pi(\alpha(a))h. \end{aligned}$$

Setting $a = e_\lambda$ in the equation (4.4), $V^n\pi(e_\lambda)(V^*)^n = \pi(\alpha^n(e_\lambda))$, which is a projection. Therefore, $V^n\pi(e_\lambda)\psi(\bar{\alpha}(1))(V^*)^{n-1}h = V^n\psi(\alpha(e_\lambda))(V^*)^{n-1}h$ converges to $V^n(V^*)^n h$. So it is a projection and then V^n is a partial isometry; V is power partial isometry. $\square$

Proof of Theorem 3.1. Let $(A \times_\alpha^{iso} \mathbb{N}, i_A, i_{\mathbb{N}})$ be a crossed product of the system (A, α) . By [4, Proposition 1.3] it is enough to show that (i_X, i_A) is a Toeplitz representation of the Hilbert bimodule $X := \bar{\alpha}(1)A$ where $i_X(x) := i_{\mathbb{N}}(1)^*i_A(x)$ for all $x \in X$ which satisfies the following:

a) for every Toeplitz representation (ψ, π) of X on a Hilbert space $\mathcal{H}$, there is homomorphism $\psi \times \pi$ from $A \times_{\alpha}^{piso} \mathbb{N}$ to $B(\mathcal{H})$ with $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$,

b) $A \times_{\alpha}^{piso} \mathbb{N}$ is generated by $i_X(X) \cup i_A(A)$.

For each $x = \overline{\alpha}(1)a, y \in X$ and $b \in A$,

$$i_X(x \cdot b) = i_X((\overline{\alpha}(1)a) \cdot b) = i_{\mathbb{N}}(1)^* i_A(\overline{\alpha}(1)ab) = i_{\mathbb{N}}(1)^* i_A(\overline{\alpha}(1)a) i_A(b) = i_X(x) i_A(b),$$

$$\begin{aligned} i_X(b \cdot x) &= i_X(\alpha(b)\overline{\alpha}(1)a) = i_{\mathbb{N}}(1)^* i_A(\alpha(b)\overline{\alpha}(1)a) = i_{\mathbb{N}}(1)^* i_A(\alpha(b)) i_A(\overline{\alpha}(1)a) \\ &= i_A(b) i_{\mathbb{N}}(1)^* i_A(\overline{\alpha}(1)a) = i_A(b) i_X(x). \end{aligned}$$

Suppose (e_{λ}) is an approximate identity in A . So,

$$\begin{aligned} i_A(\langle \alpha(e_{\lambda})a, y \rangle_A) &= i_A(\alpha(e_{\lambda})a)^* y = i_A(a^*) i_A(\alpha(e_{\lambda})) i_A(y) = i_A(a^*) i_A(\alpha(e_{\lambda})) i_A(y) \\ (4.5) \quad &= i_A(a^*) i_{\mathbb{N}}(1) i_A(e_{\lambda}) i_{\mathbb{N}}(1)^* i_A(y) = i_A(a^*) i_A(\alpha(e_{\lambda})) i_{\mathbb{N}}(1) i_X(y) \\ &= (i_{\mathbb{N}}(1)^* i_A(\alpha(e_{\lambda}) i_A(a))^* i_X(y) = (i_{\mathbb{N}}(1)^* i_A(\alpha(e_{\lambda})a))^* i_X(y). \end{aligned}$$

Since $\alpha(e_{\lambda})a$ converges to $\overline{\alpha}(1)a$, by 4.5

$$i_A(\langle x, y \rangle_A) = (i_{\mathbb{N}}(1)^* i_A(x))^* i_X(y) = i_X(x)^* i_X(y).$$

Thus (i_X, i_A) is a Toeplitz representation of X on $A \times_{\alpha}^{piso} \mathbb{N}$. Assume (ψ, π) is a Toeplitz representation of X on a Hilbert space $\mathcal{H}$. Then Lemma 4.3 says $(\pi, V := \psi(\overline{\alpha}(1))^*)$ is a covariant partial-isometry of the system (A, α) on $\mathcal{H}$. This implies that there is a nondegenerate representation $\pi \times V : A \times_{\alpha}^{piso} \mathbb{N} \rightarrow B(\mathcal{H})$ such that $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_{\mathbb{N}} = V$. But $(\pi \times V) \circ i_A = \pi$ and for all $a \in A$,

$$\begin{aligned} (\pi \times V) \circ i_X(\overline{\alpha}(1)a) &= (\pi \times V)(i_{\mathbb{N}}(1)^* i_A(\overline{\alpha}(1)a)) = V^* \pi(\overline{\alpha}(1)a) \\ &= \psi(\overline{\alpha}(1)) \pi(\overline{\alpha}(1)a) = \psi(\overline{\alpha}(1)a). \end{aligned}$$

To verify condition (b), note that $i_{\mathbb{N}}(1)$ belongs to the C^* -algebra generated by $i_X(X) \cup i_A(A)$ because for an approximate identity (e_{λ}) in A ,

$$i_X(\alpha(e_{\lambda})) = i_{\mathbb{N}}(1)^* i_A(\alpha(e_{\lambda})) = i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(1) i_A(e_{\lambda}) i_{\mathbb{N}}(1)^*$$

converges to

$$i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(1) \overline{i_A(1)} i_{\mathbb{N}}(1)^* = i_{\mathbb{N}}(1)^* i_{\mathbb{N}}(1) i_{\mathbb{N}}(1)^* = i_{\mathbb{N}}(1)^*$$

and hence, $i_{\mathbb{N}}(1)$ is in the C^* -algebra generated by $i_X(X) \cup i_A(A)$.

Now, let $(i_A, i_{\mathbb{N}})$ be the universal isometric covariant representation of the crossed product $A \times_{\alpha}^{iso} \mathbb{N}$. Since $i_{\mathbb{N}}$ is an isometric representation, so it is partial isometry as well. Consequently, setting $i_X(x) := i_{\mathbb{N}}(1)^* i_A(x)$ for all $x \in X$, (i_X, i_A) is a Toeplitz representation of X on $A \times_{\alpha}^{iso} \mathbb{N}$ satisfying (a), (b) and (c).

5. INVARIANT IDEALS

In this section, we assume an ideal I is invariant under an extendible endomorphism α on a A . In this case, we have the crossed products $I \times_{\alpha} \mathbb{N}$ and $(A/I) \times_{\tilde{\alpha}} \mathbb{N}$. It is investigated how they are related to each other.

Let I be an ideal in A and X be Hilbert bimodule over A . Then

$$X_I := \{x \in X : \langle x, y \rangle_A \in I \text{ for all } y \in X\}$$

is a Hilbert bimodule over I . One can show that

$$X_I = XI := \{x \cdot i : x \in X, i \in I\}$$

is closed.

Definition 5.1. Let X be a Hilbert bimodule over a C^* -algebra A with φ as its left action. An ideal I in A is called X -invariant if $\varphi(I)X \subseteq XI$

In order to construct the quotient Hilbert bimodule X/XI , the ideal I need to be X -invariant. So, if I is an X -invariant ideal, then according to [5, Lemma 2.3] X/XI is a Hilbert bimodule over A/I with

$$(x + XI) \cdot (a + I) := x \cdot a + XI \text{ for } x \in X \text{ and } a \in A, \text{ and}$$

$$\langle x + XI, y + XI \rangle_{A/I} := \langle x, y \rangle_A + I \text{ for } x, y \in X,$$

and the left action is given by the homomorphism $\varphi_{A/I} : A/I \rightarrow \mathcal{L}(X/XI)$ satisfying

$$\varphi_{A/I}(a + I)(x + XI) = a \cdot x + XI.$$

Lemma 5.1. *If I is an extendibly α -invariant ideal in A , and $X = \overline{\alpha}(1)A$, then XI is a Hilbert bimodule over I and $XI = \overline{\alpha}(1_{M(A)})I = \overline{\alpha}(1_{M(I)})I$. Moreover, I is X -invariant.*

Proof. As we know $XI = \{x \cdot i : x \in X, i \in I\} = \{\overline{\alpha}(1_{M(A)})ai : i \in I, a \in A\} \subseteq \overline{\alpha}(1_{M(A)})I$.

Suppose (e_λ) is an approximation identity in A . For $\overline{\alpha}(1_{M(A)})i$ in $\overline{\alpha}(1_{M(A)})I$,

$$\overline{\alpha}(1_{M(A)})i = \lim_{\lambda} \overline{\alpha}(1_{M(A)})\alpha(e_\lambda)i$$

is in XI , because XI is closed; $\overline{\alpha}(1_{M(A)})I \subseteq XI$. Since I is α -extendible, there is a nondegenerate homomorphism $\psi : A \rightarrow M(I)$ such that $\overline{\psi}(\overline{\alpha}(1_{M(A)})) = \overline{\alpha}|_I(1_{M(I)})$, so for all $i \in I$,

$$(5.1) \quad \overline{\alpha}|_I(1_{M(I)})i = \overline{\psi}(\overline{\alpha}(1_{M(A)}))i = \overline{\psi}(\overline{\alpha}(1_{M(A)}))i = \psi(\overline{\alpha}(1_{M(A)}))i = \overline{\alpha}(1_{M(A)})i$$

This implies that $XI = \overline{\alpha}(1_{M(A)})I = \overline{\alpha}(1_{M(I)})I$.

For each $i \in I, a \in A$, $\varphi(i)\overline{\alpha}(1_{M(A)})a = \alpha(i)\overline{\alpha}(1_{M(A)})a = \alpha(i)a \in XI$ because I is α -invariant. So, $\varphi(I)X \subseteq XI$ and I is X -invariant. Thus XI is a Hilbert bimodule over I . $\square$

Theorem 5.1. *Let I be an α -extendible ideal in A , $X := \overline{\alpha}(1)A$, $(A \times_{\alpha}^{piso} \mathbb{N}, i_A, i_{\mathbb{N}})$ be a crossed product for (A, α) , and $(I \times_{\alpha}^{piso} \mathbb{N}, j_I, j_{\mathbb{N}})$ be a crossed product for $(I, \alpha|_I)$. Then there is a homomorphism $\rho : I \times_{\alpha}^{piso} \mathbb{N} \rightarrow A \times_{\alpha}^{piso} \mathbb{N}$ which is injective and maps $I \times_{\alpha}^{piso} \mathbb{N}$ onto $\mathcal{J}(I)$ the ideal generated by $i_A(I)$ in $A \times_{\alpha}^{piso} \mathbb{N}$.*

Proof. We know by Theorem 3.1 the Toeplitz algebras $\mathcal{T}_X$ and $\mathcal{T}_{XI}$ are $(A \times_{\alpha}^{piso} \mathbb{N}, i_X, i_A)$ and $(I \times_{\alpha}^{piso} \mathbb{N}, j_{XI}, j_I)$ respectively where $i_X(x) := i_{\mathbb{N}}(1)^* i_A(x), x \in X$, and $j_{XI}(x) := j_{\mathbb{N}}(1)^* j_I(x), x \in XI$. Observe that $(i_X|_{XI}, i_A|_I)$ is a Toeplitz representation of XI in $A \times_{\alpha}^{piso} \mathbb{N}$. So, there is a homomorphism $\rho : I \times_{\alpha}^{piso} \mathbb{N} \rightarrow A \times_{\alpha}^{piso} \mathbb{N}$ such that $\rho \circ j_X = i_X|_{XI}$, $\rho \circ j_I = i_A|_I$. In addition, as in [5, the proof of Proposition 3.4], ρ is injective. We claim that $Imp \rho = \mathcal{J}(I)$. Since $\rho(j_{XI}(x)) = i_X(x), \rho(j_I(i)) = i_A(i)$, and $I \times_{\alpha}^{piso} \mathbb{N}$ is generated by $j_{XI}(XI) \cup j_I(I)$, so the range of ρ is generated by $i_X(XI) \cup i_A(I)$ and hence $Imp \rho \subseteq \mathcal{J}(I)$. Since $\mathcal{J}(I)$ is generated by $i_A(I)$, so for all $i \in I, a \in A$ and $x \in X$, the elements of the form

$$\begin{aligned} i_X(x)i_A(i) &= i_X(xi) = \rho(j_{XI}(xi)), \\ i_A(i)i_X(x) &= i_X(\alpha(i)x) = \rho(j_{XI}(\alpha(i)x)), \\ i_X(x)^* i_A(i) &= (i_A(i^*)i_X(x))^*, \\ i_A(i)i_X(x)^* &= (i_X(x)i_A(i^*))^* \end{aligned}$$

and $i_A(a)i_A(i)$ in $\mathcal{J}(I)$, belong to $Imp \rho$. It follows that $\mathcal{J}(I) \subseteq Imp \rho$. $\square$

Corollary 5.1. *If I is an α -extendible ideal in A , $(A \times_{\alpha}^{piso} \mathbb{N}, i_A, i_{\mathbb{N}})$ is a crossed product for the dynamical system (A, α) , and $X = \overline{\alpha}(1)A$, then the ideal $\mathcal{J}(I)$ generated by $i_A(I)$ in $\mathcal{T}_X$ is isomorphic to $\mathcal{T}_{XI}$.*

Proof. It follows by the fact that $XI = \overline{\alpha|_I}(1_{M(I)})I$ is a Hilbert bimodule over I and by Theorem 3.1 $\mathcal{T}_{XI} \cong I \times_{\alpha}^{piso} \mathbb{N}$. $\square$

Lemma 5.2. *Suppose (A, α) and (B, β) are two isomorphic dynamical systems in the sense that there is an isomorphism $\phi : A \rightarrow B$ such that $\phi \circ \alpha = \beta \circ \phi$. If $X := \overline{\alpha}(1)A$ is the Hilbert bimodule over A and Y is the Hilbert bimodule $\overline{\beta}(1)B$ over B , then X is isomorphic to Y as Hilbert bimodules.*

Proof. Define $T : X \rightarrow Y$ by $T(\overline{\alpha}(1)a) = \overline{\beta}(1)\phi(a)$. If $\overline{\alpha}(1)a = \overline{\alpha}(1)b$, for an approximate identity (e_λ) in A , $\alpha(e_\lambda)a$ and $\alpha(e_\lambda)b$ converge to $\overline{\alpha}(1)a$ and $\overline{\alpha}(1)b$ respectively, furthermore,

$$\phi(\alpha(e_\lambda)a) = \phi(\alpha(e_\lambda))\phi(a) = \beta(\phi(e_\lambda))\phi(a)$$

converges to $\overline{\beta}(1)\phi(a)$ because ϕ is onto and $(\phi(e_\lambda))$ is an approximate identity in B . Continuity of ϕ implies that

$$\overline{\beta}(1)\phi(b) = \phi(\overline{\alpha}(1)b) = \phi(\overline{\alpha}(1)a) = \overline{\beta}(1)\phi(a)$$

which says T is well-defined. Suppose $\overline{\alpha}(1)a \in \ker T$. So, $\overline{\beta}(1)\phi(a) = 0$, i.e $\beta(b_\lambda)\phi(a) \rightarrow 0$ where $b_\lambda = \phi(e_\lambda)$, it follows that

$$\beta(\phi(e_\lambda))\phi(a) = \phi(\alpha(e_\lambda)a) \rightarrow 0$$

, $\phi(\overline{\alpha}(1)a) = 0$ and hence $\overline{\alpha}(1)a = 0$. Since ϕ is onto so is T . To show that T is isometric and then is bounded, we observe that for an approximate identity (e_λ) in A , $(b_\lambda) := (\phi(e_\lambda))$ is an approximate identity in B , so

$$\begin{aligned} \|T(\overline{\alpha}(1)a)\| &= \|\overline{\beta}(1)\phi(a)\| = \lim_{\lambda} \|\beta(b_\lambda)\phi(a)\| \\ &= \lim_{\lambda} \|\phi \circ \alpha \circ \phi^{-1}(b_\lambda)\phi(a)\| = \lim_{\lambda} \|\phi(\alpha(e_\lambda))\phi(a)\| \\ &= \lim_{\lambda} \|\phi(\alpha(e_\lambda)a)\| = \lim_{\lambda} \|\alpha(e_\lambda)a\| \\ &= \|\overline{\alpha}(1)a\|. \end{aligned}$$

$\square$

Lemma 5.3. *Let X, Y be two Hilbert bimodule over a C^* -algebra A and $T : X \rightarrow Y$ be an onto A -linear homomorphism, in the sense that $T(a \cdot x) = a \cdot T(x)$ and $T(x \cdot a) = T(x) \cdot a$ for all $a \in A$, $x \in X$, and $\langle Tx, Ty \rangle_A = \langle x, y \rangle_A$ for all $x, y \in X$. Then $\mathcal{T}_X$ is isomorphic to $\mathcal{T}_Y$.*

Proof. Since by definition X and Y are unitary equivalent, the proof is straightforward. $\square$

Proposition 5.1. Let I be an α -extendible ideal in a C^* -algebra A , and $X = \overline{\alpha}(1)A$ the Hilbert bimodule with $x \cdot a = xa$, $a \cdot x = \alpha(a)x$ and $\langle x, y \rangle_A = x^*y$. Then the Toeplitz algebra $\mathcal{T}_{X/XI}$ is isomorphic to the crossed product $A/I \times_{\alpha^{A/I}}^{iso} \mathbb{N}$.

Proof. Since α is extendible, so $\alpha^{A/I} : A/I \rightarrow A/I$ defined by $\alpha^{A/I}(a+I) = \alpha(a)+I$ is extendible. Suppose $Y := \overline{\alpha^{A/I}}(1)A/I$. By Theorem 3.1 $\mathcal{T}_Y \cong A/I \times_{\alpha^{A/I}}^{iso} \mathbb{N}$. According to Lemma 5.3, it is enough to show that there is an isomorphism $T : X/XI \rightarrow Y$. Define $T : X/XI \rightarrow Y$ by

$$\overline{\alpha}(1)a + XI \mapsto \overline{\alpha^{A/I}}(1)(a + I).$$

For an approximate identity (e_λ) in A , $(e_\lambda + I)$ is an approximate identity in A/I . This implies that for all $a \in A$, $\overline{\alpha}(1)a + I = \overline{\alpha^{A/I}}(1)(a + I)$ because

$$\overline{\alpha}(1)a + I = \lim_{\lambda} \alpha(e_\lambda)a + I = \lim_{\lambda} (\alpha(e_\lambda) + I)(a + I) = \overline{\alpha^{A/I}}(1)(a + I).$$

Indeed, $T(x + XI) = x + I$. If $x + XI = y + XI$, then $x - y \in XI$. Since $XI \subseteq I$, so $x + I = y + I$, and hence $T(x + XI) = T(y + XI)$. Thus T is well-defined and obviously is onto. For each $a \in A$ and $x \in X$,

$$\begin{aligned} T((x + XI) \cdot (a + I)) &= T(xa + XI) = xa + I \\ &= (xb + I)(a + I) = (x + I) \cdot (a + I) = T(x + XI) \cdot (a + I), \end{aligned}$$

moreover,

$$\begin{aligned} T((a + I) \cdot (x + XI)) &= T(\alpha(a)x + XI) = \alpha(a)x + I \\ (5.2) \quad &= (\alpha(a) + I)(x + I) = \alpha^{A/I}(a + I)(x + XI) \\ &= (a + I) \cdot T(x + XI). \end{aligned}$$

This shows that T is A/I -linear. Note that for all $x = \overline{\alpha}(1)a, y = \overline{\alpha}(1)b \in X$,

$$\begin{aligned} (5.3) \quad \langle T(x + XI), T(y + XI) \rangle_{A/I} &= \langle \overline{\alpha^{A/I}}(1)(a + I), \overline{\alpha^{A/I}}(1)(b + I) \rangle_{A/I} \\ &= \langle x + I, y + I \rangle_{A/I} \\ &= \langle x, y \rangle_A + I = \langle x + XI, y + XI \rangle_{A/I}, \end{aligned}$$

so T preserves the inner product. $\square$

The main result comes in the next corollary.

Corollary 5.2. *Let α be an endomorphism of a C^* -algebra A , and I be an extendibly α -invariant ideal in A . Then we have the exact sequences*

$$0 \rightarrow I \times_{\alpha}^{iso} \mathbb{N} \rightarrow A \times_{\alpha}^{iso} \mathbb{N} \rightarrow A/I \times_{\alpha_{A/I}}^{iso} \mathbb{N} \rightarrow 0$$

Proof. It was shown in Theorem 5.1 that there is an injective homomorphism $\rho : I \times_{\alpha}^{iso} \mathbb{N} \rightarrow A \times_{\alpha}^{iso} \mathbb{N}$ which its image is the ideal $\mathcal{J}(I)$ generated by $i_A(I)$ in $A \times_{\alpha}^{iso} \mathbb{N}$, then we have the natural quotient map $q : A \times_{\alpha}^{iso} \mathbb{N} \rightarrow A \times_{\alpha}^{iso} \mathbb{N} / \mathcal{J}(I)$. But [5, Corollary 3.2] gives an isomorphism $A \times_{\alpha}^{iso} \mathbb{N} / \mathcal{J}(I) \cong \mathcal{T}_{X/XI}$, and by Proposition 5.1 $\mathcal{T}_{X/XI} \cong A/I \times_{\alpha_{A/I}}^{iso} \mathbb{N}$, so we can get the first exact sequence. To get the second exact sequence, similar argument and change piso by iso together with [5, Corollary 3.3] will be enough. $\square$

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