COEXISTENCE OF ANTI–PHASE AND COMPLETE SYNCHRONIZATION IN A CHAOTIC FINANCE SYSTEM

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Abstract. In this paper, the coexistence of anti–phase and complete synchronization in a three–dimensional chaotic finance system is investigated. Linear feedback control method and the adaptive feedback control method are presented. In both cases, sufficient conditions for the synchronization are obtained analytically. Numerical simulations show the validity and feasibility of the analytical results.

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1. Introduction

Chaos describes a kind of behaviour in which any uncertainty in the initial state is amplified exponentially with time, so that the system behaves is as unpredictable. Such behaviour is found in many areas of physics, in the presence of nonlinearity and feedback. Financial markets also show nonlinearity and feedback, and the often wild fluctuations of share prices could be due to a kind of chaotic dynamics. Both

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academic and applied researchers studying financial markets and other economic series have become interested in the topic of chaotic dynamics. The possibility of chaos in financial markets opens important questions for both economic theorists as well as financial market participants [3].

It has now been almost ten years since economists began searching for chaotic dynamics in economic time series. This search has yielded deeper understandings of the dynamics of many different series, and has led to the development of several useful tests for nonlinear structure. However, the direct evidence for deterministic chaos in many economic series remains weak. The possibilities of chaos in economic systems brought an enormous amount of initial interest. The concepts of limited forecastability and complex dynamical properties has very strong intuitive appeal for economics. From forecasting movements in foreign exchange and stock markets, to understanding international business cycles, chaos in economics had a broad range of potential applications. This led to an explosion of empirical work searching for possible chaos in all types of economic and financial time series [2, 4, 16].

Chaos synchronization has been studied since the pioneering work of Pecora and Carroll [12] was published. It has attracted great attention due to it superior potential applications, for examples, in communication, laser physics and optics, mechanics, and chemistry. Many methods have been developed for synchronizing of chaos such as, linear control [17, 14], LMI–based approach [10, 9], backstepping design [5], sliding control [13], adaptive control [1], active control [11], and passivity–based control [8].

However, most of the existing works focus on investigating the same kind synchronization in a given system, i.e., all the states of the response system have the same kind synchronization to the corresponding states of the drive system. For example, when we say that two systems are complete synchronized (or lag synchronized, or something else) with each other, it means that each pair of the states between the interactive systems is complete synchronous (or lag synchronous, or something else). More recently, Zhang et al. [18] discussed a class of new synchronization phenomenon, two kinds of synchronization coexist in the same system, i.e., one part states of the interactive systems is anti-phase synchronous, and the other part is complete synchronous under certain parameter region.

In this paper, we consider a three–dimensional chaotic finance system and we investigate the coexistence of anti–phase and complete synchronization of two identical finance system. This paper is organized as follows. In Section 2, the dynamical system is introduced. In section 3, linear feedback control method is presented. In section 4, the adaptive feedback control method is designed. Numerical simulations are given in section 5. We conclude the paper in section 6.

2. System description

Chaos in economics brought deep impact into western mainstream economics, because the chaos in economics–system means the inherent instability in macroeconomics. In this section, we consider the following three–dimensional continuous autonomous finance system:

$$\begin{aligned} \dot{x}_1 &= x_3 + (x_2 - a)x_1, \\ \dot{x}_2 &= 1 - bx_2 - x_1^2, \\ \dot{x}_3 &= -x_1 - cx_3 \end{aligned} \tag{1}$$

where x_1 denotes the interest rate, x_2 denotes the investment demand and x_3 denotes the price index. The parameter $a \geq 0$ denotes the saving amount, $b \geq 0$ denotes the investment cost and $c \geq 0$ denotes the commodities demand elasticity.

- Interest rate is a rate which is charged or paid for the use of money. An interest rate is often expressed as an annual percentage of the principal. It is calculated by dividing the amount of interest by the amount of principal.
- Investment demand refers to the demand by businesses for physical capital goods and services used to maintain or expand its operations. Think of it as the office or factory space, machinery, computers, desks, etc, used to operate a business.
- Price index is Percentage number that shows the extent to which a price (or a 'basket' of prices) has changed over a period (month, quarter, year) as compared with the price(s) in a certain year (base year) taken as a standard.
- Saving is the simple process of putting aside a part of your earnings, usually in the from of cash in hand or a savings account, or in the from of some highly liquid (and safe) instruments such as govermant issued treasury bills.

- Investment costs are those program costs required beyond the development phase to introduce into operational use a new capability; to procure initial, additional, or replacement equipment for operational forces; or to provide for major modifications of an existing capability. They exclude research, development, test and evaluation, military personnel, and operation and maintenance appropriation costs.
- Demand elasticity, also known as price elasticity of demand, is a concept economists use to measure price sensitivity. In principle, demand elasticity is an economic term defined as the percentage change in quantity demanded, divided by the percentage change in price.

System (1) is a chaotic finance system based on production, currency, securities and labour force. The authors of Ref. [15] applied adaptive control to study the synchronization of the system (1). The authors of Ref. [6] and Ref. [7] show us the bifurcation topological structure and the global complicated character of the system (1).

The system (1) has the only fixed point $P = (0, 1/b, 0)$, if $c - b - abc < 0$ and it has three fixed points $Q^{\pm} = (\pm\sqrt{(c - b - abc)/c}, (1 + ac)/c, \mp 1/c\sqrt{(c - b - abc)/c})$ and $P = (0, 1/b, 0)$, if $c - b - abc \geq 0$. When $b \in [0, 3]$ varies, the corresponding Lyapunov exponent spectrum of system (1) are shown in Figure 1. The bifurcation diagram with respect to $b \in [0, 3]$ is given in Figure 2. It can be observed that the bifurcation diagram well coincides with the spectrum of Lyapunov exponents. For the parameter vector $(a, b, c) = (0.9, 0.2, 1.2)$, the Lyapunov exponents are $\lambda_1 = 0.0938 > 0$, $\lambda_2 = -0.0024$ and $\lambda_3 = -0.7045$. We can observe that there is a positive Lyapunov exponent, which implies that this system is chaotic. Figure 3 shows the corresponding chaotic attractor.

Throughout the paper, the drive system is chosen as the finance system (1) and the response system is

$$
\begin{aligned}
\dot{y}_1 &= y_3 + (y_2 - a)y_1 + u_1, \\
\dot{y}_2 &= 1 - by_2 - y_1^2 + u_2, \\
\dot{y}_3 &= -y_1 - cy_3 + u_3
\end{aligned}
\tag{2}
$$

where u_1, u_2 and u_3 are controllers to be designed to achieve the coexistence of anti–phase and complete synchronization between systems (1) and (2), i.e., the first

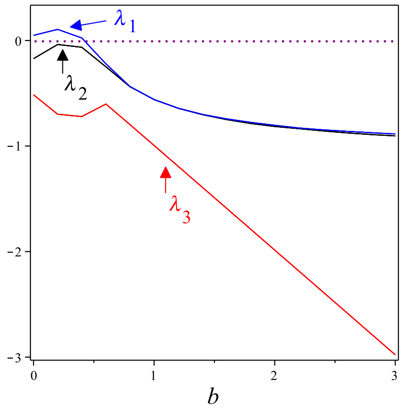

FIGURE 1. Lyapunov exponents of system (1) with $a = 0.9$, $c = 1.2$ and $b \in [0, 3]$.

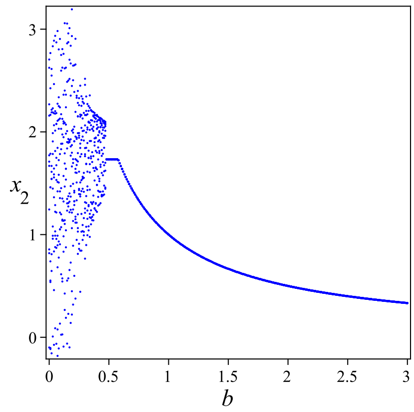

FIGURE 2. Bifurcation diagram of system (1) with $a = 0.9$, $c = 1.2$ and $b \in [0, 3]$.

and the third pairs of state variables between the interactive systems are anti–phase synchronous, while the second pair of state variable is complete synchronous. To this end, we define the error variable as

$$e = (e_1, e_2, e_3)^T = (x_1 + y_1, x_2 - y_2, x_3 + y_3)^T.$$

Then the error system is given by

$$\begin{aligned} \dot{e}_1 &= e_3 + x_1 e_2 + x_2 e_1 - e_1 e_2 - a e_1 + u_1, \\ \dot{e}_2 &= -b e_2 + e_1^2 - 2 x_1 e_1 - u_2, \\ \dot{e}_3 &= -e_1 - c e_3 + u_3. \end{aligned} \tag{3}$$

Next, we will show that the control laws u_1, u_2 and u_3 can be chosen such that the zero solution of the error system (3) is globally exponentially stabilized. Thus, systems (1) and (2) are globally exponentially synchronized.

3. Linear feedback control method

Many methods and techniques have been developed to synchronize the chaotic systems, but the linear feedback control method is one of the most popular methods because it is simple for the theoretical analysis and implementation. In this section,

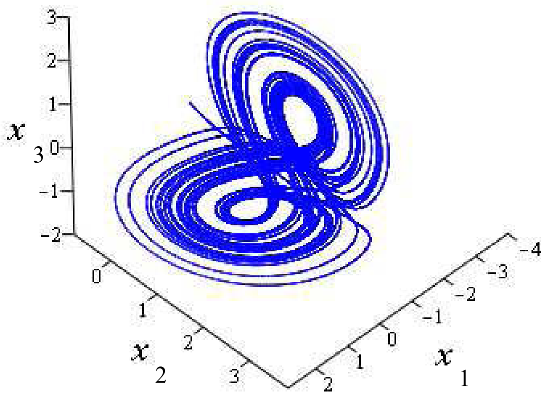

FIGURE 3. The phase portrait of the system (1) with parameters $a = 0.9$, $b = 0.2$ and $c = 1.2$.

we consider the linear feedback method to study the coexistence of anti–phase and complete synchronization of two identical finance system. We take three linear feedback control laws as follows:

$$u_1 = -k_1e_1, \quad u_2 = -k_2e_2, \quad u_3 = -k_3e_3 \tag{4}$$

where k_1, k_2, k_3 are the feedback gain coefficients to be specified in the following theorem.

Theorem 1. *The anti–phase synchronization and complete synchronization coexistent in the system* (1) *and system* (2) *if the controllers are selected as* (4) *and the following conditions holds:*

(1) $k_1 > N - a$,
(2) $k_3 > -c$,
(3) $(a + k_1 - N)(b - k_2) > \frac{M^2}{4}$

where $|x_1| < M$ *and* $|x_2| < N$ *such that* M *and* N *are upper bounds of the absolute variables* x_1 *and* x_2*, respectively.*

Proof. Choose a positive definite Lyapunov function V for the system (3) as follows

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{5}$$

The time derivative of V along solutions of system (3) gives

$$\begin{aligned}
\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 \\
&= e_1^2(x_2 - a - k_1) + e_2^2(-b + k_2) + e_3^2(-c - k_3) - x_1 e_1 e_2 \\
&\leq e_1^2(N - a - k_1) + e_2^2(-b + k_2) + e_3^2(-c - k_3) + M|e_1||e_2| \\
&= -(|e_1|\ |e_2|\ |e_3|)P(|e_1|\ |e_2|\ |e_3|)^T
\end{aligned}$$

where

$$P = \begin{pmatrix} a + k_1 - N & -\frac{M}{2} & 0 \\ -\frac{M}{2} & b - k_2 & 0 \\ 0 & 0 & c + k_3 \end{pmatrix}.$$

Obviously, P is positive definite under the conditions of the theorem 1, which implies that $\dot{V}$ is negative definite. Consequently, according to the Lyapunov stability theorem, the error system (3) can converge to the origin asymptotically. This implies the result of the theorem. □

4. Adaptive feedback control method

Although the linear feedback method is easy in the form, it is not precise because the upper bounds M and N in the finance system are not easy to determine and it is very difficult to determine the upper bound of the gains k_i, $i = 1, 2, 3$. People usually take the gain big enough to ensure synchronization and this leads to a waste in practice. In the following, we take the adaptive feedback technique in which we need not to know the upper bound of the finance system. To this end, we take the controllers as

$$u_1 = -k_1(t)e_1, \quad u_2(t) = -k_2(t)e_2 \quad u_3 = -k_3(t)e_3 \tag{6}$$

where $k_i(t)$, $i = 1, 2, 3$ are a function of t and satisfy the following equations

$$\dot{k}_1(t) = e_1^2, \quad \dot{k}_2(t) = e_2^2, \quad \dot{k}_3(t) = e_3^2. \tag{7}$$

Theorem 2. *The anti–phase synchronization and complete synchronization coexistent in the system* (1) *and system* (2) *if the controllers are selected as* (6) *and the conditions* (7) *holds.*

Proof. Choose a positive definite Lyapunov function V for the system (3) as follows

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + (k_1 - k_1^*)^2 - (k_2 - k_2^*)^2 + (k_3 - k_3^*)^2) \tag{8}$$

where k_i, $i = 1, 2, 3$ are constants satisfying

$$\begin{aligned} & k_1^* > N - a, \\ & k_3^* > -c, \\ & (a + k_1^* - N)(b - k_2^*) > \frac{M^2}{4}. \end{aligned} \tag{9}$$

such that M and N are upper bounds of the absolute variables x_1 and x_2, respectively. The time derivative of V along solutions of system (3) gives

$$\begin{aligned} \dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \dot{k}_1(k_1 - k_1^*) - \dot{k}_2(k_2 - k_2^*) + \dot{k}_(k_3 - k_3^*) \\ &= e_1^2(x_2 - a - k_1^*) + e_2^2(-b + k_2^*) + e_3^2(-c - k_3^*) - x_1e_1e_2 \\ &\leq e_1^2(N - a - k_1^*) + e_2^2(-b + k_2^*) + e_3^2(-c - k_3^*) + M|e_1||e_2| \\ &= -(|e_1|\ |e_2|\ |e_3|)P(|e_1|\ |e_2|\ |e_3|)^T \\ &= -e^TPe \\ &\leq 0 \end{aligned}$$

where

$$P = \begin{pmatrix} a + k_1^* - N & -\frac{M}{2} & 0 \\ -\frac{M}{2} & b - k_2^* & 0 \\ 0 & 0 & c + k_3^* \end{pmatrix}.$$

Obviously, from the conditions (9), P is positive definite, so $\dot{V} \leq 0$. Since V is positive definite, and $\dot{V}$ is negative semi–definite in the neighbourhood of zero solution of system (3), then we have $e_i, k_i(t) \in L_\infty$, $i = 1, 2, 3$. Since $\dot{V} \leq -e^TPe$, we obtain

$$\begin{aligned} \int_0^t \Lambda_{\min}(P)\|e\|^2 &\leq \int_0^t e^TPedt \\ &\leq -\int_0^t \dot{V}dt \\ &= V(0) - V(t) \\ &\leq V(0), \end{aligned}$$

where $\Lambda_{\min}(P)$ is the smallest eigenvalue of the positive matrix P. Thus, we can easily know that $e_i \in L_2$, $i = 1, 2, 3$ and by Barbalat's lemma, we have $\lim_{t\to\infty} \|e\| = 0$. This implies the result of the theorem. □

5. Numerical simulation

Numerical simulations are conducted in this section to confirm the effectiveness of the designed controllers. The parameters of the chaotic finance system are selected as $a = 0.9$, $b = 0.2$ and $c = 1.2$, so the system exhibit chaotic behavior. The initial conditions of the drive and response system are $(3, -0.5, 3)$ and $(-2, 2, 0.4)$, respectively. In linear feedback control, we take $k_1 = 30$, $k_2 = -2$ and $k_3 = 2$. In adaptive feedback control, the initial conditions of feedback gains are $k_1(t) = 10$, $k_2(t) = -5$ and $k_3(t) = 4$. Figures 4(a)-(c) and 5(a)-(c) display state trajectories of drive system (1) and the response system (2) via linear feedback control and adaptive linear feedback control, respectively. Figures 4(d) and 5(d) show the convergence of the synchronization errors. From Figure 6, we can see that the variable feedback strength $k_i(t)$, $i = 1, 2, 3$ is automatically adapted to a suitable strength.

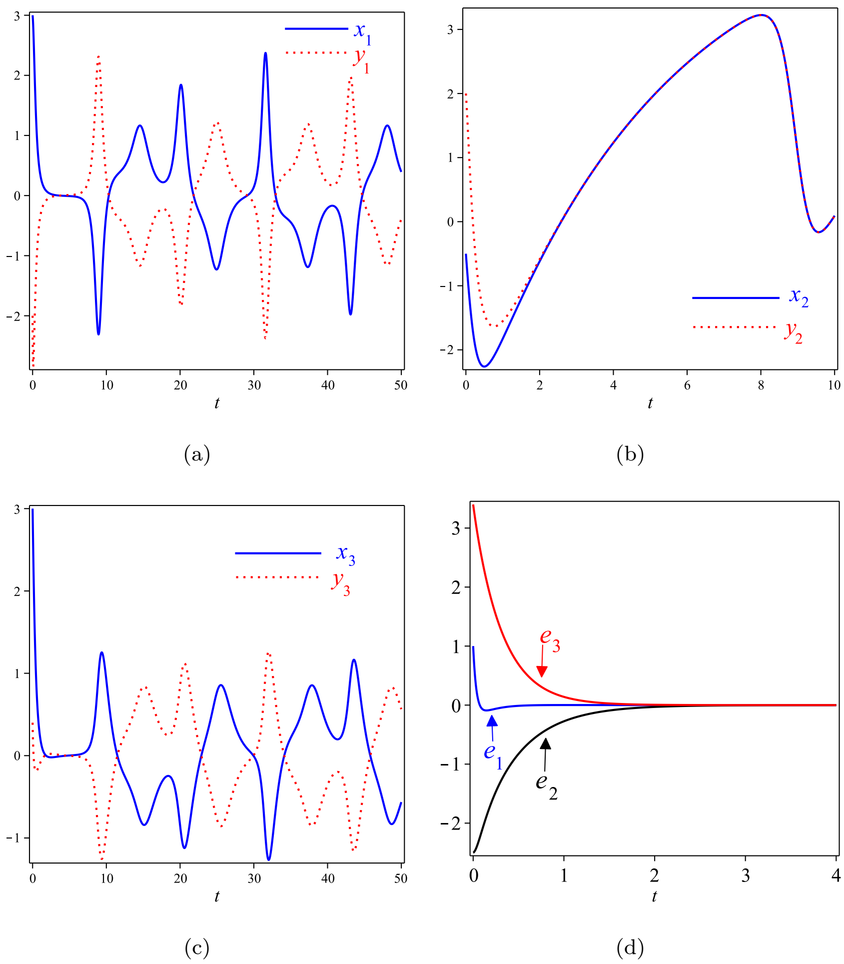

FIGURE 4. The time response of states for drive system (1) and the response system (2) via linear feedback control: (a) signals x_1 and y_1; (b) signals x_2 and y_2; (c) signals x_3 and y_3. (d) The convergence dynamics of the error system (3).

6. Conclusion

In this paper, we investigate a class of synchronization phenomenon in a three–dimensional chaotic finance system –two kinds of synchronization coexist in the

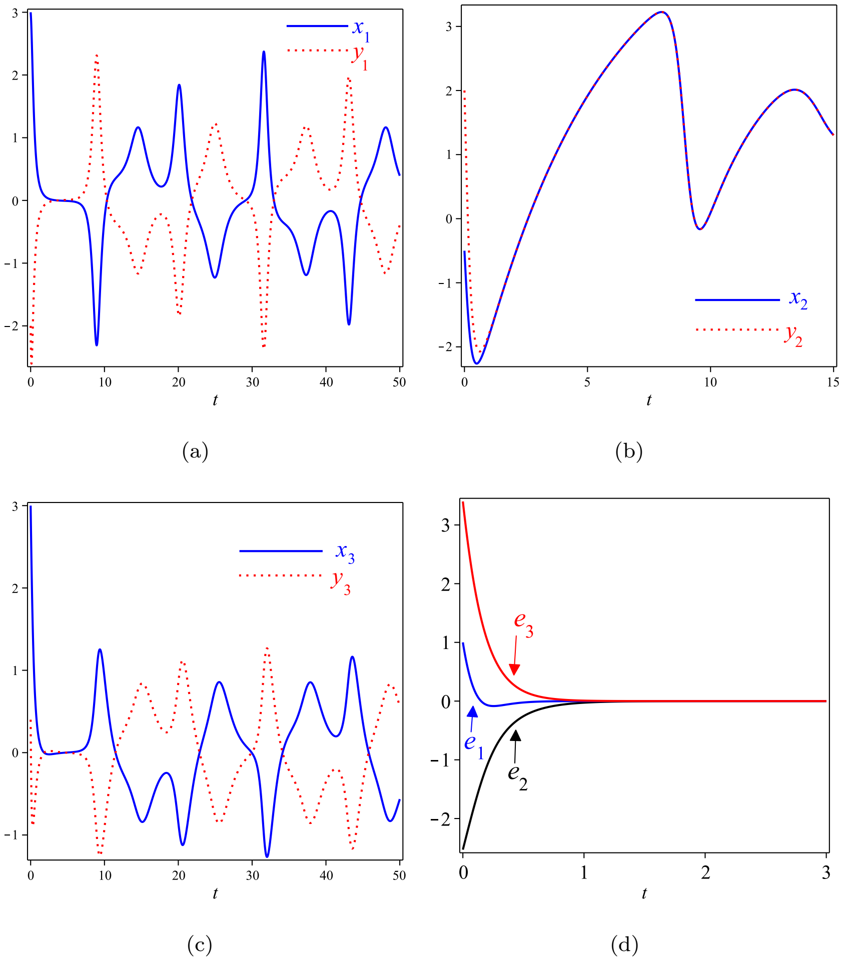

Figure 5. The time response of states for drive system (1) and the response system (2) via adaptive linear feedback control: (a) signals x_1 and y_1; (b) signals x_2 and y_2; (c) signals x_3 and y_3. (d) The convergence dynamics of the error system (3).

same system, i.e., one part states of the interactive systems is anti–phase synchronized, and the other part is complete synchronized under certain parameter region, which is quite different from conventional synchronization. All the theoretical results are verified by numerical simulations to demonstrate the effectiveness of the proposed synchronization schemes.

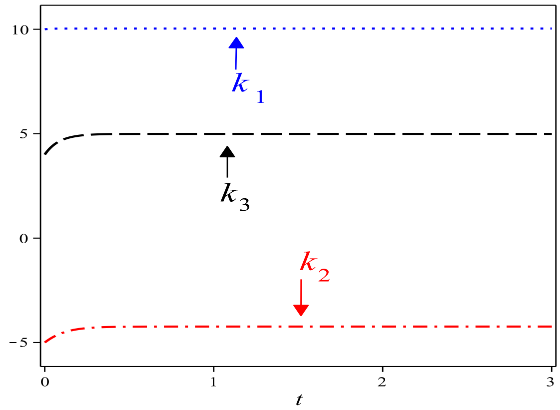

FIGURE 6. Feedback gains $k_i(t)$, $i = 1, 2, 3$.

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