

ASYMPTOTIC DYNAMICS VIA A TIME SET WITHOUT ANY ORDER

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ABSTRACT. In this paper the notion of time as an algebraic object in the system theory is considered. Quasi-limit sets for semi-dynamical systems as the invariant objects under topological conjugate relations are studied. By using of the special kind of subgroups of a topological semigroup some orbits in the quasi-limit set of a topological semi-dynamical system are detected. Complete semi-dynamical systems by using of completely simple semigroups are considered. New meaning of time in hyper dynamical systems are presented.

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1. INTRODUCTION

Let us to begin this paper by the following philosophical problem:
What does mean "time"?
We would like to answer to this question with considering the main property of time. In fact time is a reason for a motion or an evolution. To consider an evolution we need operators on a set with the special physical properties. This kind of structures has been considered by Poincare [11] via dynamical systems. More precisely let X

be a non-empty set, and $D = \{\varphi^t : \varphi^t : X \rightarrow X \text{ is a map and } t \in R\}$. Then (X, D) is called a dynamical system if:

- (a) φ^0 is the identity map;
- (b) $\varphi^t \circ \varphi^s = \varphi^{t+s}$ for all $t, s \in R$.

So in this model time is a member of the set of real numbers.

In the population dynamics we accept with a kind of dynamics which time values are members of the group Z of the integer numbers. So in the mathematical modeling of time we can call a member of a group as a time parameter. Sometimes we need only to consider the future of dynamics. In this case we replace R with a semi-group S , and the pair (X, D) only with the property (b) is called a semi-dynamical system. So according to the system theory time is a member of a semigroup.

In the next section for an abstract topological semigroup as the time set of a topological semi-dynamical system the notion of quasi-limit set is considered. This notion is an extension of the notion of attractor, and the notion of repeller. We prove that the special subgroups of a topological semigroup can create an orbit in the quasi-limit set of a topological semi-dynamical system (see theorem 1 and theorem 2). We also prove that quasi-limit sets are invariant objects under a topological conjugate relation (see theorem 3). In section 3 by using of hyper-dynamical systems a new meaning of time is presented.

2. QUASI-LIMIT SETS FOR TOPOLOGICAL SEMIGROUPS

In this section we assume that S is a topological semigroup i.e., S is Hausdorff topological space which is also a semigroup, and the product of S is a continuous map, where the topology of $S \times S$ is the product topology.

Example 2.1. The subset $T = \{(\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) : 0 < \theta < 2\pi, 0 < \phi < \pi\}$ of the sphere S^2 with the product :

$$((\sin \phi_1 \cos \theta_1, \sin \phi_1 \sin \theta_1, \cos \phi_1), (\sin \phi_2 \cos \theta_2, \sin \phi_2 \sin \theta_2, \cos \phi_2)) \longmapsto (\sin \phi_1 \cos \theta_2, \sin \phi_1 \sin \theta_2, \cos \phi_1)$$

is a topological semigroup. The topology of T is the subspace topology of R^3 .

In example 2.1 if for $t \in T$ we define the evolution operator $\varphi^t : T \rightarrow T$ by $\varphi^t(s) = ts$, then $(T, \{\varphi^t : t \in T\})$ is a topological semi-dynamical system. Generally if X is a topological space, then a semi-dynamical system $(X, D = \{\varphi^s : s \in S\})$ is called a topological semi-dynamical system if the mapping $\varphi : S \times X \rightarrow X$ defined by

$\varphi(s, x) = \varphi^s(x)$ is a continuous map.

Attractors, repellers [4, 10, 11] and fix points are the main objects for studying the behavior of semi-dynamical systems, and dynamical systems when we take their times as the members of the semigroup N , and the groups Z and R . We now would like to extend these notions for a semigroup S as a time set, by introducing the notion of "quasi-limit set" for a semi-dynamical system.

Definition 2.1. If $(X, D = \{\varphi^s : s \in S\})$ is a topological semi-dynamical system, and S is also an infinite set, then the set of all points such as $l \in X$ so that there is an infinite sequence $\{s_n\}$ in S and $x \in X - \{l\}$ such that $\lim_{n \rightarrow \infty} \varphi^{s_n}(x) = l$ is called the quasi-limit set for (X, D) .

Theorem 1. Let G be a subset of a topological semigroup S , which is also a group with the product of S , and let $(X, \{\varphi^s : s \in S\})$ be a topological semi-dynamical system with the non-empty quasi-limit set L . For given $l \in L$ if there is an infinite sequence $\{s_n\}$ in G and $x \in X - \{l\}$ so that $\lim_{n \rightarrow \infty} \varphi^{s_n}(x) = l$. Then $\varphi^s(l) \in L$ for all $s \in G$.

Proof. Let $s \in G$ be given. Since $\lim_{n \rightarrow \infty} \varphi^{s_n}(x) = l$ and φ^s is a continuous map, then $\lim_{n \rightarrow \infty} \varphi^{ss_n}(x) = \varphi^s(l)$. If s_n, s_m are two points of the sequence $\{s_n\}$ and $ss_n = ss_m$ then $s^{-1}ss_n = s^{-1}ss_m$. So $s_n = s_m$. Thus the sequence $\{ss_n\}$ is an infinite sequence. Hence $\varphi^s(l) \in L$. $\square$

Theorem 2. Let S be a compact topological semi-group which is equal to the union of the pairwise disjoint open sets G_i , $i = 1, 2, \dots, k$. Moreover let each G_i be a group with the product of S . If L is the non-empty limit set for a topological semi-dynamical system $(X, \{\varphi^s : s \in S\})$, and $l \in L$, then there is $i_0 \in \{1, 2, \dots, k\}$ such that $\varphi^s(l) \in L$ for all $s \in G_{i_0}$.

Proof. Since $S = \bigcup_{i=1}^k G_i$, and each G_i is an open set, then each G_i is also a closed set. If $l \in L$, then there is an infinite sequence $\{s_n\}$ in S and $x \in X - \{l\}$ such that $\lim_{n \rightarrow \infty} \varphi^{s_n}(x) = l$. Since $\{s_n\}$ is an infinite sequence in the compact space S then it has a convergence subsequence $\{s_{n_i}\}$ which it converges to a point $s_0 \in S$, and $\varphi^{s_0}(x) = l$. There is i_0 such that $s_0 \in G_{i_0}$. So there is a infinite subsequence $\{s'_n\}$ of $\{s_n\}$ in G_{i_0} such that $\lim_{n \rightarrow \infty} \varphi^{s'_n}(x) = l$. Thus the previous theorem implies that $\varphi^s(l) \in L$ for all $s \in G_{i_0}$. $\square$

The assumptions of theorem 1 and theorem 2 can appear in the class of topological complete semi-dynamical systems [6]. Let us to recall the definition of complete semi-dynamical system. Let S be a completely simple semigroup [3]. We use of the equivalent definition of completely simple semigroup [7] as follows:

A completely simple semigroup is a semigroup S with the following properties:

- (i) For each x in S there exists a unique z in S such that $xz = zx = x$ (we denote z by $e(x)$);
- (ii) For each x in S there exists y in S such that $xy = yx = e(x)$. We denote y by x^{-1} .

If S is a completely simple semigroup, and $(X, D = \{\varphi^s : s \in S\})$ is a semi-dynamical system, then (X, D) is called a complete semi-dynamical system [6] if for given $x \in X$ there exists $s \in S$ such that $\varphi^s(x) = x$.

One can find many examples for the theorems 1 and 2 in topological complete semi-dynamical systems [5].

We now would like to show that quasi-limit sets are invariant under a topological conjugate relation. Let S and S' be two topological semigroups. Let $(X, \{\varphi^s : s \in S\})$ and $(X, \{\psi^{s'} : s' \in S'\})$ be two topological semi-dynamical systems. Then we say that these two systems are topologically conjugate if there is a homeomorphism $h : X \rightarrow X$, and $f : S \rightarrow S'$ such that for given $s \in S$ the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\varphi^s} & X \\ h \downarrow & & \downarrow h \\ X & \xrightarrow{\psi^{f(s)}} & X \end{array}$$

Theorem 3. *Let $(X, \{\varphi^s : s \in S\})$ and $(X, \{\psi^{s'} : s' \in S'\})$ be two topologically conjugate semi-dynamical systems with the homeomorphisms $h : X \rightarrow X$ and $f : S \rightarrow S'$. Moreover let L be the quasi-limit set for $(X, \{\varphi^s : s \in S\})$. Then $h(L)$ is the quasi-limit set for $(X, \{\psi^{s'} : s' \in S'\})$.*

Proof. If $l \in L$ then there is an infinite sequence $\{s_n\}$ in S and $x \in X - \{l\}$ such that $\lim_{n \rightarrow \infty} \varphi^{s_n}(x) = l$. So $\lim_{n \rightarrow \infty} h(\varphi^{s_n}(x)) = h(l)$. Hence $\lim_{n \rightarrow \infty} \psi^{f(s_n)}(h(x)) = h(l)$. Since f is a homeomorphism, then $\{f(s_n)\}$ is an infinite sequence in S' . These imply that $h(l)$ is a member of the quasi-limit set L' of $(X, \{\psi^{s'} : s' \in S'\})$.

The similar argument imply that if $l' \in L'$, then $h^{-1}(l') \in L$. So $l' = h(h^{-1}(l')) \in h(L)$. Thus $h(L) = L'$. $\square$

3. NEW CONCEPT OF TIME IN HYPER-DYNAMICAL SYSTEMS

There is a novel approach to the meaning of time in the notion of hyper-dynamical system. The notion of hyper-dynamical system as a new kind of dynamics has been introduced by this author in 2003 [8]. Let us to explain this notion by recalling the definition of a hypergroup [1, 2, 12]. If H is a nonempty set, then a hyper-operation or join operation is a mapping $\cdot : H \times H \rightarrow P_*(H)$, where $P_*(H)$ is the set of all nonempty subsets of H . This join operation creates an operation $\cdot : P_*(H) \times P_*(H) \rightarrow P_*(H)$ which is defined by $A.B = \bigcup_{(a,b) \in A \times B} a.b$. We denote $a.b$ and $A.B$ by ab and AB respectively. We also denote $\{a\}B$ by aB .

A nonempty set H with a join operation $\cdot$ is called a hypergroup if it satisfies the following axioms:

- (i) $aH = H = Ha$, for all $a \in H$;
- (ii) $a(bc) = (ab)c$, for all $a, b, c \in H$.

Let $(H, \cdot)$ be a hypergroup, X be a nonempty set and $D = \{\varphi^h : \varphi^h : X \rightarrow X \text{ is a map and } h \in H\}$. Then (M, D) is called a hyper-dynamical system if it satisfies the following condition:

- (i) If $g, h \in H$, and $x \in X$, then $\varphi^g \circ \varphi^h(x) \in \varphi^{gh}(x)$, where $\varphi^{gh}(x) = \{\varphi^u(x) : u \in gh\}$.

Example 3.1. If $H = R^3$, then H with the join operation defined by $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = \{(a_1 + b_1, a_2 + b_2, z) : z \in R\}$ is a hypergroup. Let D be the set of mappings $\varphi^{(a,b,c)} : R^3 \rightarrow R^3$ defined by $\varphi^{(a,b,c)}(x, y, z) = (x + a, y + b, |z|^{(c-a)})$, where $(a, b, c) \in H$. Then (R^3, D) is a hyper-dynamical system.

The dynamics presented in example 3.1 can not create by any semi-dynamical system. Because the product deduced by this dynamics on R^3 is:

$$(a', b', c')(a, b, c) = (a' + a, b' + b, (c - a)(c' - a'))$$

and this product is not a semigroup product.

Now we are in the level which we can determine the meaning of time. According to our point of view based on the property of time: which is the reason of motion, time is a member of a hypergroup.

Now we would like to consider quasi-limit sets for topological hyper-dynamical systems.

We assume that H is a Hausdorff topological space, and $(H, .)$ is a hypergroup. The join operation $.$ is called a continuous join operation [9] if for given $a, b \in H$, and for all open set W with $W \cap (ab) \neq \emptyset$ there exist open neighborhoods U of a and V of b and a subset $Z \subseteq UV$ such that

- i) $W \cap Z \neq \emptyset$, and;
- ii) $(cd) \cap Z \neq \emptyset$ for all $c \in U$ and $d \in V$.

A hyper-dynamical system $(X, \{\varphi^h : h \in H\})$ with times in H is called a topological hyper-dynamical system if

- i) H is a Hausdorff topological space and its join operation is a continuous join operation;
- ii) X is a topological space;
- iv) The mapping $\varphi : H \times X \rightarrow X$ defined by $\varphi(h, x) = \varphi^h(x)$ is a continuous map.

Definition 3.1. Let $(X, D = \{\varphi^h : h \in H\})$ be a topological hyper-dynamical system, and let H be an infinite set. Then the set of all points such as $l \in X$ so that there is an infinite sequence $\{h_n\}$ in H and $x \in X - \{l\}$ such that $\lim_{n \rightarrow \infty} \varphi^{h_n}(x) = l$ is called the quasi-limit set for (X, D) .

Two topological hyper-dynamical systems $(X, \{\varphi^h : h \in H\})$ and $(X, \{\psi^{h'} : h' \in H'\})$ are called topologically conjugate hyper-dynamical systems if there exist homeomorphisms $f : X \rightarrow X$ and $g : H \rightarrow H'$ such that $f \circ \varphi^h = \psi^{g(h)} \circ f$ for all $h \in H$ (see the following diagram).

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi^h \downarrow & & \downarrow \psi^{g(h)} \\ X & \xrightarrow{f} & X. \end{array}$$

The following theorem is the similar version of theorem 3 for hyper-dynamical systems.

Theorem 4. Assume that $(X, \{\varphi^h : h \in H\})$ and $(X, \{\psi^{h'} : h' \in H'\})$ are topologically conjugate hyper-dynamical systems with the homeomorphisms $f : X \rightarrow X$ and $g : H \rightarrow H'$, and quasi-limit sets L and L' respectively. Then $f(L) = L'$.

Proof. Let $l \in L$ be given. Then there is an infinite sequence $\{h_n\}$ in H and $x \in X - \{l\}$ such that $\lim_{n \rightarrow \infty} \varphi^{h_n}(x) = l$. Thus $\lim_{n \rightarrow \infty} f(\varphi^{h_n}(x)) = f(l)$. So $\lim_{n \rightarrow \infty} \psi^{g(h_n)}(f(x)) = f(l)$. Moreover $\{g(h_n)\}$ is an infinite sequence in H' . So $f(l) \in L'$.

If $l' \in L'$, then by the same method $f^{-1}(l') \in L$. Thus $l' = f(f^{-1}(l')) \in f(L)$. So $f(L) = L'$. $\square$

Theorem 5. Suppose that $(X, \{\varphi^h : h \in H\})$ and $(X', \{\psi^{h'} : h' \in H'\})$ are two topological hyper-dynamical systems with the nonempty quasi-limit sets L and L' respectively. For given $(h, h') \in H \times H'$ let $\theta^{(h, h')} : X \times X' \rightarrow X \times X'$ be defined by $\theta^{(h, h')}(x, x') = (\varphi^h(x), \psi^{h'}(x'))$. Then $(X \times X', \{\theta^{(h, h')} : (h, h') \in H \times H'\})$ is a topological hyper-dynamical system with the quasi-limit set $(\varphi^H(X) \times L') \cup (L \times \psi^{H'}(X')) \cup (L \times L')$, where $\varphi^H(X) = \{\varphi^h(x) : h \in H \text{ and } x \in X\}$, $\psi^{H'}(X') = \{\psi^{h'}(x') : h' \in H' \text{ and } x' \in X'\}$, and the topologies of the product spaces are the product topologies.

Proof. The straightforward calculations show that

$(X \times X', \{\theta^{(h, h')} : (h, h') \in H \times H'\})$ is a topological hyper-dynamical system. Let M be its quasi-limit set, $l \in L$ and $l' \in L'$. Then there exist infinite sequences $\{h_n\}$ in H , $\{h'_n\}$ in H' , and $x \in X - \{l\}$, $x' \in X' - \{l'\}$ such that $\lim_{n \rightarrow \infty} \varphi^{h_n}(x) = l$ and $\lim_{n \rightarrow \infty} \psi^{h'_n}(x') = l'$. So $(x, x') \in (X \times X') - \{(l, l')\}$, and for infinite sequence $\{(h_n, h'_n)\}$ in $H \times H'$ we deduce $\lim_{(n, n') \rightarrow \infty} \theta^{(h_n, h'_n)}(x, x') = \lim_{(n, n') \rightarrow \infty} (\varphi^{h_n}(x), \psi^{h'_n}(x')) = (l, l')$. Moreover if $(h, h') \in H \times H'$ and $(y, y') \in Y \times Y'$, then $\{(h, h'_n)\}$ and $\{(h_n, h')\}$ are infinite sequences in $H \times H'$, and $\lim_{n' \rightarrow \infty} \theta^{(h, h'_n)}(y, x') = \lim_{n' \rightarrow \infty} (\varphi^h(y), \psi^{h'_n}(x')) = (\varphi^h(y), l')$ and $\lim_{n \rightarrow \infty} \theta^{(h_n, h')}(x, y') = \lim_{n \rightarrow \infty} (\varphi^{h_n}(x), \psi^{h'}(y')) = (l, \psi^{h'}(y'))$.

So $(\varphi^H(X) \times L') \cup (L \times \psi^{H'}(X')) \cup (L \times L') \subseteq M$. By the similar argument we can prove that $M \subseteq (\varphi^H(X) \times L') \cup (L \times \psi^{H'}(X')) \cup (L \times L')$. So $M = (\varphi^H(X) \times L') \cup (L \times \psi^{H'}(X')) \cup (L \times L')$. $\square$

Example 3.2. Let $H = R^3$ be the hypergroup of example 3.1, and let D be the set of mappings $\varphi^{(a, b, c)} : R^2 \times (R - \{0\}) \rightarrow R^2 \times (R - \{0\})$ defined by $\varphi^{(a, b, c)}(x, y, z) = (x + a, y + b, |z|^{(c-a)})$, where $(a, b, c) \in H$. Then $(R^2 \times (R - \{0\}), D)$ is a topological hyper-dynamical system where the topology of R^3 is the topology generated by Euclidean metric. $R^2 \times \{-1, 1\}$ is the quasi-limit set of $(R^2 \times (R - \{0\}), D)$.

4. CONCLUSION

In this paper we used of some algebraic notions to determine the meaning of time. Consideration of the future and the past of the orbit of a point is one of the main topics in dynamical systems. For consideration the future and the past of an orbit we need a time order. The concept of time as a member of a semigroup or as a member of a hypergroup indicate to some time sets without any order. For the case of time sets without any order we can use of quasi-limit sets as the invariant objects of a dynamics.

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