ANTI-SYNCHRONIZATION OF THE FOUR-DIMENSIONAL HYPERCHAOTIC SYSTEM WITH UNCERTAIN PARAMETERS VIA ADAPTIVE CONTROL

AZIZEH JABBARI, HOSSEIN KHEIRI
DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF TABRIZ, TABRIZ, IRAN
AZIZEH.JABBARI@GMAIL.COM
H-KHEIRI@TABRIZU.AC.IR

(Received: 10 June 2011, Accepted: 12 December 2011)

Abstract. In this paper, an adaptive control scheme is presented to study the anti-synchronization behavior between two identical four-dimensional hyperchaotic systems with unknown parameters. Anti-synchronization can be characterized by the vanishing of the sum of relevant variables. Based on Lyapunov stability theory, an adaptive anti-synchronization controller is designed, and the analytic expression of the controller and the adaptive laws of parameters are developed. Numerical simulations are performed for four-dimensional hyperchaotic systems to demonstrate the effectiveness of the proposed control strategy.

AMS Classification: 37-xx, 37D45.
Keywords: Four-dimensional hyperchaotic systems; Anti-synchronization; Adaptive control; Lyapunov stability theory.

1. Introduction

Since the idea of synchronizing chaotic systems was introduced by Pecora and Carroll [15] in 1990, chaos synchronization has received increasing attention due

JOURNAL OF DYNAMICAL SYSTEMS & GEOMETRIC THEORIES
VOL. 10, NUMBER 1 (2012) 1-12.

to its theoretical challenge and its great potential applications in secure communication, chemical reaction and biological systems [2]. Recently several different types of synchronizations have been proposed in the literature, for example, generalized synchronization [1]-[6], phase synchronization[10, 11], lag synchronization [16]-[17], anti-synchronization [12]-[14] and so on .The generalized synchronization implies the establishment of functional relations between drive and response systems. Phase synchronization is characterized in that the phase difference between two chaotic systems are locked within 2π, while their amplitudes remain chaotic and uncorrelated. Lag synchronization is described as the coincidence of two chaotic trajectories with a constant time lag. The state vectors of synchronized systems have the same absolute values but opposite signs, i.e. the sum of the output signals of two systems can converge to zero, called the anti-synchronization(AS) method. Most of the works mentioned so far are involved mainly with low-dimensional identical chaotic systems with only one positive Lyapunov exponent. Hyperchaotic system, possessing at least two positive Lyapunov exponents, has more complex behavior and abundant dynamics than chaotic system. How to realize synchronization of hyperchaotic systems is an interesting and challenging work. Many attempts have been made to control hyperchaos and achieve synchronization of hyperchaos in hyperchaotic systems [9]-[21]. In this paper, an adaptive control scheme is presented to study the anti-synchronization behavior between two identical novel four-dimensional hyperchaotic system with unknown parameters [8]. Based on the Lyapunov stability theory, an adaptive anti-synchronization controller and adaptive laws of parameters are developed.

2. Anti-synchronization

The drive system and the response system are defined as

$$\dot{x} = f(x), \tag{1}$$

$$\dot{y} = f(y) + u, \tag{2}$$

where $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ are the state vectors of the systems (1) and (2) respectively; $f : \mathbb{R}^n \to \mathbb{R}^n$ defines a vector field in n-dimensional space and $u = (u_1, \ldots, u_n)^T$ is an n-dimensional control signals.

Let $e = x + y$ is the anti-synchronization error vector. The goal is to design an appropriate controller u such that the trajectory of the response system (2) with initial conditions y_0 can approach asymptotically the drive system (1) with initial conditions x_0, in this sense, that is

$$\lim_{t\to\infty} \parallel e \parallel = \lim_{t\to\infty} \parallel x(t, x_0) + y(t, y_0) \parallel = 0,$$

where $\parallel \cdot \parallel$ is the Euclidean norm. At this point, it means that the drive system (1) and the response system (2) are anti-synchronized under the controller u as time t tends to infinity.

3. System description

Consider the original chaotic dynamical equation which is given by [8]:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx + xz, \\ \dot{z} &= -xy - cz, \end{aligned} \tag{3}$$

where a, b, c are fixed parameters and x, y, z are the dynamic variables of system (3). This system was found to be chaotic in a wide parameter range. For example, when a = 10, b = 35, c = 1.4, the system (3) has a chaotic attractor and has many interesting complex dynamical behaviors. The Lyapunov exponent spectrum of system (3) is

$$\lambda_1 = 0.8653, \quad \lambda_2 = 0, \quad \lambda_3 = -12.2677.$$

Apparently, system (3) only has one positive Lyapunov exponent, thus it is a three dimensional autonomous chaotic system. Here, on the basis of system (3), a new hyperchaotic system is described by the following four-dimensional autonomous differential equations [7]:

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx + xz - w, \\ \dot{z} &= -xy - cz + w, \\ \dot{w} &= dx + y, \end{aligned} \tag{4}$$

Furthermore, we also let a = 10, b = 35, c = 1.4, and d = 5 be the constant parameters. Meanwhile, the dynamical system (4) can be characterized with its Lyapunov exponents that are computed numerically by the Wolf method [19]:

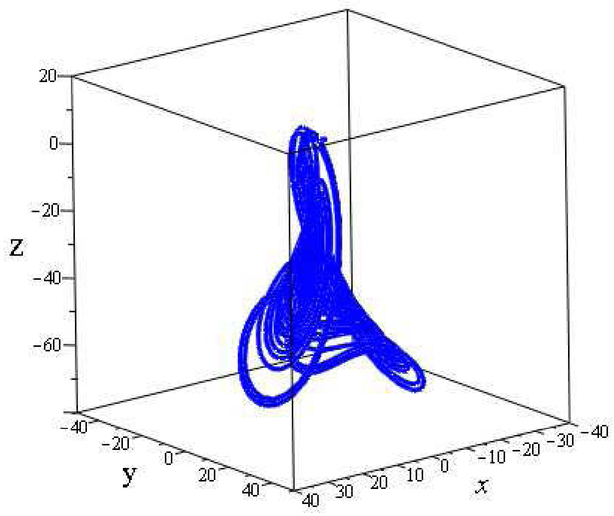

FIGURE 1. Three-dimensional view of the system (4)

- 1. For periodic orbits, there are one zero and three negative Lyapunov exponents.
- 2. For quasi-periodic orbits, there are two zeros and two negative Lyapunov exponents.
- 3. For chaotic orbits, there are one positive, one zero, and two negative Lyapunov exponents.
- 4. For hyperchaotic orbits, there are two positive values in the four Lyapuvov exponents.

The Lyapunov exponents of system (4) can be obtained as

$$\lambda_1 = 0.8058, \quad \lambda_2 = 0.0981, \quad \lambda_3 = 0, \quad \lambda_4 = -12.3058.$$

The three-dimensional view of the hyperchaotic strange attractor is shown in Fig. 1. Fig. 2 shows some dynamical behavior in different planes for system (4).

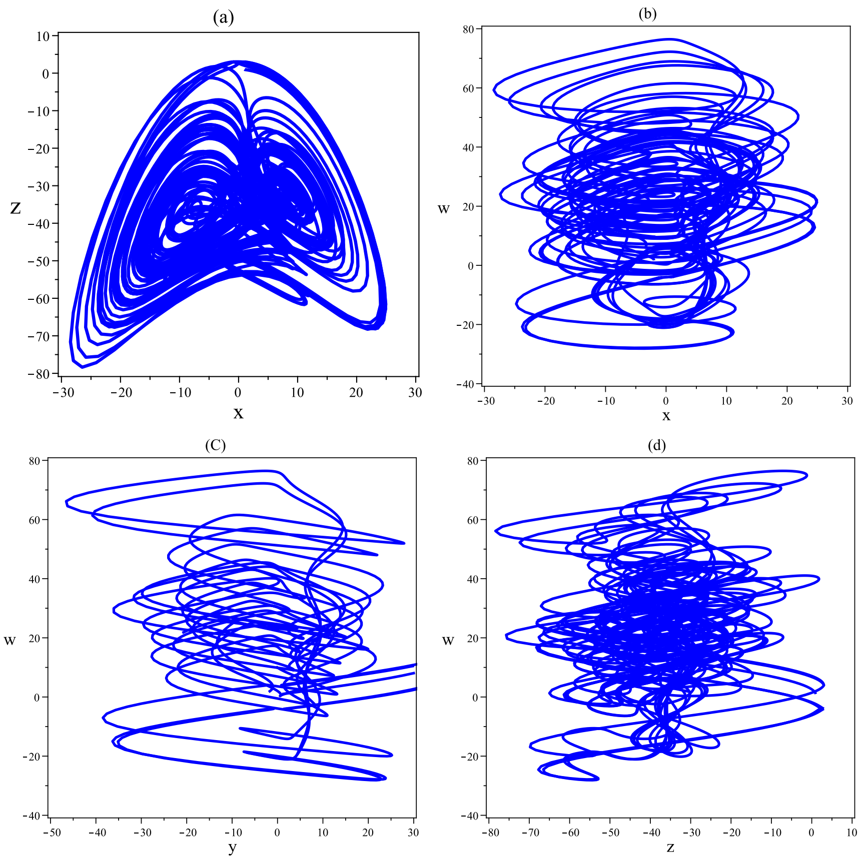

FIGURE 2. Phase portrait of the hyperchaotic system (4) in the (a) x - z, (b) x - w, (c) y - w, and (d) z - w plane with a = 10, b = 35, c = 1.4, and d = 5.

4. Adaptive anti-synchronization of two identical four-dimensional hyperchaotic systems

In this section, we will study the anti-synchronization of two identical four-dimensional hyperchaotic systems with unknown parameters. For system (4), the drive system and the response system are defined as below, respectively,

$$\begin{aligned} \dot{x_1} &= a(y_1 - x_1), \\ \dot{y_1} &= bx_1 + x_1 z_1 - w_1, \\ \dot{z_1} &= -x_1 y_1 - c z_1 + w_1, \\ \dot{w_1} &= dx_1 + y_1, \end{aligned} \tag{5}$$

and

$$\begin{aligned} \dot{x_2} &= a(y_2 - x_2) + u_1, \\ \dot{y_2} &= bx_2 + x_2 z_2 - w_2 + u_2, \\ \dot{z_2} &= -x_2 y_2 - c z_2 + w_2 + u_3, \\ \dot{w_2} &= dx_2 + y_2 + u_4, \end{aligned} \tag{6}$$

where a, b c and d are unknown parameters and u_1, u_2, u_3 and u_4 are three control functions to be designed. In the following, an effective controller is designed to achieve the chaos anti-synchronization between these systems. We add (6) to (5) and get the following error system

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + u_1, \\ \dot{e}_2 &= be_1 + x_1 z_1 + x_2 z_2 - e_4 + u_2, \\ \dot{e}_3 &= -x_1 y_1 - x_2 y_2 - ce_3 + e_4 + u_3, \\ \dot{e}_4 &= de_1 + e_2 + u_4, \end{aligned} \tag{7}$$

where $e_1 = x_1 + x_2$, $e_2 = y_1 + y_2$, $e_3 = z_1 + z_2$ and $e_4 = w_1 + w_2$. Our goal is to find proper control functions u_i $(i = 1, 2, 3, 4)$ and parameter update laws, such that the response system (6) globally and asymptotically anti-synchronizes the drive system (5), i.e., $\lim_{t\to\infty} \parallel e \parallel = 0$ where $e = [e_1, e_2, e_3, e_4]^T$. For two identical chaotic systems without controllers u_1, u_2, u_3 and u_4, the trajectories of the two identical systems will quickly separate and can not anti-synchronize on the condition that the initial values $(x_1(0), y_1(0), z_1(0), w_1(0)) \neq (x_2(0), y_2(0), z_2(0), w_2(0))$. However, with appropriate control schemes, the two systems anti-synchronize for any initial value.

The following **Theorem** shows that system (5) and system (6) can be effectively anti-synchronized in the situation of uncertain parameters.

Theorem 1. *The drive system (5) and the response system (6) can be anti- synchronized globally and asymptotically for any different initial condition with following adaptive controller*

$$\begin{aligned} u_1 &= -\hat{a}(e_2 - e_1) - e_1, \\ u_2 &= -\hat{b}e_1 - x_1 z_1 - x_2 z_2 + e_4 - e_2, \\ u_3 &= x_1 y_1 + x_2 y_2 + \hat{c}e_3 - e_4 - e_3, \\ u_4 &= -\hat{d}e_1 - e_2 - e_4, \end{aligned} \tag{8}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are the estimates of a, b, c and d, respectively, and the adaptive parameter update laws are chosen as

$$\begin{aligned} \dot{\hat{a}} &= -\dot{\tilde{a}} = e_1(e_2 - e_1) + \tilde{a}, \\ \dot{\hat{b}} &= -\dot{\tilde{b}} = e_1 e_2 + \tilde{b}, \\ \dot{\hat{c}} &= -\dot{\tilde{c}} = -{e_3}^2 + \tilde{c}, \\ \dot{\hat{d}} &= -\dot{\tilde{d}} = -e_1 e_4 + \tilde{d}, \end{aligned} \tag{9}$$

where $\tilde{a} = a - \hat{a}$, $\tilde{b} = b - \hat{b}$, $\tilde{c} = c - \hat{c}$ and $\tilde{d} = d - \hat{d}$.

Proof. Substituting (8) into (7) leads to the following error system

$$\begin{aligned} \dot{e}_1 &= \tilde{a}(e_2 - e_1) - e_1, \\ \dot{e}_2 &= \tilde{b}e_1 - e_2, \\ \dot{e}_3 &= -\tilde{c}e_3 - e_3. \\ \dot{e}_4 &= \tilde{d}e_1 - e_4. \end{aligned} \tag{10}$$

Consider the following Lyapunov function

$$V = \frac{1}{2}(e^T e + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2).$$

Then the time derivative of V along the solutions Eqs. (9) and (10) gives

$$\begin{aligned} \dot{V} &= \dot{e}^T e + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} \\ &= e_1[\tilde{a}(e_2 - e_1) - e_1] + e_2[\tilde{b}e_1 - e_2] \\ &\quad + e_3[-\tilde{c}e_3 - e_3] + e_4[\tilde{d}e_1 - e_4] + \tilde{a}[-e_1(e_2 - e_1) - \tilde{a}] \\ &\quad + \tilde{b}[-e_1e_2 - \tilde{b}] + \tilde{c}[{e_3}^2 - \tilde{c}] + \tilde{d}[e_1e_4 - \tilde{d}] \\ &= -e^T e - \tilde{a}^2 - \tilde{b}^2 - \tilde{c}^2 - \tilde{d}^2 \\ &< 0. \end{aligned}$$

It is clear that V is positive definite and $\dot{V}$ is negative definite. According to the Lyapunov stability theorem, the error system (10) can converge to the origin asymptotically. Therefore, the drive system (5) and the response system (6) can be asymptotically and globally anti-synchronized. This completes the proof. □

4.1. **Numerical simulations.** In this section, numerical simulations are presented to verify the effectiveness of the proposed method. Fourth-order Runge-Kutta method is used to solve the systems with time step size 0.001. The parameters are chosen as $a = 10$, $b = 35, c = 1.4$ and $d = 5$ in the simulations such that the system (4) exhibits chaotic behavior. The initial values of drive system and the response system are chosen as $(x_1(0), y_1(0), z_1(0), w_1(0)) = (1, -2, 6, 3)$ and $(x_2(0), y_2(0), z_2(0), w_2(0)) = (4, 1, -5, -7)$, respectively. Hence, the error system has the initial values $e_1(0) = 5$, $e_2(0) = -1$, $e_3(0) = 1$ and $e_4(0) = -4$. Moreover, the initial values of the estimated parameters are chosen as $\hat{a}(0) = 1$, $\hat{b}(0) = 10$, $\hat{c}(0) = 4$ and $\hat{d}(0) = 2$. Fig. 3 display state trajectories of drive system (5) and the response system (6). Fig 4 display the anti-synchronization errors between systems (5) and (6). Fig 5 shows that the estimated values of the unknown parameters converge to $a = 10$, $b = 35, c = 1.4$ and $d = 5$ as $t \to \infty$, respectively.

5. Conclusion

In this paper, we investigated chaos anti-synchronization of four-dimensional hyperchaotic systems via adaptive control. Sufficient conditions for the anti-synchronization were obtained analytically. Finally, numerical simulations were provided to show the effectiveness of the proposed method.

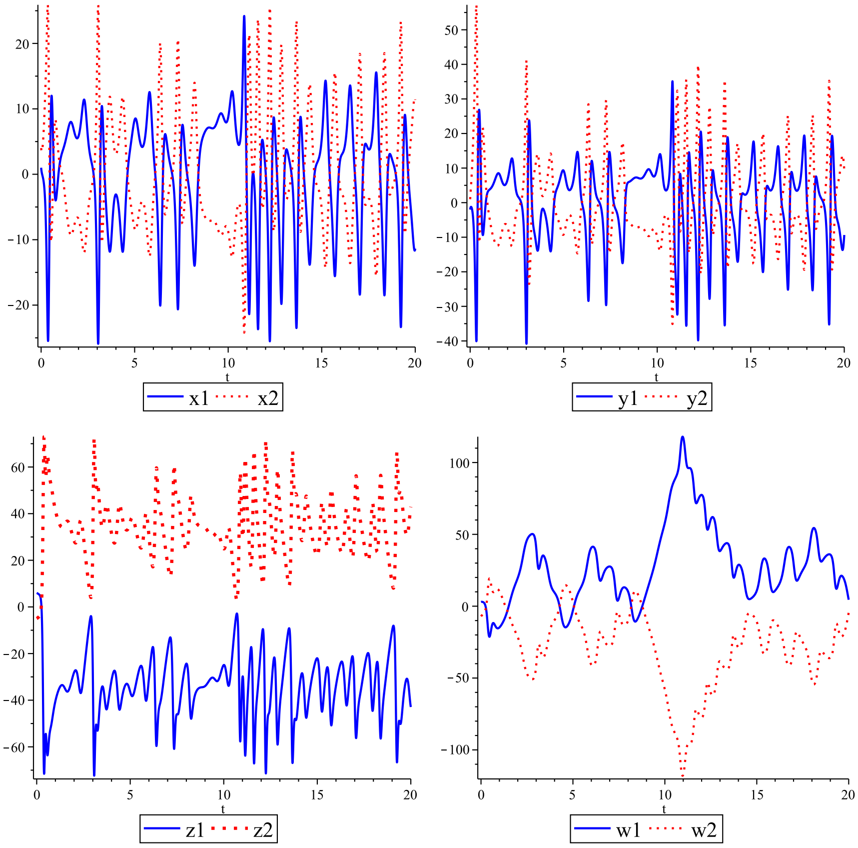

FIGURE 3. The time response of states for drive system (5) and the response system (6) : (a) signals x1 and x2; (b) signals y1 and y2; (c) signals z1 and z2; (d) signals w1 and w2.

REFERENCES

[1] H. Chen, M. Sun, Generalized projective synchronization of the energy resource system, International Journal of Nonlinear Science, Vol. 2, Number 3, 166-170 (2006).

[2] G. Chen, X. Dong, Form chaos to order: methodologies, perspectives and applications, Singapore: World Scientific, 1998.

[3] Y. Chen, X. Chen, S. Gu, Lag synchronization of structurally nonequivalent chaotic systems with time delays, Nonlinear Analysis, Vol. 66, Number 9, 1929-1937 (2007).

[4] J.W. Feng, S.H. Chen, C.P. Wang, Adaptive synchronization of uncertain hyperchaotic systems based on parameter identification, Chaos, Solitons and Fractals, Vol. 26, Number 4, 1163-1169 (2005).

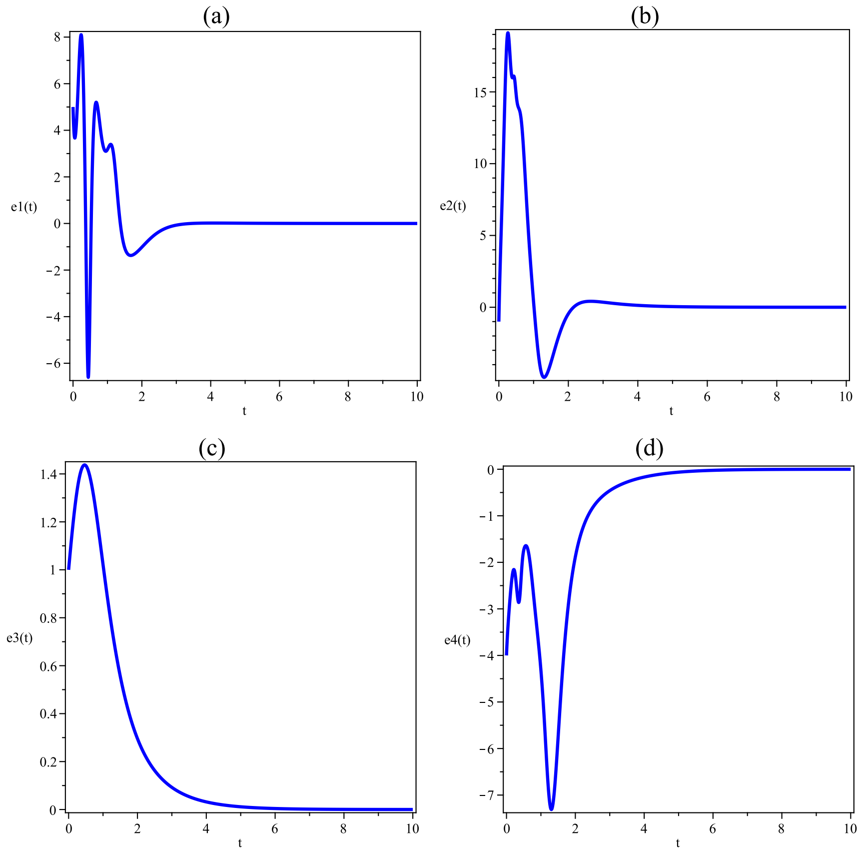

FIGURE 4. Synchronization errors e1, e2, e3 and e4 of the drive system (5) and the response system (6).

[5] C. Hua, X. Guan, P. Shi, Adaptive feedback control for a class of chaotic systems, Chaos, Solitons and Fractals, Vol. 23, Number 3, 757-765 (2005).

[6] L. Kocarev, U. Parlitz, Gneralized synchronization, predictability, and equivalence of unidirectionally coupled dynamical systems, Physical Review Letters, Vol. 76, Number 11, 1816-1819 (1996).

[7] L. Liu, Ch. Liu, Y. Zhang, Analysis of a Novel Four-Dimensional Hyperchaotic System, Chinese journal of physics, Vol. 46, Number 4, 386-393 (2008).

[8] L. Liu, Y.C. Su, C.X. Liu, A modified Lorenz system, International Journal on Nonlinear Science and Numerical Simulation, Vol. 7, Number 2, 187-190 (2006).

[9] Y. Li, W.K.S. Tang, G. Chen, Generating hyperchaos via state feedback control, International Journal of Bifurcation and Chaos, Vol. 15, Number 10, 3367-3375 (2005).

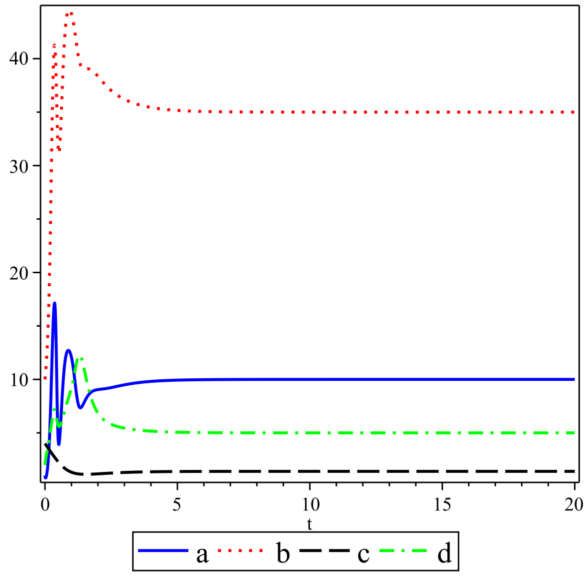

FIGURE 5. Changing parameters a, b, c and d of the drive system (5) and the response system (6) with time t.

[10] Y. Li, J. Zhang, Z. Liu, Circadian Oscillators and Phase Synchronization under a Light-Dark Cycle, International Journal of Nonlinear Science, Vol. 3, Number 1, 131-138 (2006).

[11] G.R. Michael, S.P. Arkady, K. Jrgen, Phase synchronization of chaotic oscillators, Physical Review Letters, Vol. 76, Number 11, 1804-1807 (1996).

[12] M. Mossa Al-sawalha, M.S.M. Noorani, Anti-synchronization of chaotic systems with uncertain parameters via adaptive control, Physics Letters A, Vol. 373, Number 32, 2852-2857 (2009).

[13] M. Mossa Al-sawalha, M.S.M. Noorani, On anti-synchronization of chaotic systems via nonlinear control, Chaos, Solitons and Fractals, Vol. 42, Number 1, 170-179 (2009).

[14] M. Mossa Al-sawalha, M.S.M. Noorani, Adaptive reduced-order anti-synchronization of chaotic systems with fully unknown parameters, Communications in Nonlinear Science and Numerical Simulation, Vol. 15, Number 10, 3022-3034 (2010).

[15] L.M. Pecora, T.L. Carroll, Synchronization in chaotic systems, Physical Review Letters, Vol. 64, Number 8, 821-824 (1990).

[16] M. Sun, L. Tian, J. Xu, Time-delayed feedback control of the energy resource chaotic system, International Journal of Nonlinear Science, Vol. 3, Number 1, 172-177 (2006).

[17] I.S. Taherion, Y.C. Lai, Observability of lag synchronization of coupled chaotic oscillators, Physical Review E, Vol. 59, Number 6, 6247-6250 (1999).

[18] X. Wang, L. Tian, Liqin Yu, Linear feedback controlling and synchronization of the Chens chaotic system, International Journal of Nonlinear Science, Vol. 1, Number 3, 43-49 (2006).

[19] A. Wolf, J.B. Swift, H.L. Swinney, J.A. Vastano, Determining Lyapunov exponents from a time series, Physica 16D, Vol. 16, 285-317 (1985).

[20] S.S. Yang, C.K. Duan, Generalized synchronization in chaotic systems, Chaos, Solitons and Fractals, Vol. 10, 1703-1707 (1998).

[21] M.T. Yassen, Adaptive chaos control and synchronization for uncertain new chaotic dynamical system, Physics Letters A, Vol. 350, Number 1-2, 36-43 (2006).