ON A GROUP OF UNITARY DEFORMATIONS

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ABSTRACT. We develop in this work an idea initially due to J.M.Levy-Leblond ([10]) for deforming through explicit homotopies, the Fourier transform into identity, the convolution product into pointwise product, differential operators into others, et cetera, and we express here our best thanks to him.
The main tools to achieve this are the Hermite expansion of any tempered distribution, due to L.Schwartz ([6]) but not much used since, and Mehler's formula on Hermite functions, which allows us an explicit computation of the kernels of all the operators involved.

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For the sake of clarity in formulae and indices, we chose to write this article in the case of the one-dimensional space $\mathbb{R}$. However,since all the spaces of functions and distributions we will consider, as well as the Hermite functions themselves, are in several dimensions tensorial products of the one-dimensional ones, the generalizations to any dimension would be straightforward.

1. THE SETTING

1.1. **Hermite expansions.** Recall that the classical "Hermite functions" (introduced in [3], and much studied since, for instance in [1],[8],[2],[5],[9],[7]...)are defined, for $p \in \mathbb{N}$ and $x \in \mathbb{C}$ by:

$$h_p(x) = \lambda_p e^{-\frac{x^2}{2}} H_p(x) \tag{1}$$

where:

$$H_p(x) = (-1)^p e^{x^2} \frac{\partial}{\partial x}^p (e^{-x^2})$$

and:

$$\lambda_p = (\pi^{n/2} 2^p p!)^{-1/2}$$

As a set of functions on $\mathbb{R}$, the $(h_p)_{p\in\mathbb{N}}$ form a Hilbertian basis of the space $\mathbb{L}^2 = \mathbb{L}^2(\mathbb{R})$ of square-integrable functions. But they belong obviously to the space $\mathcal{S} = \mathcal{S}(R)$ of L. Schwartz of rapidly decreasing functions, and this author remarked already in [6] that when a function $\varphi \in \mathcal{S}$ is developed in this basis $\varphi = \sum c_p h_p$,

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with $c_p =< \varphi, h_p >$, the sequence (c_p) is itself rapidly decreasing (and conversely).

Transposing this we get:

Let $T \in \mathcal{S}' = \mathcal{S}'(\mathbb{R})$ a tempered distribution , and for $p \in \mathbb{N}$, $t_p =< T, h_p >$. Then $(t_p)_{p\in\mathbb{N}}$ is a slowly growing sequence, and the series $\sum t_p h_p$ is an expansion of T, at least weakly convergent in $\mathcal{S}'$.

Example 1: For the Dirac measure δ, we get:

$$\delta = \sum_{p\in\mathbb{N}} (-1)^p \pi^{-1/4} \frac{((2p)!)^{1/2}}{2^p p!} h_{2p}$$

Let us note $\partial_+ = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$ and $\partial_- = \frac{1}{\sqrt{2}}\left(x - \frac{\partial}{\partial x}\right)$. Since $\{\partial_+, \partial_-\}$ is a basis of the (non-commutative) algebra of linear differential operators with polynomial coefficients, and using the relations, for all $p \in \mathbb{N}$

$$\partial_- h_p = \sqrt{p+1}\, h_{p+1} \qquad \partial_+ h_p = \sqrt{p}\, h_{p-1} \tag{2}$$

(putting $h_{-1} = 0$), we can easily compute the Hermite expansion of any tempered distribution satisfying a differential equation of this type, at least through recurrence relations on the coefficients.

Example 2: If $sgn(x) = \sum_p t_p h_p(x)$, from $(sgn(x))' = \delta$, formula (2) and example 1, we deduce

$$t_{2p+1} = (-1)^p \frac{\pi^{-1/4}((2p)!)^{1/2}}{2^{p-1} p! \sqrt{p+1}} + \sqrt{\frac{p}{p+1}}\, t_{2p-1}$$

and $t_{2p} = 0$.

But identifying as usual $\mathbb{R}$ with its dual, the Hermite functions become eigenvalues of the Fourier transform, through the well-known formula

$$\forall p \in \mathbb{N} \qquad \mathcal{F} h_p = (-i)^p h_p \tag{3}$$

easily proved by induction on p using (2).

Thus if $T = \sum_p t_p h_p \in \mathcal{S}'$ then $\widehat{T} = \sum_p (-i)^p t_p h_p$.

Example 3: Using the last remark and the preceding examples, we get for instance:

$$1 = \sum_p \frac{\pi^{-1/4}((2p)!)^{1/2}}{2^{p-1/2} p!} h_{2p} \qquad x = \sum_p \frac{\pi^{1/4}((2p+1)!)^{1/2}}{2^{p-1} p!} h_{2p+1}$$

$$\delta' = \sum_p (-1)^{p+1} \frac{\pi^{-1/4}((2p+1)!)^{1/2}}{2^p p!} h_{2p}$$

and if $vp(1/x) = \sum_p t_p h_p$ then

$$t_{2p+1} = \frac{\pi^{1/4}((2p)!)^{1/2}}{2^{p-1/2} p! \sqrt{p+1}} - \sqrt{\frac{p}{p+1}}\, t_{p-1}$$

and $t_{2p} = 0$.

1.2. **Mehler's formula.** The Hermite functions are in fact entire functions on $\mathbb{C}$, and Janssen and Eijndhoven have obtained in [4] a beautiful characterization of those functions $\varphi = \sum c_p h_p \in \mathcal{S}$ such that the coefficient sequence (c_p) is of exponential decrease (as well as of their Bargmann and Wigner transforms!): interpreting their result when one chooses their function $M(x)$ to be $x^2/2$, one gets that the functions φ above are entire functions such that

$$\exists a, b, C > 0 \quad \forall x, y \in \mathbb{R} \quad |\varphi(x+iy)| \leq C \exp(-ax^2 + by^2)$$

Thus it is possible, by transposition again, to extend these Hermite expansions to a much bigger space of analytic functionals, containing in particular all functions on $\mathbb{R}$ growing at infinity not faster than some Ce^{ax^2} $(a, C) > 0$. But we will keep here to $\mathbb{L}^2$, the nuclear space $\mathcal{S}$ and its dual $\mathcal{S}'$.

Among the many formulae involving Hermite functions, we recall here a proof, using only Fourier inversion, of Mehler's formula, most important in the following: Mehler's formula: $\forall x, y, z \in \mathbb{C}, |z| < 1$:

$$\sum_{p\in\mathbb{N}} h_p(x)h_p(y)z^p = (\pi(1-z^2))^{-1/2} \exp\left(-\frac{1+z^2}{1-z^2}\frac{x^2+y^2}{2} + \frac{2z}{1-z^2}xy\right) \tag{4}$$

Proof:
We can assume $x, y, z \in \mathbb{R}$, so that $1 - z^2 > 0$.
If $k_p(x) = (-1)^p e^{x^2}(e^{-x^2})^{(p)}$ so that $h_p = \lambda_p k_p$ according to (1), we have:

$$e^{-x^2} = \pi^{-1/2} \int_{\mathbb{R}} e^{-u^2+2ixu}\, du$$

thus

$$k_p(x) = \pi^{-1/2} e^{x^2/2} (-2i)^p \int_{\mathbb{R}} u^p e^{-u^2+2ixu}\, du$$

then

$$\sum_{p\in\mathbb{N}} h_p(x)h_p(y)z^p = \pi^{-1/2} \sum_{p\in\mathbb{N}} \frac{k_p(x)k_p(y)}{2^p p!} z^p$$

$$= \pi^{-3/2} e^{\frac{x^2+y^2}{2}} \sum_{p\in\mathbb{N}} \int\int_{\mathbb{R}^2} \frac{(-2zuv)^p}{p!} e^{-u^2-v^2+2ixu+2iyv}\, dudv$$

$$= \pi^{-3/2} e^{\frac{x^2+y^2}{2}} \int\int_{\mathbb{R}^2} e^{-u^2-v^2+2ixu+2iyv-2zuv}\, dudv$$

$$= \pi^{-1} e^{\frac{x^2-y^2}{2}} \int_{\mathbb{R}} e^{(-(1-z^2)u^2+2i(x-zy)u)}\, du$$

$$= \pi^{-1/2} e^{\frac{x^2-y^2}{2}} (1-z^2)^{-1/2} \exp\left(-\frac{(x-yz)^2}{1-z^2}\right)$$

□

The right-hand term in (4) is an entire function of x and y, and analytic in z in a natural domain of $\ln(1-z^2)$, for instance $\mathbb{C} - (]-\infty, -1] \cup [1, \infty[)$. In particular it extends continuously to $\{z \in \mathbb{C} \mid |z| = 1, z \neq \pm 1\}$.
Taking $z = e^{-i\theta}$ $\quad (\theta \notin \pi\mathbb{Z})$, (4) gives as a corollary:

$$\forall x, y \in \mathbb{C}, \ \theta \in \mathbb{R} - \pi\mathbb{Z}$$

$$\sum_{p\in\mathbb{N}} h_p(x)h_p(y)e^{-ip\theta} = \pi^{-1/2}(1-e^{-2i\theta})^{-1/2} \exp\left(\tfrac{i}{2}\cot\theta(x^2+y^2) - \tfrac{i}{\sin\theta}xy\right)$$

2. Dephasing

2.1. **The dephasing group Γ.** For $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, let us define an operator Φ_α on the basis of Hermite functions by

$$\forall p \in \mathbb{N} \quad \Phi_\alpha h_p = e^{-ip\alpha} h_p \tag{5}$$

Reminding proposition 1.1, Φ_α is clearly an isomorphism of $\mathcal{S}'$, whose restriction to $\mathbb{L}^2$ is an unitary operator, and to $\mathcal{S}$ again an isomorphism. Furthermore, as $\Phi_\alpha \circ \Phi_\beta = \Phi_{\alpha+\beta}$, $\alpha \mapsto \Phi_\alpha$ is a group isomorphism of $\mathbb{R}/2\pi\mathbb{Z}$ onto a compact subgroup Γ of the algebra of operators of $\mathcal{S}'$ (or $\mathbb{L}^2$, or $\mathcal{S}$), and any reasonable topology on these algebras (weak or strong, inductive or projective limits) will, restricted to Γ, collapse by compacity into the usual topology of the circle.
Denoting by $\vee$ the Cech operator (defined by $\vee f(x) = f(-x)$), we get from (3):

$$\Phi_0 = id, \quad \Phi_{\frac{\pi}{2}} = \mathcal{F}, \quad \Phi_\pi = \vee, \quad \Phi_{\frac{3\pi}{2}} = \mathcal{F}^{-1} \tag{6}$$

In particular, for $t \in [0, \pi/2]$, Φ_t is a homotopy of the Fourier transform onto identity.

Through (2), the infinitesimal generator of the dephasing group Γ can be identified as a differential operator:

$$\frac{\partial}{\partial t}\Big|_{t=0} \Phi_t h_p = -iph_p = -i\partial_-\partial_+ h_p = -\frac{i}{2}\left(x^2 - \frac{\partial^2}{\partial x^2} - 1\right) h_p$$

Now $\Theta = x^2 - \frac{\partial^2}{\partial x^2} = \partial_-\partial_+ + \partial_+\partial_- = 2\partial_+\partial_- - 1 = 2\partial_-\partial_+ + 1$ is the classical Hermite operator ($\forall p \in \mathbb{N}, \quad \Theta h_p = (2p+1)h_p$) so that Γ solves the Cauchy problem for the Schrödinger equation of the harmonic oscillator:

$$\left\{ \begin{array}{l} \left(\frac{1}{i}\frac{\partial}{\partial t} - \Theta - 1\right) U_{t,x} = 0 \\ U_{0,x} = T_x \in \mathcal{S}' \end{array} \right\} \Leftrightarrow U_{t,x} = \Phi_t T_x \tag{7}$$

Thus we can write:

$$\Phi_t = e^{\frac{it}{2}} \sum_{n=0}^{\infty} \frac{(-i)^n t^n}{2^n n!} \Theta^n \tag{8}$$

where the series is weakly convergent on functions of $\mathcal{S}$.
A simple calculation using (2) and (5) gives

$$\Phi_\alpha \circ \frac{1}{i}\partial_+ \circ \Phi_{-\alpha} = \sin\alpha x + \cos\alpha \frac{1}{i}\frac{\partial}{\partial x} \tag{9}$$

$$\Phi_\alpha \circ x \circ \Phi_{-\alpha} = \cos\alpha x - \sin\alpha \frac{1}{i}\frac{\partial}{\partial x}$$

Thus conjugating by Φ_α means rotating of α the couple of self adjoint operators $\{x, \frac{1}{i}\frac{\partial}{\partial x}\}$ and this conjugation is, according to the remark in example 1, an isomorphism of the Weyl algebra of linear differential operators with polynomial coefficients.

We end this section with three remarks

Remark 1: Γ (which would become a torus in several dimensions), is of course a very small compact subgroup of the simplectic group. But its explicit definition in terms of Hermite expansion will allow exact computations, without a jump into the cotangent bundle.

Remark 2: The eigenvalues of the dephasing operators Φ_α are the $e^{-ip\alpha}$ $(p \in \mathbb{N})$ all distinct as soon as $\frac{\alpha}{\pi} \notin \mathbb{Q}$. Thus the Fourier transform appears as a very degenerate case of a dephasing operator. But there are many others, some entire power of which reduces to identity.

Remark 3: Let us call $\mathcal{C}$ the unit circle in $\mathbb{C}$. By proposition 1.1, the correspondence:

$$\mathcal{S}'(\mathbb{R}) \ni T = \sum_p c_p h_p \mapsto \sum_p c_p e^{ip\theta} \in \mathcal{D}'(\mathcal{C})$$

is an isomorphism of $\mathcal{S}'$ onto the space of analytic functions on the unit disk (by $(c_p) \mapsto \sum c_p z^p$) which admit a distribution boundary value on $\mathcal{C}$, and Φ_α becomes then the convolution operator by $\sum_p e^{-ip\alpha} e^{ip\theta} = \delta_\alpha$, that is to say the rotation of α on $\mathcal{C}$.

2.2. **Computing the kernels.** Since $\mathcal{S}'(\mathbb{R}^2)$ is a nuclear space, the operators Φ_α have a kernel, that is to say, there exists $K_\alpha(x,y) \in \mathcal{S}''(\mathbb{R}^2)$ such that:

$$\forall \phi \in \mathcal{S} \quad \Phi_\alpha \phi(x) = < K_\alpha(x,y), \phi(y) >$$

Of course by (6), we have:

$$K_0(x,y) = \delta(x-y), \quad K_\pi(x,y) = \delta(x+y), \quad K_{\pm\frac{\pi}{2}}(x,y) = (2\pi)^{-1/2} e^{\mp ixy} \tag{10}$$

This kernel is the sum of the weakly convergent series

$$K_\alpha(x,y) = \sum_{p \in \mathbb{N}} e^{-ip\alpha} h_p(x) h_p(y)$$

By using proposition 1.2 and the remark above, we can compute the kernel for any $\alpha \notin \pi\mathbb{Z}$:

$$K_\alpha(x,y) = c_\alpha \exp \frac{i}{2} \left(\cot\alpha (x^2+y^2) - \frac{2xy}{\sin\alpha} \right) \tag{11}$$

where:

$$c_\alpha = (2\pi|\sin\alpha|)^{-1/2} e^{-\frac{i}{2}(\alpha - (-1)^{\lfloor \frac{\alpha}{\pi} \rfloor}\pi/2)}$$

This can be rewritten:

$$K_\alpha(x,y) = c_\alpha \exp \frac{i}{4} \left[\cot\frac{\alpha}{2}(x-y)^2 - \tan\frac{\alpha}{2}(x+y)^2 \right] \tag{12}$$

making it clear that none of the dephasing operators (but for $\Phi_0 = id$) is a convolution operator, and that the phase of Φ_α is always quadratic, except for $\alpha = \frac{\pi}{2}$ (modπ), that is for $\mathcal{F}$ and $\mathcal{F}^{-1}$, where it is linear.
Yet another writing of the kernel, which we study further in the next section, is, as soon as $\alpha \notin (\pi/2)\mathbb{Z}$:

$$K_\alpha(x,y) = c_\alpha \exp \frac{i}{2} \left[-(\tan\alpha)x^2 + \cot\alpha \left(y - \frac{x}{\cos\alpha} \right)^2 \right] \tag{13}$$

Of course many identities can be derived from these computations.
Here are a few examples. Already, for $\alpha = 0, \pi$ we get:

$$\delta(x-y) = \sum_{p\in\mathbb{N}} h_p(y)h_p(x) \quad \text{and} \quad \delta(x+y) = \sum_{p\in\mathbb{N}} (-1)^p h_p(y)h_p(x)$$

these series being strongly convergent in $\mathcal{S}'$, so that for instance; for$x, y \in \mathbb{R}$

$$\sum_{p\in\mathbb{N}} h_p(x)h_p(y) = 0 \quad \text{for } x \neq y \text{ and } \sum_{p\in\mathbb{N}} h_p^2(x) = \infty$$

For $\alpha = \frac{\pi}{2}$, we get the weak identity in $\mathcal{S}'(\mathbb{R}^2)$:

$$e^{ixy} = \sqrt{2\pi} \sum_{p\in\mathbb{N}} i^p h_p(x)h_p(y) \tag{14}$$

For any $a \in \mathbb{R}$ this gives the Hermite expansion of plane waves

$$e^{iax} = \sqrt{2\pi} \sum_{p\in\mathbb{N}} i^p h_p(a)h_p(x)$$

which we might have got from the obvious:

$$\delta_a = \sqrt{2\pi} \sum_{p\in\mathbb{N}} h_p(a)h_p(x)$$

by Fourier transform. But by restriction of (14) to the diagonal (the right side series are only simply convergent, but uniformly on any bounded set):

$$\cos x^2 = \sqrt{2\pi} \sum_{p\in\mathbb{N}} (-1)^p h_{2p}^2(x) \quad \text{and} \quad \sin x^2 = \sqrt{2\pi} \sum_{p\in\mathbb{N}} (-1)^p h_{2p+1}^2(x) \tag{15}$$

3. Deformation

3.1. **Fresnel functions and Fresnel averages.** We call Fresnel function any function from $\mathbb{R}$ into $\mathbb{C}$ of the form:

$$F_{z,c}(x) = \exp \frac{iz}{2}(x-c)^2$$

with $c \in \mathbb{R}$ and $z \in \mathbb{C}$, $Im(z) \geq 0$, $z \neq 0$.
Since $Re(iz) = -Im(z) \leq 0$, Fresnel functions define tempered distributions, and belong in fact to $\mathcal{S}$ as soon as $Im(z) > 0$.
The Fourier transform of a Fresnel function is another one, with a factor and dephased: a simple residue calculation gives:

$$\widehat{F_{z,c}}(x) = \frac{e^{i(\pi/4 - cx)}}{\sqrt{z}} F_{-\frac{1}{z},0}(x) \tag{16}$$

And we call Fresnel average around $c \in \mathbb{R}$ the tempered distribution:

$$\varphi \longmapsto \int_{\mathbb{R}} F_{z,c}(x)\varphi(x)\, \mathrm{d}x$$

Clearly these averages take much more into account the behaviour of φ in a neighborhood of c than anywhere else: for instance ,by the stationary phase lemma, if $z = Re^{i\theta}$, the last integral is equivalent to $\sqrt{\frac{\pi}{2}} e^{i\frac{\pi}{4}} R^{-1/2}\varphi(c)$ as $R \to \infty$.
For $\varphi \in \mathcal{S}$ and $\alpha \notin (\pi/2)\mathbb{Z}$, (13) gives:

$$\Phi_\alpha\varphi(x) = c_\alpha \exp\left(-\frac{i}{2}\tan\alpha . x^2\right) \int_{\mathbb{R}} \exp\left[\frac{i}{2}\cot\alpha\left(y - \frac{x}{\cos\alpha}\right)^2\right] \varphi(y)\, dy \tag{17}$$

meaning that to $\Phi_\alpha\varphi(x)$ is a Fresnel average of φ around $\frac{x}{\cos\alpha}$ (which tends to $\varphi(x)$ as $\alpha \to 0$. But since $\Phi_\alpha = \Phi_{\alpha-\frac{\pi}{2}} \circ \mathcal{F}$),

$$(18)\quad \Phi_\alpha\varphi(x) = c_{\alpha-\frac{\pi}{2}} \exp\left(\frac{i}{2}\cot\alpha.x^2\right)\int_{\mathbb{R}} \exp\left[-\frac{i}{2}\tan\alpha\left(y - \frac{x}{\sin\alpha}\right)^2\right]\widehat{\varphi}(y)\,dy$$

showing that $\Phi_\alpha\varphi(x)$ is a Fresnel average of $\widehat{\varphi}$ just as well.

Example 4: Let us for instance give an explicit formula for the square root of the Fourier transform, $\mathcal{F}^{1/2} = \Phi_{\pi/4}$. By (17) and(18), we have, for $\varphi \in \mathcal{S}$:

$$(19)\quad \mathcal{F}^{1/2}\varphi(x) = c_{\frac{\pi}{4}} e^{\left(-\frac{i}{2}x^2\right)}\int_{\mathbb{R}} \exp\left[\frac{i}{2}(y - x\sqrt{2})^2\right]\varphi(y)\,dy$$

$$= c_{-\frac{\pi}{4}} e^{\left(\frac{i}{2}x^2\right)}\int_{\mathbb{R}} \exp\left[-\frac{i}{2}(y + x\sqrt{2})^2\right]\widehat{\varphi}(y)\,dy$$

with $c_{\frac{\pi}{4}} = (\pi\sqrt{2})^{-1/2}e^{-i\frac{\pi}{8}}$ and $c_{-\frac{\pi}{4}} = \overline{c_{\frac{\pi}{4}}}$.
Example 5: Formulae (11) and (12) allows us to compute $\Phi_\alpha T$ for many distributions $T \in \mathcal{S}'$, those with compact support at least.
For instance for the Dirac measure δ_a at a point $a \in \mathbb{R}$

$$(20)\quad \Phi_\alpha\delta_a = K_\alpha(x,a) = c_\alpha \exp i[\cot\alpha.\frac{a^2}{2} - \frac{ax}{\sin\alpha} + \cot\alpha\frac{x^2}{2}]$$

as soon as $\alpha \notin \frac{\pi}{2}\mathbb{Z}$, and it gives the Hermite expansion of Fresnel functions, reminding that $K_\alpha(x,a) = \sum_p e^{ip\alpha}h_p(a)h_p(x)$.
Remark 4: Formulae (9) generalize to the Fresnel operators $\partial_z = x + \frac{i}{z}\frac{\partial}{\partial x}$ ($z \in \mathbb{C}, Imz \geq 0, z \neq 0$, thus for instance $\partial_i = \sqrt{2}\,\partial_+$): they all have a one dimensional kernel in $\mathcal{S}'$, generated by $F_{z,0}$ and using (9)

$$\Phi_\alpha \circ \partial_z \circ \Phi_{-\alpha} = \left(\cos\alpha - \frac{\sin\alpha}{z}\right)\partial_{z'}$$

for $z \neq \tan\alpha$, with $z' = \frac{z\cos\alpha - \sin\alpha}{z\sin\alpha + \cos\alpha} = g_\alpha(z)$, where g_α is a hyperbolic rotation of the Poincaré half plane. Thus $\Phi_\alpha(\ker\partial_z) = \ker\partial_{g_\alpha(z)}$ and $\Phi_\alpha(e^{iz\frac{x^2}{2}}) = a_\alpha e^{ig_\alpha(z)\frac{x^2}{2}}$, where $a_\alpha = \Phi_\alpha(e^{iz\frac{x^2}{2}})(0) = e^{i\alpha/2}(z\sin\alpha + \cos\alpha)^{-1/2}$ from (17), if $0 < \alpha < \pi$.

3.2. **Deformation of products.** Every operator $A : \mathcal{S}' \to \mathcal{S}'$ at least weakly continuous, has a matrix $(a_{pq})_{p,q\in\mathbb{N}}$ defined by $a_{pq} =< Ah_p, h_q >$, which characterises it, and its conjugate $\Phi_\alpha \circ A \circ \Phi_{-\alpha}$ has , by (5) and since $\Phi_{-\alpha} = \Phi_\alpha^*$, the matrix $(e^{i\alpha(p-q)}a_{pq})_{pq}$. Mehler's formula and its corollaries will allow us to compute the kernel of the deformed operator $\Phi_\alpha \circ A \circ \Phi_{-\alpha}$, knowing that of A.
As a last example , we will compute here the kernel of the deformed product of two functions: let us define a bilinear operation from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{S}$, for $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$, by

$$\forall\phi,\psi \in \mathcal{S}(\mathbb{R}) \quad \phi \overset{\alpha}{*} \psi = \Phi_\alpha(\Phi_{-\alpha}\phi * \Phi_{-\alpha}\psi)$$

Clearly $\overset{\alpha}{*}$ is continuous, $\overset{0}{*}$ is the usual convolution product while $\overset{\frac{\pi}{2}}{*}$ is the pointwise product and $[0,\pi/2] \ni \alpha \longmapsto \overset{\alpha}{*}$ is a homotopy between the two classical products. Since $\Phi_\pi = \vee$, one checks easily that $\overset{\alpha+\pi}{*} = \overset{\alpha}{*}$.

But $\overset{\alpha}{*}$ is a nuclear operator: for $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ and $x \in \mathbb{R}$ there exists $K_\alpha(x,.,.) \in \mathcal{S}'(\mathbb{R}^2)$ such that for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^2)$:

$$(\phi \overset{\alpha}{*} \psi)(x) = < K_\alpha(x,y,z), \phi(y)\psi(z) >$$

In $\mathbb{R}^2_{x,y}$, denoting D_x the straight line of equation $y+z=x$, and δ_{D_x} her canonical measure. Clearly $K_0(x) = \delta_{D_x}$ and $K_{\frac{\pi}{2}}(x) = \delta_{x,x}$. In the general case, we can compute the kernel

$$\phi \overset{\alpha}{*} \psi(x) = \iiiint K_\alpha(x,y)K_{-\alpha}(z,u)K_{-\alpha}(y-z,v)\phi(u)\psi(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}z$$

$$\begin{aligned} = \quad & c_\alpha c_{-\alpha}^2 \iiiint \exp\frac{i}{2}\left[-z^2\cot\frac{\alpha}{2} + z^2\tan\frac{\alpha}{2} + yz\cot\frac{\alpha}{2} - uz\cot\frac{\alpha}{2} - yz\tan\frac{\alpha}{2}\right. \\ & \left. -uz\tan\frac{\alpha}{2} + vz\cot\frac{\alpha}{2} + vz\tan\frac{\alpha}{2}\right] \\ & \exp\frac{i}{4}\left[-2xy\cot\frac{\alpha}{2} + x^2\cot\frac{\alpha}{2} - x^2\tan\frac{\alpha}{2} - 2xy\tan\frac{\alpha}{2} - u^2\cot\frac{\alpha}{2}\right. \\ & \left. +2uy\cot\frac{\alpha}{2} + u^2\tan\frac{\alpha}{2} + 2uy\tan\frac{\alpha}{2} - v^2\cot\frac{\alpha}{2} + v^2\tan\frac{\alpha}{2}\right]\phi(u)\psi(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}z \end{aligned}$$

Integrating with respect to z, and using (16), and later with respect to y, we get finally:

$$K_\alpha(x,u,v) = \frac{c_{-\alpha}}{2\pi\cos\alpha}\exp\frac{i\cot\alpha}{2}\left[\frac{2xv+2xu-2uv}{\cos^2\alpha} + x^2 - u^2 - v^2\right].$$

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