

CONVEX SETS AND POLYGONS

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ABSTRACT. In this paper the properties of convex set and its inscribed polygons in a two dimensional Euclidean space are studied. Consequently we applied our results in solving a problem in geometry of Banach space.

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Let E_2 be a two dimension Euclidean space, K be a bounded central symmetric convex set with origin o as its center and interior point, and $C = \partial K$ be the boundary of K . For vectors x , and y in E_2 , we use $[x, y]$ to denote the line segment connecting x and y , $|x|$ the usual Euclidean norm, and $||[x, y]||$ or $|y - x|$ the usual Euclidean length of $[x, y]$ in E_2 .

1. INVERTIBLE LINEAR MAPPING IN E_2

Let T be an invertible linear mapping from E_2 to E_2 , then T maps a line segment to a line segment; a pair of parallel lines to a pair of parallel lines [4].

Proposition 1: T does not change the ratio of lengths of two parallel line segments.

Proof: Suppose $[p_1, q_1] \parallel [p_2, q_2]$, then $q_2 - p_2 = c(q_1 - p_1)$ for some real number c . So $T(q_2) - T(p_2) = c(T(q_1) - T(p_1))$. Therefore $\frac{|T(q_2) - T(p_2)|}{|T(q_1) - T(p_1)|} = \frac{|q_2 - p_2|}{|q_1 - p_1|} = |c|$.

Let F be the set of all ellipses centered at the origin of E_2 .

Proposition 2: F is invariant under T .

Proof: Let $F : (x, y)A\begin{pmatrix} x \\ y \end{pmatrix} = c$, where A is a 2×2 matrix with $\det(A) > 0$, be an ellipse in E_2 . Let $T\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$, where M is a 2×2 invertible matrix representing T under standard coordinate system of E_2 . Then $TF : (x, y)M'AM\begin{pmatrix} x \\ y \end{pmatrix} = c$, where $\det(M'AM) = (\det(M))^2 \det(A) > 0$, is again an ellipse.

If F^1 , and F^2 are two ellipses in E_2 .

It is easy to see via mapping both F^1 and F^2 to the unit circle S of E_2 : $T_1(F^1) = T_2(F^2) = S$ that there exists an invertible mapping $T = T_2^{-1}T_1$ such that $T(F^1) = F^2$.

2. DIRECTIONAL DISTANCE

We use n -gon to denote the polygon with n sides.

Definition 1: The directional distance or C -length of line segment $[x, y]$ with respect to $C = \partial K$ in E_2 , which is denoted by $\|y - x\|_C$ is the ratio of $|y - x|$ and $|r|$, where $(y - x) \parallel r$, and $r \in C$.

Definition 2: A n -gon in E_2 is called regular n -gon with respect to C if all n -sides have the same C -lengths.

Remark: If C is unit circle of E_2 , regular n -gon becomes normal n -gon.

A continuous mapping $x(t)$ from a closed interval $[a, b]$ to a metric space X is called a curve in X : $D = x(t)$, $a \leq t \leq b$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$.

The concept of the C -length of a curve in E_2 resembles the same concept in usual case.

Definition 3: For curve $D = x(t)$, let P stands for a partition $a = t_0 < t_1 < t_2 < \dots < t_i < \dots < t_n = b$ of interval $[a, b]$ and $l(D, P) = \sum_{i=1}^n \|x(t_i) - x(t_{i-1})\|_C$, where $x_i(t)$, $i = 0, 1, 2, \dots, n$ are called partition points on D , then the C -length $l(D)$ of curve $D = x(t)$, $a \leq t \leq b$, is defined as the least upper bound of $l(D, P)$ for all possible partitions of $[a, b]$: $l(D) = \sup_P \{l(D, P)\}$. If $l(D)$ is finite the curve is called rectifiable.

For a normed linear space X , we use X_2 to denote a two dimensional subspace of X , then $S(X_2) = \{x : \|x\| = 1\}$ is a simple closed central symmetric curve and unique up to orientation.

Theorem 1: Let E_2 , K , and C be as above. Let H_1, H_2 be closed convex subsets of E_2 , and have nonvoid interiors. If $H_1 \subseteq H_2$, then $l(\partial(H_1)) \leq l(\partial(H_2))$, where $l(\partial(H_i))$ denote the C -lengths of the circumferences of H_i , $i = 1, 2$.

Theorem 2: Let E_2 , K , and C be as above and $2L$ be the C -length of C itself, then $6 \leq 2L \leq 8$.

For definition 3, theorem 1 and theorem 2 see [2] and [3].

For a vector $x \in C$, let κ be one of arcs of C from x to $-x$, K' be one half of K containing κ , and let $g(s) : [0, L] \rightarrow \kappa$ be the standard representation of κ in terms of arc length, where L is the C -length of κ , $g(0) = x$, and $g(L) = -x$.

Theorem 3:

- (i) $\varphi(s) = \|g(s) - x\|_C$ is a non-decreasing continuous function from $[0, L]$ to $[0, 2]$.
- (ii) If C does not contain any line segment, then $\varphi(s)$ is a strictly increasing function.
- (iii) If C does not contain any line segment with C -length greater than 1, then $\varphi(s)$ is a strictly increasing function from $[0, L_1]$ to $[0, 1]$, where L_1 is some number between 0 and L .

Proof: The first part (i) follows from theorem 4F of [3].

To prove (ii), let $y_1 = g(s_1)$, $y_2 = g(s_2) \in \kappa$ with $0 \leq s_1 < s_2 \leq L$, $y_1 \neq y_2$ but $\|y_1 - x\|_C = \|y_2 - x\|_C = a$. Let $u_1 = (y_1 - x)/a$ and $u_2 = (y_2 - x)/a$, then $u_1, u_2 \in \kappa$ and $[u_1, u_2] \parallel [y_1, y_2]$ as shown in Figure 1. So the four points y_1, y_2, u_1 , and u_2 are co-linear by the convexity of K' . Therefore the whole line segment $[u_2, y_1] \subseteq \kappa$.

To prove (iii), let y_1, y_2, u_1 , and u_2 be as in (ii), and $\|y_1 - x\|_C = \|y_2 - x\|_C = a$, where $0 < a \leq 1$.

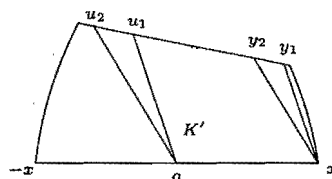


FIGURE 1

If $a = 1$, $\|u_2 - y_1\|_C = \|u_2 - y_2\|_C + \|y_2 - y_1\|_C = 1 + \|y_2 - y_1\|_C > 1$.

If $0 < a < 1$ and $\|u_2 - y_1\|_C \leq 1$, let $v = y_1 - u_2$ then $v \in K$. Since $[x, y_1] \parallel [o, u_1]$, $[v, y_1] \parallel [o, u_2]$, and $|y_1 - x|/|u_1| = |y_2 - x|/|u_2| = a < 1$, we have $x \in \Delta oy_1 v \setminus \partial(\Delta oy_1 v)$ as shown in Figure 2. But $\Delta oy_1 v \subseteq K$. This implies that $x \in$

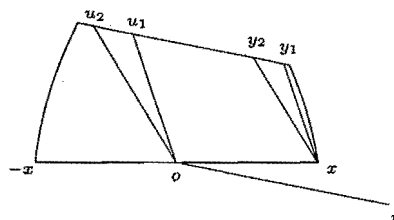


FIGURE 2

$K \setminus C$, which contradicts $x \in C$. So $\|u_2 - y_1\|_C > 1$. This contradicts the hypothesis.

Theorem 4:

- (i) If C does not contain any line segment, $n \geq 3$, and $x \in C$, then there exists a unique inscribed regular n -gon of C with x as its one of n vertices.
- (ii) If C does not contain any line segment with C -length greater than 1, $n \geq 8$, and $x \in C$, then there exists a unique inscribed regular n -gon of C with x as its one of n vertices.

Proof: Proof of (i): For a vector $x \in C$, let $g(s) : [0, 2L] \rightarrow C$ be the standard representation of C in terms of arc length, where $2L$ is the C -length of C . Let $n+1$ vectors: $x = x_1 = g(0)$, $y_2 = g(s_1)$, $y_3 = g(s_2)$, ..., $y_{n-1} = g(s_{n-2})$, $y_n = g(s_{n-1})$ and $y_{n+1} = g(s_n)$, where $0 < s_1 < s_2 < \dots < s_{n-2} < s_{n-1} < s_n \leq 2L$, are arranged counterclockwise on C such that $\|y_2 - x_1\|_C = \|y_3 - y_2\|_C = \dots = \|y_n - y_{n-1}\|_C = \|y_{n+1} - y_n\|_C$. From theorem 3, s_n is a strict increasing and continuous function of $\|y_2 - x_1\|_C$. Hence there exists a unique vector $x_2 \in C$ and consecutive vectors $x_3, \dots, x_n$ and x_{n+1} on C such that $\|x_2 - x_1\|_C = \|x_3 - x_2\|_C = \dots = \|x_n - x_{n-1}\|_C =$

$\|x_{n+1} - x_n\|_C$, and $x_{n+1} = g(2L)$. This implies $x_{n+1} = x_1 = x$. The polygon with vertices $x = x_1, x_2, x_3, \dots, x_{n-1}$ and x_n is an inscribed regular n -gon of C with x as its one of n vertices.

The part (ii) would be proved by using theorem 1, theorem 2, (iii) of theorem 3, and same argument as in the proof of part (i).

3. INSCRIBED REGULAR POLYGON IN C

If S is the unit circle in E_2 , then every inscribed regular n -gon becomes normal n -gon and has side length of $2 \sin \frac{\pi}{2n}$. By using an invertible mapping from a given ellipse F to the unit circle, every inscribed regular n -gon in F has side F -length of $2 \sin \frac{\pi}{2n}$ from proposition 1 and 2 in section 1.

Theorem 5: Every inscribed regular n -gon in an ellipse F has side F -length of $2 \sin \frac{\pi}{2n}$.

But is the converse true? That is: let K be a bounded central symmetric convex set in E_2 and $C = \partial K$, if every inscribed regular n -gon in C has side C -length of $2 \sin \frac{\pi}{2n}$, then C must be an ellipse?

Lemma 1: If the curve C is invariant under a rotation by α , where $0 < \alpha \leq 45^\circ$, then C does not contain any line segment with C -length greater than 1.

Proof: This is a direct result of theorem 2.

Theorem 6:

- (i) If the curve C is invariant under a rotation by $\frac{\pi}{4n}$, and C does not contain any line segment, then for any $x \in C$ the unique inscribed regular $4n$ -gon with x as its $4n$ vertices has side C -length of $2 \sin \frac{\pi}{4n}$.
- (ii) If $n \geq 2$, and C is invariant under a rotation by $\frac{\pi}{4n}$, the above result is still true.

Proof: In (ii), $n \geq 2$ implies that $\frac{\pi}{4n} < 45^\circ$ and $4n \geq 8$, so in both cases (i) and (ii) the existence and uniqueness of such inscribed regular polygon is shown in theorem 4. Suppose $x = x_1 = x_{4n+1}, x_2, x_3, \dots, x_{4n}$ be the consecutive $4n$ vertices of the inscribed regular $4n$ -gon as shown in Figure 3. Then $\angle x_2 x_1 o = \frac{(2n-1)\pi}{4n}$,

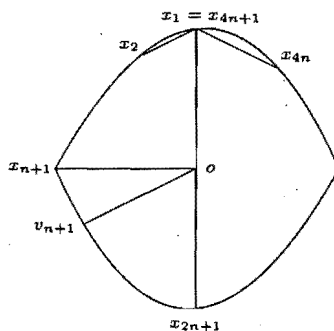


FIGURE 3

and $\angle x_1 o x_{n+1} = \frac{\pi}{2}$. Let $v_{n+1} \in C$, and $\angle x_{n+1} o v_{n+1} = \frac{\pi}{4n}$ then $|v_{n+1}| = |x_1|$, and $(x_2 - x_1) \parallel v_{n+1}$. So $\|x_2 - x_1\|_C = \frac{|x_2 - x_1|}{|v_{n+1}|} = \frac{|x_2 - x_1|}{|x_1|} = 2 \sin \frac{\pi}{4n}$.

Theorem 7:

- (i) If the curve C is invariant under a rotation by $\frac{\pi}{2n+1}$, and C does not contain any line segment, then for any $x \in C$ the unique inscribed regular $(4n+2)$ -gon with x as its $4n+2$ vertices has side C -length of $2 \sin \frac{\pi}{4n+2}$.
- (ii) If $n \geq 2$, and C is invariant under a rotation by $\frac{\pi}{2n+1}$ the above result is still true.

Proof: In (ii), $n \geq 2$ implies that $\frac{\pi}{2n+1} < 45^\circ$ and $4n+2 \geq 8$, so in both cases (i) and (ii) the existence and uniqueness of such inscribed regular polygon comes from theorem 4. Suppose $x = x_1 = x_{4n+3}, x_2, x_3, \dots, x_{4n+2}$ be the consecutive $4n+2$ vertices of the inscribed regular $(4n+2)$ -gon as shown in Figure 4. Then $\angle x_2 x_1 o = \frac{n\pi}{2n+1}$, and $\angle x_1 o x_{n+2} = \frac{(n+1)\pi}{2n+1}$. So $(x_2 - x_1) \parallel x_{n+2}$. Therefore $\|x_2 - x_1\|_C = \frac{|x_2 - x_1|}{|x_{n+2}|} = \frac{|x_2 - x_1|}{|x_1|} = 2 \sin \frac{\pi}{4n+2}$.

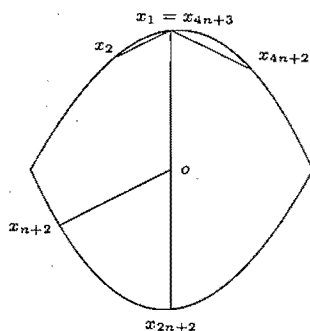


FIGURE 4

Theorem 8:

- (i) If the curve C is invariant under a rotation by $\frac{\pi}{2(2n+1)}$, and C does not contain any line segment, then for any $x \in C$ the unique inscribed regular $(2n+1)$ -gon with x as its $2n+1$ vertices has side C -length of $2 \sin \frac{\pi}{2n+1}$.
- (ii) If $n \geq 4$, and C is invariant under a rotation by $\frac{\pi}{2(2n+1)}$ the above result is still true.

Proof: In (ii), $n \geq 4$ implies that $\frac{\pi}{2n+1} < 45^\circ$ and $2n+1 \geq 8$, so in both cases (i) and (ii) the existence and uniqueness of such inscribed regular polygon is shown in theorem 4. Suppose $x = x_1 = x_{2n+2}, x_2, x_3, \dots, x_{2n+1}$ be the consecutive $2n+1$ vertices of the inscribed regular $(2n+1)$ -gon as shown in Figure 5. Then $\angle x_2 x_1 o = \frac{(2n-1)\pi}{2(2n+1)}$. Let $v_n \in C$ with $\angle x_1 o v_n = \frac{(2n+3)\pi}{2(2n+1)}$ then $|v_n| = |x_1|$, and $(x_2 - x_1) \parallel v_n$. So $\|x_2 - x_1\|_C = \frac{|x_2 - x_1|}{|v_n|} = \frac{|x_2 - x_1|}{|x_1|} = 2 \sin \frac{\pi}{2n+1}$.

By using an invertible mapping from E_2 to E_2 and proposition 1, we can enhance the above three theorems a step further:

Theorem 9:

- (i) If the curve C is an image under an invertible mapping of a curve C' which is invariant under a rotation by $\frac{\pi}{4n}$, and C does not contain any line segment, then for any $x \in C$ the unique inscribed regular $4n$ -gon with x as its $4n$ vertices has side C -length of $2 \sin \frac{\pi}{4n}$.

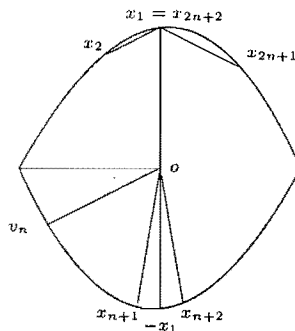


FIGURE 5

- (ii) If $n \geq 2$, and C is an image under an invertible mapping of a curve C' which is invariant under a rotation by $\frac{\pi}{4n}$, then the above result is still true.

Theorem 10:

- (i) If the curve C is an image under an invertible mapping of a curve C' which is invariant under a rotation by $\frac{\pi}{2n+1}$, and C does not contain any line segment, then for any $x \in C$ the unique inscribed regular $(4n+2)$ -gon with x as its $4n+2$ vertices has side C -length of $2 \sin \frac{\pi}{4n+2}$.
- (ii) If $n \geq 2$, and C is an image under an invertible mapping of a curve C' which is invariant under a rotation by $\frac{\pi}{2n+1}$, then the above result is still true.

Theorem 11:

- (i) If the curve C is an image under an invertible mapping of a curve C' which is invariant under a rotation by $\frac{\pi}{2(2n+1)}$, and C does not contain any line segment, then for any $x \in C$ the unique inscribed regular $(2n+1)$ -gon with x as its $2n+1$ vertices has side C -length of $2 \sin \frac{\pi}{2n+1}$.
- (ii) If $n \geq 4$, and C is an image under an invertible mapping of a curve C' which is invariant under a rotation by $\frac{\pi}{2(2n+1)}$, then the above result is still true.

It is easy to construct a nonelliptical bounded central symmetric convex set satisfying the conditions of any one of the above six theorems. So the inverse of theorem 5 is not true.

4. APPLICATIONS TO GEOMETRY OF BANACH SPACES

Proposition 3:

- (i) Let X_2 be a two dimensional Banach space, then $U(X_2) = \{x : \|x\| \leq 1\}$, the unit ball of X_2 , is a bounded central symmetric convex set;
- (ii) For any bounded central symmetric convex set K with the origin o as its interior point in E_2 there is a unique two dimensional Banach space with K as its unit ball.
- (iii) X_2 is a Hilbert space if and only if $S(X_2)$ is an ellipse.

Proof: The part (i) comes directly from definition of norm.

To prove part (ii), for any $x \in E_2$ we define a norm as the C -length, where $C = \partial K$ the boundary of K . We only need to show how to get the triangle inequality. Let $u, v \in E_2$, without loss of generality we assume $\|u\|_C = 1$, and $\|v\|_C \geq 1$. Let $v' = [o, v] \cap C$, $w = av$, where $a > 0$ such that $|w - v| = |u|$ as shown in Figure 6. Let $f = [u, v'] \cap [o, u + v]$ and $g = C \cap [o, u + v]$, then $\|o, f\| \leq \|o, g\|$ by the convexity of K . Therefore $\|u + v\|_C = \frac{|u+v|}{|g|} \leq \frac{|u+v|}{|f|} = \frac{|w|}{|v'|} = \|w\|_C = \|v\|_C + 1 = \|u\|_C + \|v\|_C$.

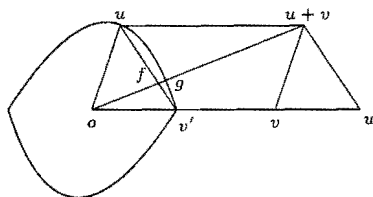


FIGURE 6

The part (iii) comes from proposition 1 and 2 in section 1.

The theorems 6-11 show that for a Banach space X , the condition that every inscribed regular n -gon of $S(X)$ has side C -length of $2 \sin \frac{\pi}{2n}$ does not implies that X becomes an Euclidean space.

It is easy to see that if X is an inner product space, then for any fixed $\lambda > 0$, $\|x\| = \|y\| = 1$, $\|x + \lambda y\| = \|x - \lambda y\|$ which implies that $\|x + \lambda y\|^2 = \|x - \lambda y\|^2 = \|x\|^2 + \lambda^2 \|y\|^2 = 1 + \lambda^2$. In 1980, Borwein and Keener [1] asked for a Banach space X whether the existence of such λ implies that X is an inner product space. In two dimensional Banach space X_2 , let $\lambda = 1$ and $C = S(X_2)$, Borwein and Keener's question becomes: if every inscribed regular polygon with 4 sides, rhombus, of $S(X_2)$ has side C -length of $2 \sin \frac{\pi}{4} = \sqrt{2}$, must X_2 be a two dimensional Euclidean space?

Let $C : \rho = (|\cos 4\theta|^{\frac{3}{2}} + |\sin 4\theta|^{\frac{3}{2}})^{-\frac{2}{3}}$, $0 \leq \theta \leq 2\pi$, then C is a central symmetric nonelliptic curve, invariant under rotation of $\frac{\pi}{4}$ and has no line segment. Let X_2 be a two dimensional Banach space with C as its unit sphere, then every inscribed regular rhombus of $S(X_2)$ has side C -length of $2 \sin \frac{\pi}{4} = \sqrt{2}$ from (i) of theorem 6, but X_2 is not Euclidean space by (iii) of proposition 3.

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