

Fuzzy line graphs and their intuitionistic extension : Mathematical properties and applications

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Abstract

Fuzzy line graphs extend the classical line-graph construction so that edge adjacency is recorded as a graded membership value rather than a binary fact, and the resulting construction, together with its basic transformation and connectivity theory, was developed by Mohammad [1]. That framework, however, records only a single degree of confidence for each link and has no place for an explicit degree of doubt. Many of the systems it is meant to model—unreliable sensor links, contested survey data, roads whose condition reports conflict—are more naturally described by an intuitionistic fuzzy graph, in which every vertex and edge carries both a membership degree and an independent non-membership degree, leaving a hesitation margin between them. This paper extends the fuzzy line graph construction of [1] to this intuitionistic setting. We define the intuitionistic fuzzy line graph (IFLG), prove that the construction is well defined and order-preserving in both the membership and the non-membership component, and prove that fuzzy connectivity is preserved under the extended construction. We then prove a Hesitation Interpolation Lemma, a result with no counterpart in the purely fuzzy case, showing that the hesitation degree of every edge produced by the construction lies between the hesitation degrees of the two edges it joins, so that uncertainty about a link is neither manufactured nor erased by the transformation. The framework is illustrated on a worked example and on a transportation network in which each road is assigned both a confidence and a doubt value, and the discussion compares the intuitionistic construction with its fuzzy predecessor, showing the fuzzy line graph to be the special case in which hesitation is uniformly zero.

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1. Introduction

A classical graph records only whether two vertices are linked. Fuzzy graph theory [2], [3] replaces this binary fact with a membership degree in $[0, 1]$, and the line graph, the classical device that turns adjacency between edges into adjacency between vertices has been extended to this graded setting by Mohammad [1], who gave the edge-based transformation rule, proved that it is well defined and order-preserving, and established a connectivity-preservation theorem together with supporting lemmas and a transportation-network case study. A single membership degree, however, cannot distinguish “we are fairly confident this link holds” from “we simply have not gathered enough evidence to say either way.” Intuitionistic fuzzy sets [4] address exactly this by attaching to each element both a membership degree and an independent non-membership degree, whose sum need not reach 1; the gap is the hesitation degree, a direct measure of unresolved uncertainty. Intuitionistic fuzzy graphs, built on this idea, have been studied in their own right [5], [6].

Line graphs have a long history in both the crisp and the fuzzy settings. Fuzzy line graphs were first studied by Mordeson [7], building on the classical line graph [8] and on Whitney’s [9] classical result that, with one small exception, a connected graph is determined by its line graph. The standard monograph on fuzzy graphs and hypergraphs is [10]. On the intuitionistic side, Akram and Dudek [11] developed intuitionistic fuzzy hypergraphs, and Akram, Dudek, and Yousaf [12] introduced line graphs for vague graphs, whose truth and falsity memberships play a role closely related to that of the membership and non-membership degrees used here; vague sets are known to be mathematically equivalent to intuitionistic fuzzy sets [13]. More recent work has studied matrix representations of intuitionistic fuzzy graphs [14] and dominating vertex sets in them [15]. What has not, to our knowledge, been examined is how the *hesitation degree* behaves under the line-graph transformation, and whether the connectivity theory of [1] survives it. That is the gap this paper addresses. Chaudhari et al. [16] investigated the bit error rate (BER) performance of different error-correction techniques with particular emphasis on a concatenated Reed–Solomon (RS) and convolutional coding scheme. Salunkhe et al. [17] proposed a hybrid framework for secure image transmission by integrating encryption and digital watermarking on a reconfigurable FPGA platform.

2. Background: Fuzzy Graphs and the Fuzzy Line Graph

We summarise the definitions and results of [1] needed below; full statements, proofs and worked examples appear there. A fuzzy graph is a triple $G = (V, \sigma, \mu)$, where V is a finite non-empty vertex set, $\sigma: V \rightarrow [0, 1]$ assigns each vertex a membership degree, and $\mu: V \times V \rightarrow [0, 1]$ assigns each edge a membership degree subject to $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$ [3]. Given such a G , its fuzzy line graph $L_\mu(G)$ has one vertex for each edge $e = (u, v)$ of G , with membership $\sigma_L(e) = \mu(u, v)$; two such vertices are adjacent exactly when the corresponding edges of G share an endpoint, and the connecting edge is assigned membership $\mu_L(e_1, e_2) = \min\{\mu(u, v), \mu(v, w)\}$. Mohammad [1] proves that this map is well defined, single-valued and order-preserving (his Theorem 4.1), that fuzzy connectivity of G with every vertex incident to at least two positive-membership edges implies fuzzy connectivity of $L_\mu(G)$, and gives a supporting lemma, a completeness corollary, and a five-city road-network illustration. We build directly on this apparatus; the extension below adds a second, independent component to every membership value. When every degree is 0 or 1, the fuzzy line graph reduces to the classical line graph of the support graph [7]; in either case the line graph has one vertex per edge of G and $\sum_v C(d(v), 2)$ edges, where $d(v)$ is the degree of v [8].

3. Intuitionistic Fuzzy Graphs and the Intuitionistic Fuzzy Line Graph

3.1 Preliminaries

An intuitionistic fuzzy set A on a universe X assigns to each $x \in X$ a membership degree $\mu_A(x) \in [0, 1]$ and a non-membership degree $\nu_A(x) \in [0, 1]$ with $\mu_A(x) + \nu_A(x) \leq 1$ [4]. The hesitation degree $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \geq 0$ is the residual uncertainty not accounted for by either commitment. An intuitionistic fuzzy graph (IFG) is a pair $G = (V, \sigma, \mu)$, where $\sigma = (\sigma_1, \sigma_2): V \rightarrow [0, 1]^2$ assigns each vertex a membership and a non-membership degree with $\sigma_1(v) + \sigma_2(v) \leq 1$, and $\mu = (\mu_1, \mu_2): V \times V \rightarrow [0, 1]^2$ assigns each edge a membership and a non-membership degree subject to, for all $u, v \in V$,

$$\mu_1(u, v) \leq \min\{\sigma_1(u), \sigma_1(v)\} \text{ and } \mu_2(u, v) \geq \max\{\sigma_2(u), \sigma_2(v)\}. \quad (1)$$

This is the standard IFG edge condition of Parvathi and Karunambigai [5]; the notion of an intuitionistic fuzzy graph itself goes back to Shannon and Atanassov [6]. As in the fuzzy case, the edge hesitation degree $\pi(u, v) = 1 - \mu_1(u, v) - \mu_2(u, v)$ is well defined whenever (1) holds, since it forces $\mu_1(u, v) + \mu_2(u, v) \leq 1$.

3.2 Definition of the Intuitionistic Fuzzy Line Graph

Definition 3.1: Let $G = (V, \sigma, \mu)$ be an IFG. The intuitionistic fuzzy line graph of G , written $L^1(G)$, is constructed by mapping every edge $e = (u, v)$ of G to a vertex of $L^1(G)$ carrying the pair

$$\tau(e) = (\tau_1(e), \tau_2(e)) = (\mu_1(u, v), \mu_2(u, v)). \quad (2)$$

Two such vertices $e_1 = (u, v)$ and $e_2 = (v, w)$ sharing the endpoint v are joined by an edge carrying

$$\eta(e_1, e_2) = (\eta_1, \eta_2) = (\min\{\tau_1(e_1), \tau_1(e_2)\}, \max\{\tau_2(e_1), \tau_2(e_2)\}). \quad (3)$$

Equation (3) is the natural intuitionistic analogue of the fuzzy rule of [1]: the membership component still aggregates by minimum, but the non-membership component must aggregate by maximum — a connecting link can be trusted no more than the more doubted of the two edges it joins. Discarding the second component of every pair recovers exactly the fuzzy construction of [1]. The pair (min, max) is also the standard intersection of two intuitionistic fuzzy sets [4], which gives (3) a principled justification beyond convenience.

3.3 Well-Definedness of the Construction

Theorem 3.2: (Intuitionistic Transformation Theorem). *Let $G = (V, \sigma, \mu)$ be an IFG, and construct $L^1(G)$ as in (2)–(3). Then $L^1(G)$ is itself an intuitionistic fuzzy graph: the map is well defined and single-valued, and every vertex and edge pair (τ_1, τ_2) and (η_1, η_2) satisfies $\tau_1 + \tau_2 \leq 1$, $\eta_1 + \eta_2 \leq 1$, and $\eta_1(e_1, e_2) \leq \min\{\tau_1(e_1), \tau_1(e_2)\}$, $\eta_2(e_1, e_2) \geq \max\{\tau_2(e_1), \tau_2(e_2)\}$, i.e. the IFG edge condition (1) holds in $L^1(G)$.*

Proof: Each edge $e = (u, v)$ of G has a unique pair $(\mu_1(u, v), \mu_2(u, v))$ by definition of μ , and (2) sends it to exactly one vertex of $L^1(G)$, so the vertex assignment is single-valued; $\tau_1(e) + \tau_2(e) = \mu_1(u, v) + \mu_2(u, v) \leq 1$ because G is an IFG. Adjacency of e_1, e_2 in $L^1(G)$ holds exactly when the corresponding edges of G share an endpoint, a well-defined and symmetric relation, so η is well defined and single-valued. That $\eta_1 \leq \min\{\tau_1(e_1), \tau_1(e_2)\}$ and $\eta_2 \geq \max\{\tau_2(e_1), \tau_2(e_2)\}$ is immediate from (3), since the minimum of two numbers never exceeds either of them and the maximum is never below either of them; this is exactly the IFG edge condition (1) applied to $L^1(G)$. Finally, writing $a_1 = \tau_1(e_1)$, $b_1 = \tau_2(e_1)$, $a_2 = \tau_1(e_2)$, $b_2 = \tau_2(e_2)$, we have $a_1 + b_1 \leq 1$ and $a_2 + b_2 \leq 1$, and $\eta_1 + \eta_2 = \min\{a_1, a_2\} + \max\{b_1, b_2\}$. If $a_1 \leq a_2$, this equals $a_1 + \max\{b_1, b_2\}$, which is at most $a_1 + \max\{1 - a_1, 1 - a_2\} = a_1 + 1 - \min\{a_1, a_2\} = a_1 + 1 - a_1 = 1$, using $b_1 \leq 1 - a_1$ and $b_2 \leq 1 - a_2$ and $a_1 \leq a_2$;

the case $a_2 \leq a_1$ is symmetric. Hence $\eta_1 + \eta_2 \leq 1$ always, so $L^I(G)$ is a well-defined intuitionistic fuzzy graph.

4. Connectivity and the Hesitation Interpolation Lemma

4.1 Connectivity Preservation

Call an edge (u, v) of an IFG positive if $\mu_1(u, v) > 0$, and call G intuitionistic fuzzy connected if every two vertices are joined by a path of positive edges, exactly as in the fuzzy case of [1] but restricted to the membership component.

Theorem 4.1: (Intuitionistic Connectivity Preservation). *Let $G = (V, \sigma, \mu)$ be an IFG that is intuitionistic fuzzy connected, and suppose every vertex of G is incident with at least two positive edges. Then $L^I(G)$ of Theorem 3.2 is intuitionistic fuzzy connected.*

Proof: The argument is that of Theorem 4.2 in [1], applied to the membership component τ_1, η_1 , which by (2)–(3) coincides exactly with the fuzzy construction of [1] applied to μ_1 . Since G is intuitionistic fuzzy connected and every vertex meets at least two positive edges, no vertex of $L^I(G)$ is isolated and every walk in G between edges sharing an endpoint yields a path in $L^I(G)$ each of whose links has η_1 equal to the minimum of two positive numbers, hence positive. So every two vertices of $L^I(G)$ are joined by a path of positive edges, i.e. $L^I(G)$ is intuitionistic fuzzy connected.

4.2 The Hesitation Interpolation Lemma

Connectivity alone does not use the second, non-membership component of the construction; the next result does, and has no counterpart in [1], since a purely fuzzy edge carries no hesitation degree to speak of. Write $\pi(e) = 1 - \tau_1(e) - \tau_2(e)$ for the hesitation degree of a vertex e of $L^I(G)$, and $\pi_L(e_1, e_2) = 1 - \eta_1(e_1, e_2) - \eta_2(e_1, e_2)$ for that of a connecting edge.

Lemma 4.2: (Hesitation Interpolation Lemma). *For any two adjacent edges e_1, e_2 of an IFG G , the hesitation degree of the connecting edge in $L^I(G)$ lies between the hesitation degrees of e_1 and e_2 :*

$$\min\{\pi(e_1), \pi(e_2)\} \leq \pi_L(e_1, e_2) \leq \max\{\pi(e_1), \pi(e_2)\}. \quad (4)$$

Proof: Write $a_1 = \tau_1(e_1)$, $b_1 = \tau_2(e_1)$, $\pi_1 = 1 - a_1 - b_1$, and a_2, b_2, π_2 likewise for e_2 . By (3), $\pi_L = 1 - \min\{a_1, a_2\} - \max\{b_1, b_2\}$. Without loss of generality suppose $a_1 \leq a_2$, so $\min\{a_1, a_2\} = a_1$; a symmetric argument covers $a_2 \leq a_1$.

If $b_1 \geq b_2$, then $\max\{b_1, b_2\} = b_1$, so $\pi_L = 1 - a_1 - b_1 = \pi_1$, which trivially lies between π_1 and π_2 .

If $b_1 < b_2$, then $\max\{b_1, b_2\} = b_2$, so $\pi_L = 1 - a_1 - b_2$. Comparing with $\pi_1 = 1 - a_1 - b_1$: since $b_2 > b_1$, $\pi_L = \pi_1 - (b_2 - b_1) < \pi_1$, so $\pi_L \leq \pi_1$. Comparing with $\pi_2 = 1 - a_2 - b_2$: since $a_1 \leq a_2$, $\pi_L = \pi_2 + (a_2 - a_1) \geq \pi_2$, so $\pi_L \geq \pi_2$. Hence $\pi_2 \leq \pi_L \leq \pi_1$ in this case. In both cases π_L lies between π_1 and π_2 , and in particular satisfies (4).

Lemma 4.2 says the transformation neither manufactures new uncertainty nor silently discards it: a confidently characterised link joined to a poorly characterised one produces a connecting link whose hesitation sits between the two, rather than jumping to either extreme.

Remark 4.3: (Geometry and sharpness). Represent each edge by its point (μ_1, μ_2) in the admissible triangle $\mu_1 + \mu_2 \leq 1$ Figure 1 (a). Rule (3) sends e_1, e_2 to the corner of the rectangle they span with the smaller abscissa and larger ordinate; since $\pi = 1 - \mu_1 - \mu_2$ is linear, that corner's hesitation lies between those of the other two corners e_1 and e_2 , which is Lemma 4.2 in one line. The bounds are sharp: if $a_1 \leq a_2$ and $b_1 \geq b_2$ then $\pi_L = \pi_1$, and if $a_1 = a_2$ and $b_1 < b_2$ then $\pi_L = \pi_2$. In particular, $\pi_1 = \pi_2$ forces π_L to equal the common value, the degenerate case of Section 5.1.

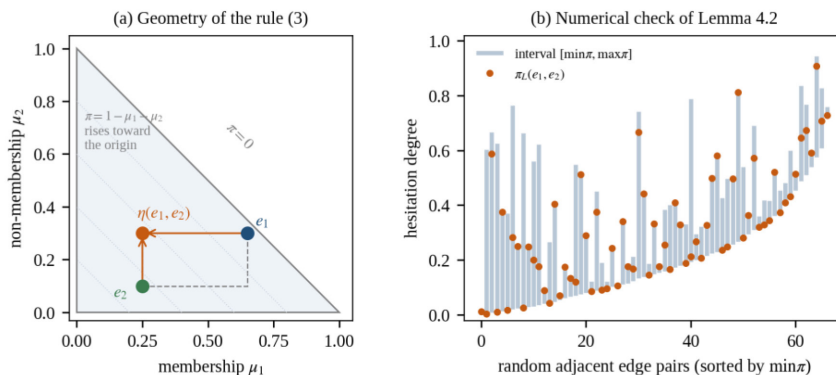


Figure 1

(a) Rule (3) in the admissible triangle: η is the corner of the rectangle spanned by e_1 and e_2 with the smaller μ_1 and larger μ_2 . (b) For 67 random adjacent pairs, the connecting hesitation π_L (dots) always lies inside the interval spanned by the two source hesitations (bars).

As an independent numerical check, 100,000 random pairs of admissible edge values, drawn uniformly from the triangle $\mu_1 + \mu_2 \leq 1$, were passed through (3); none violated $\eta_1 + \eta_2 \leq 1$ or inequality (4), and 67 of them are plotted in Figure 1.

Corollary 4.4: (Fuzzy Line Graph as a Special Case). *If $\mu_2(u, v) = 1 - \mu_1(u, v)$ for every edge of G , so that every edge has zero hesitation, then $\tau_2(e) = 1 - \tau_1(e)$ and $\eta_2(e_1, e_2) = 1 - \eta_1(e_1, e_2)$ for every vertex and edge of $L^l(G)$, so $L^l(G)$ also has zero hesitation everywhere and the membership component (τ_1, η_1) coincides exactly with the fuzzy line graph $L_\mu(G)$ of [1].*

Proof: Direct substitution: $\eta_2 = \max\{1 - a_1, 1 - a_2\} = 1 - \min\{a_1, a_2\} = 1 - \eta_1$, using $\tau_2(e_i) = 1 - \tau_1(e_i)$ for $i = 1, 2$.

5. Conclusion

Building on the fuzzy line graph construction proposed by Mohammad [1], this paper develops the intuitionistic fuzzy line graph, extending the conventional fuzzy representation by incorporating an independent non-membership degree for every vertex and edge in addition to its membership degree. This extension provides a more expressive mathematical framework for representing uncertainty, ambiguity, and incomplete information in graph-based systems. In an intuitionistic fuzzy graph, the membership degree indicates the strength or degree of association, whereas the non-membership degree represents the degree to which that association is not supported. The proposed intuitionistic fuzzy line graph construction transforms the relationships among the edges of an original intuitionistic fuzzy graph into corresponding vertices and establishes new intuitionistic fuzzy edges according to their incidence relationships. The paper first establishes that the proposed construction is well defined and satisfies the required intuitionistic fuzzy graph edge condition, as formally demonstrated in Theorem 3.2. This result confirms that the membership and non-membership assignments generated during the line graph transformation remain mathematically consistent with the fundamental requirements of intuitionistic fuzzy graph theory. The study subsequently investigates the connectivity properties of the proposed construction.

References

- [1] S. I. Mohammad, "Fuzzy line graphs and their extensions: Mathematical properties and applications," *Int. J. Appl. Math.*, vol. 38, no. 7s, pp. 519–540 (2025), doi: 10.12732/ijam.v38i7s.498.
- [2] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353 (Jun. 1965).
- [3] A. Rosenfeld, "Fuzzy graphs," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, L. A. Zadeh, K. S. Fu, K. Tanaka, and M. Shimura, Eds. New York, USA: Academic, pp. 77–95 (1975).
- [4] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96 (Aug. 1986).
- [5] R. Parvathi and M. G. Karunambigai, "Intuitionistic fuzzy graphs," in *Computational Intelligence, Theory and Applications*, Advances in Soft Computing, vol. 38. Berlin, Germany: Springer, pp. 139–150 (2006), doi: 10.1007/3-540-34783-6_15.
- [6] A. Shannon and K. T. Atanassov, "A first step to a theory of the intuitionistic fuzzy graphs," in *Proc. 1st Workshop Fuzzy Based Expert Syst.*, Sofia, Bulgaria, pp. 59–61 (Sep. 1994).
- [7] J. N. Mordeson, "Fuzzy line graphs," *Pattern Recognit. Lett.*, vol. 14, no. 5, pp. 381–384 (May 1993).
- [8] F. Harary, *Graph Theory*. Reading, MA, USA: Addison-Wesley (1969).
- [9] H. Whitney, "Congruent graphs and the connectivity of graphs," *Amer. J. Math.*, vol. 54, no. 1, pp. 150–168 (Jan. 1932).
- [10] J. N. Mordeson and P. S. Nair, *Fuzzy Graphs and Fuzzy Hypergraphs*, 2nd ed. Heidelberg, Germany: Physica-Verlag (2001).
- [11] M. Akram and W. A. Dudek, "Intuitionistic fuzzy hypergraphs with applications," *Inf. Sci.*, vol. 218, pp. 182–193 (Jan. 2013).
- [12] M. Akram, W. A. Dudek, and M. M. Yousaf, "Regularity in vague intersection graphs and vague line graphs," *Abstr. Appl. Anal.*, vol. 2014, Art. no. 525389 (2014).
- [13] H. Bustince and P. Burillo, "Vague sets are intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 79, no. 3, pp. 403–405 (May 1996).
- [14] A. A. S. Mohammad, N. Yogeesh, S. I. S. Mohammad, N. Raja, Lingaraju, P. William, A. Vasudevan, N. Alshdaifat, and M. F. A. Hunitie, "An intuitionistic fuzzy graph model: Matrix representations and ap-

- plications," *Cybern. Inf. Technol.*, vol. 25, no. 4, pp. 3–19 (2025), doi: 10.2478/cait-2025-0030.
- [15] N. Nazir, T. Shaheen, L. Jin, and T. Senapati, "An improved algorithm for identification of dominating vertex set in intuitionistic fuzzy graphs," *Axioms*, vol. 12, no. 3, Art. no. 289 (2023), doi: 10.3390/axioms12030289.
- [16] K. R. Chaudhari, A. O. Mulani, M. P. Gajare, V. Jadhav, P. Yawle, and A. V. Bang, "Bit error rate analysis of various error correction codes with concatenated RS-convolutional codes," *J. Discrete Math. Sci. Cryptogr.*, vol. 29, no. 1, pp. 199–214 (2026).
- [17] S. Salunkhe, A. O. Mulani, D. Shahane, M. Rana, S. M. Shukla, and M. M. Jadhav, "Secure image transmission using chaotic encryption and DWT watermarking on reconfigurable platform," *J. Discrete Math. Sci. Cryptogr.*, vol. 29, no. 1, pp. 349–361 (2026).

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