

Sum 3-equitable labelling in wheel graph

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Abstract

In this paper, we investigate a MATLAB program to demonstrate the behavior of wheel graphs under sum 3-equitable labeling. A vertex labeling ζ of a graph W , defined as a bijection from the vertex set $V(W)$ to the set $\{0, 1, 2, \dots, m-1\}$, induces an edge labeling ζ^* given by:

$\zeta^*(qr) = \{0 \text{ if } \zeta(q) + \zeta(r) \equiv 0(\text{mod}3) \text{ } 1 \text{ if } \zeta(q) + \zeta(r) \equiv 1(\text{mod}3) \text{ } 2 \text{ if } \zeta(q) + \zeta(r) \equiv 2(\text{mod}3)$
and $|e_\zeta(l) - e_\zeta(k)| \leq 1$ for all $0 \leq l, k \leq 2$. A graph which preserves sum 3-equitable labeling is called a sum 3- equitable graph.

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1. Introduction

Graph labeling has emerged as an important area of graph theory because of its extensive applications in communication networks, coding theory, and combinatorial optimization. A detailed survey of graph labeling methods was presented by Gallian [1], which highlights the development of numerous labeling techniques and their applications. The foundational concepts of graph theory relevant to labeling problems are well documented in the classical works of Rosa [2], West [3], Gross and Yellen [4], Harary [5], and Bondy and Murty [6].

Among the many graph classes studied, the wheel graph W_n is of particular interest. A wheel graph is formed by adjoining a new central vertex, called the hub, to every vertex of a cycle graph C_{n-1} . Thus, W_n combines the structural properties of both cycle and star graphs, making it an attractive candidate for investigating equitable labeling problems. Since a wheel graph contains two distinct edge sets—cycle edges forming the outer ring and hub edges connecting the center to outer vertices—it provides an interesting framework for studying balanced edge label distributions. Applications of equitable graph labeling extend to communication systems, load balancing, and distributed computing. Stallings [7] discussed the importance of efficient communication and resource allocation in network systems, while Kairouz and McMahan [8] highlighted similar balancing challenges in federated learning frameworks, where distributed nodes collaboratively train models without centralized data sharing. In such systems, hub-and-spoke structures analogous to wheel graphs frequently arise, making sum 3-equitable labeling relevant for workload classification and balanced resource management.

In sum 3-equitable labeling, vertex labels are assigned from the set $\{0, 1, 2, m-1\}$, and each edge receives an induced label based on the sum of labels of its end vertices modulo 3. The main objective is to ensure that the induced edge labels are partitioned into three classes with nearly equal cardinalities. Similar equitable and balanced labeling concepts have been investigated by Vaidya and Kothari [9] and Germina and Sudev [10], who studied several graph families under different labeling constraints.

For wheel graphs, the presence of both circular and spoke edges makes the balancing of induced edge labels more challenging and mathematically appealing. The equitable distribution of labels across these two edge categories ensures fairness and uniformity, preventing any one label from being overrepresented. Such balanced structures are useful in practical scenarios involving resource allocation and network optimization.

Motivated by these theoretical and practical considerations, the present work investigates sum 3-equitable labeling of wheel graphs. The results contribute to the broader literature on equitable graph labeling while also offering insights into balanced network structures and their applications.

2. Terminology and Notation

2.1 Definition-Ternary Vertex Labelling

In a graph W , each vertex can be assigned a label from the set $\{0,1,2\}$ by a procedure called ternary vertex labeling which is determined by a function $\zeta: E(V) \rightarrow \{0,1,2\}$, where $\zeta(a)$ means the label that is assigned to the vertex a . For each edge $b=q*r$ that connects two vertices, it also receives a label according to a derived function $\zeta^*: E(V) \rightarrow \{0,1,2\}$. The sets of vertices labelled 0, 1, and 2 will be denoted $a_\zeta(0)$, $a_\zeta(1)$, and $a_\zeta(2)$ respectively. Likewise, the sets of edges labelled 0, 1, and 2 are denoted $b_\zeta(0)$, $b_\zeta(1)$, and $b_\zeta(2)$.

2.2 Definition- 3-Equitable Labeling

A graph V is said to have a *3-equitable labeling* if it satisfies the following two conditions:

- (a) $|d_\zeta(a) - d_\zeta(b)| \leq 1$ and
- (b) $|c_\zeta(a) - c_\zeta(b)| \leq 1$ for all $0 \leq a, b \leq 2$.

for all $0 \leq a, b \leq 2$, where $d_\zeta(j)$ and $c_\zeta(j)$ represent the number of vertices and edges labeled j , respectively, under the labeling ζ . A graph that admits such a labeling is referred to as a *3-equitable labeled graph*.

2.3 Definition- Sum 3-Equitable Labeling

Let $E(G)$ be the vertex set of a graph G . A labeling ζ is defined as a bijection from $E(G)$ to the set $\{0,1,2,\dots,m-1\}$.

$$\zeta^*(q * r) = \begin{cases} 0 & \text{if } \zeta(q) + \zeta(r) \equiv 0 \pmod{3} \\ 1 & \text{if } \zeta(q) + \zeta(r) \equiv 1 \pmod{3} \\ 2 & \text{if } \zeta(q) + \zeta(r) \equiv 2 \pmod{3} \end{cases}$$

Additionally, the labeling satisfies $|c_\zeta(a) - c_\zeta(b)| \leq 1$ for all $0 \leq a, b \leq 2$. A graph that admits such a labeling is referred to as a *sum 3-equitable graph*.

2.4 Definition- Wheel Graphs

A wheel graph is created by linking one central vertex to every vertex of a cycle graph, enabling the assignment of vertex and edge labels in both circular and radial arrangements.

3. Algorithm

To implement the theorem “The wheel graph preserves sum 3-equitable labeling” in MATLAB, open the MATLAB editor window and enter the corresponding code.

Save the file with the name `wheel_sum3equitable_labeling.m`. When executed, the program will generate the wheel graph as shown in Figure 1 and Figure 2, also the programming determine whether the given cycle graph satisfies the conditions of a sum 3-equitable graph.

```
function wheel_sum3equitable_labeling(m)
```

```

        if m < 4
            error('m must be at least 4 for a valid
wheel graph');
        end

v = ones(1, m-1);
e = 2:m;
G = graph(v, e);

for i = 2:m-1
    G = addedge(G, i, i+1);
end

G = addedge(G, m, 2);

h = plot(G);
labels = [0, 1:m-1];
labelnode(h, 1:m, labels);

%Outer labeling
a = [0, 1:m-1];
s = 1;
t = 1;
b=0;
c=0;
d=0;

for n = 1:(m - 1)
    u = s + 1;
    if u > m - 1
        u = 1;
    end

    if rem(a(s+1) + a(u+1), 3) == 0
        labeledge(h, t+1, u+1, '0')
        b = b + 1;
    elseif rem(a(s+1) + a(u+1), 3) == 1
        labeledge(h, t+1, u+1, '1')
        c = c + 1;
    else
        labeledge(h, t+1, u+1, '2')
        d = d + 1;
    end

    s = s + 1;

```

```

        t = t + 1;
    end

    %Inner labeling
    s = 1;
    t = 2;

    for n = 1:(m - 1)
        if mod(a(s) + a(t), 3) == 0
            labeledge(h, s, t, '0');
        elseif mod(a(s) + a(t), 3) == 1
            labeledge(h, s, t, '1');
        elseif mod(a(s) + a(t), 3) == 2
            labeledge(h, s, t, '2');
        end

        t = t + 1;
    end
end
end

```

Example 1: wheel_sum3equitable_labeling (7)

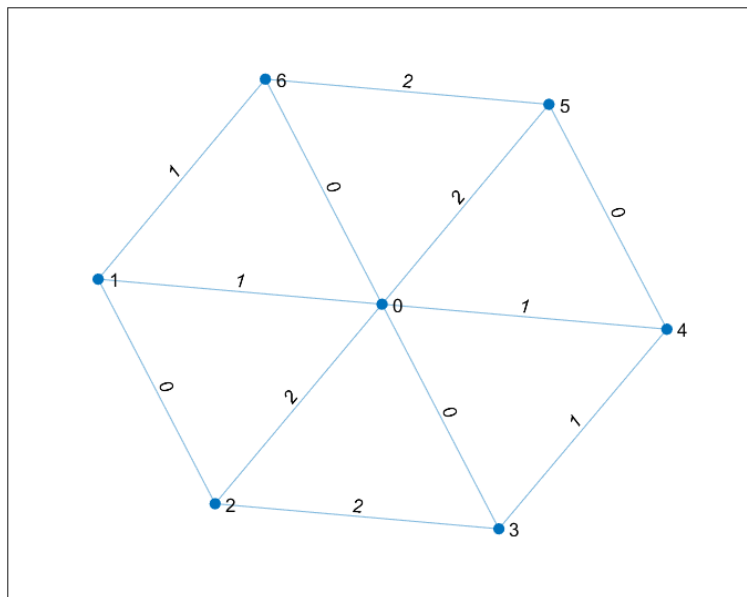


Figure 1
wheel_sum3equitable_labeling (7)

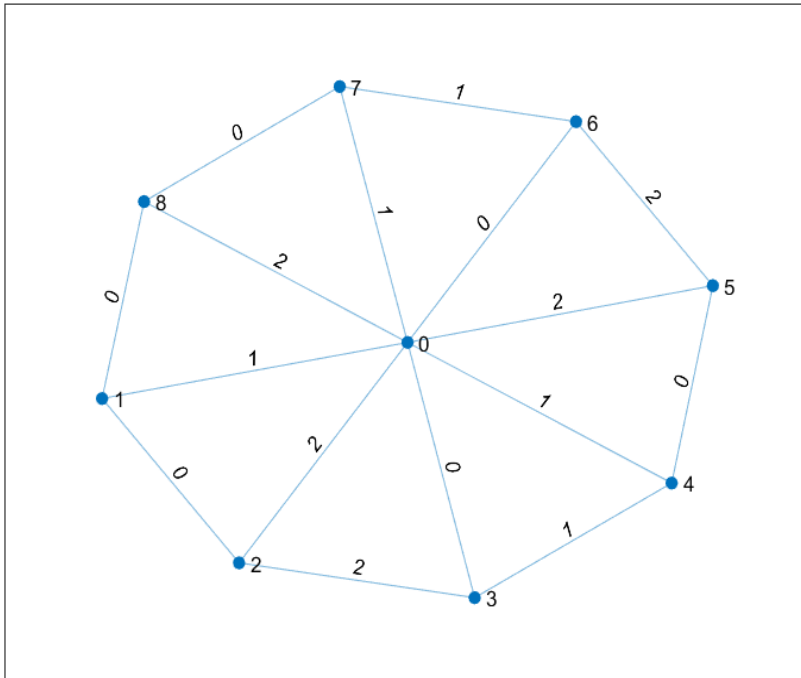
Example 2: wheel_sum3equitable_labeling (9)

Figure 2
wheel_sum3equitable_labeling (9)

4. Conclusion

Wheel graphs and sum 3-equitable labeling offers a mathematically sound, practical alternative for improving or optimizing wireless communications in a centralized format found in some wireless communication infrastructures such as smart traffic roundabouts. As shown in the MATLAB implementation, all communication channels can be appropriately allocated with minimal interference while utilizing some of the natural structure and symmetry of a wheel graph. The practical real-world example was introduced to elucidate the principle of labeling such as the one used, and its possible application in the context of modern infrastructure. In summary, the protocol is not only considered to improve reliability in such systems but also scalability due to the protocol itself being more fundamental than previous wireless communication protocols in networking. Overall, this work lays a challenge for future research to implement even more complex graph-based solutions in the context of optimization in networks and wireless communication.

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