

Local and strong metric dimension of families of alternate snake graphs

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Abstract

Consider $G = (V, E)$ be an undirected, non-trivial, simple, and connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset $\mathcal{R} \subseteq V(G)$ is called a local resolving set for G if, for every pair of adjacent vertices say $u, v \in V(G) \setminus \mathcal{R}$, there exists a vertex $t \in \mathcal{R}$ such that $d(u, t) \neq d(v, t)$. A vertex t is said to be strongly resolve two vertices u and v if, either v belongs to a shortest path connecting u and t , or u belongs to a shortest path connecting v and t . The minimum cardinality of a local resolving set is called the local metric dimension (LMD) of G , while the minimum cardinality of a strong resolving set is known as the strong metric dimension (SMD) of G . It is well known that the problem of determining these resolving sets is NP-complete and plays an important role in graph theory with many real-world applications, including transportation systems, robot navigation, and network analysis. In this paper, we determine the local and strong resolving sets for the families of alternate snake

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graphs and prove that each family possesses an independent resolving set with a cardinality of two.

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1. Introduction

Graph theory is a branch of discrete mathematics that deals with the study of objects through vertices and edges. It has a wide range of applications in various real-world problems, including robot navigation [21], network analysis [3], and fraud detection [32]. Distance-based tools are particularly important in graph theory for solving many complex problems in computer science [30], communication systems [27], transportation networks [6], and many more [19].

In this paper, we consider a non-trivial, simple, connected graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge sets $E(G)$. The cardinality of vertex and edge set is denoted as $|V(G)|$ and $|E(G)|$, respectively. The shortest path connecting two vertices u and v in G is the distance between them, represented as $d(u, v)$. Let $\mathcal{R} = \{t_1, t_2, \dots, t_k\} \subseteq V(G)$ be an ordered subset of $V(G)$. Then the representation of vector for vertex v with respect to $\mathcal{R}$ is defined as the ordered k -tuple $(d(v, t_1), d(v, t_2), \dots, d(v, t_k))$ and is represented as $r(v \mid \mathcal{R})$. This ordered set $\mathcal{R}$ is called a resolving set (RS) if, for every pair of distinct vertices u, v of G has different vector representation with respect to $\mathcal{R}$, i.e., $r(u \mid \mathcal{R}) \neq r(v \mid \mathcal{R})$.

The concept of a RS in a connected graph G was first introduced by Slater [31], and is was referred as the location number. Later, in 1976, Harary and Melter [10] investigated a similar concept and introduced the term metric basis and its minimum cardinality is known as metric dimension (MD) of the graph G . Since the introduction of the MD, many researchers have extensively studied the MD for various families of graphs and explored its applications in different fields [14, 29]. Recently, Hayat et al. [9] computed the various metric invariants of geometric networks with their potential applications in various fields [2, 8, 12, 26, 34].

In 2004, Sebő and Tannier [28] introduced another metric invariant and further refined by Oellermann and Peters-Fransen [24] known as the Strong metric dimension (SMD). This parameter is based on the concept that every pair of vertices in a graph G can be distinguished by a vertex belonging to a shortest path connecting them. A set of vertices satisfying this property is called a strong resolving set (SRS), and the minimum cardinality of such a set is referred to as the strong metric dimension (SMD) of the graph. In recent years, many researchers have investigated and computed the SMD for various families of graphs, including corona product graphs [16] and cactus graph [17]. Studies on different graph classes have shown that the SMD is highly sensitive to structural

properties of graphs, such as the diameter, cut vertices, and related characteristics readers may refer to [13].

In 2010, Okamoto, Phinezy, and Zhang [25] introduced another variant of the MD known as the local metric dimension (LMD) of graph G . This concept focuses on selecting the smallest set of vertices in a connected graph such that every pair of adjacent vertices can be distinguished based on their distances to the selected vertices and such set is known as local resolving set. The minimum cardinality of LRS is called LMD of G , denoted as $\dim_\ell(G)$. LRS play an important role in distinguishing neighboring vertices, which assists in analyzing structural characteristics of graphs and complex networks [5]. In recent years, several researchers have investigated the LMD for different families of graphs. For instance, Lal and Bhat [18], and Yang et al. [33] studied the LMD of various classes of planar graphs and explored their potential applications. For more recent work related to LMD, one can refer [1, 7, 15].

Recently, Mohamed and Amin [22] investigated the MD and the secure metric dimension [23] for alternate families of snake graphs. The study of different metric-based parameters, such as LMD and SMD, for various graph families plays a significant role in understanding the structural and distance-based properties of networks. These parameters are particularly useful in the design of efficient and reliable network systems, as they enhance our understanding of graph resolvability and its applications in network navigation and monitoring [5].

From the existing literature, it can be observed that the LMD and SMD for alternate families of snake graphs have not yet been investigated. In this paper, we aim to determine the LMD and SMD of these graph families and analyze their structural characteristics. Such an investigation not only contributes to the theoretical development of metric-based graph parameters but also provides further insight into the relationship between graph structure and distance-based resolvability. Consequently, the results obtained may help in developing improved analytical approaches for studying complex network structures.

The structure of this paper is as follows: Section 2 presents the fundamental definitions and existing results related to the LMD and SMD of graph. The LMD and SMD of alternate families of snake graphs are computed in Section 3. Lastly, Section 4 provides the conclusion and future scope of work.

2. Preliminaries

This section introduces the fundamental definitions and already established results that are required for the development of this article.

Definition 1: [25] Given an ordered subset of $V(G)$, let $\mathcal{R} = \{t_1, t_2, \dots, t_k\} \subseteq V(G)$. The ordered k tuple is the metric representation of a vertex $v \in V(G)$ with respect to the ordered set $\mathcal{R}$, represented as $r(v \mid \mathcal{R}) = (d(v, t_1), d(v, t_2), \dots, d(v, t_k))$.

Definition 2: [25] An ordered set $\mathcal{R} \subseteq V(G)$ is called a LRS if for every pair of neighbouring vertices $u, v \in V(G) \setminus \mathcal{R}$, there exist a vertex $t \in \mathcal{R}$ such that $r(u \mid \mathcal{R}) \neq r(v \mid \mathcal{R})$, and its minimum cardinality known as LMD of G , denoted by $\dim_\ell(G)$.

Definition 3: [28] A vertex $t \in V(G)$ is said to strongly resolve two vertices u and v if either $d(u, t) = d(u, v) + d(v, t)$ or $d(v, t) = d(u, v) + d(u, t)$.

The collection of distinct $\mathcal{R} \subseteq V(G)$ is referred to as a SRS if each pair of distinct vertices of G is strongly distinguished by at least one member of $\mathcal{R}$. The least possible number of vertices in such a set is called the SMD of G .

Definition 4: [24] If $G = (V, E)$ is a connected graph and u and v are any two vertices. Then the vertex u is mutually maximal distance (MMD) from the vertex v iff for every $t \in N(u)$ satisfies $d(w, v) \leq d(u, v)$.

Definition 5: [24] The SR graph of G , denoted by G_{SR} , is defined as the graph whose vertex set is $V(G)$ and in which two vertices u and v are adjacent if and only if u and v are MMD in G .

Definition 6: [24] The set of all vertices in the graph G that are MMD with another vertex in the graph G is its boundary, and it is represented as $\partial(G)$.

Theorem 1: [25] Let G be a connected graph of order $n \geq 2$. Then

- (1) $\dim_\ell(G) = n - 1 \Leftrightarrow G = K_n$, where K_n is a complete graph with n vertices.
- (2) $\dim_\ell(G) = 1 \Leftrightarrow G$ is a bipartite graph.

Theorem 2: [24] If G is a connected graph and if G_{SR} is it is SR graph. Then $sdim(G) = \beta(G_{SR})$ where,

- (1) $sdim(G)$ is the SMD of G
- (2) $\beta(G_{SR})$ is the vertex cover number of G_{SR} .

2.1 Families of alternate snake graphs and their applications

This subsection represent the families of graphs that are considered in this article and their applications.

The alternate triangular snake graph as shown in Figure 1 and is denotes as AT_n , whose vertex set is $V = \{\alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ and the edge set is defined as

$$E(AT_n) = \left\{ \alpha_i \alpha_{i+1} \mid 1 \leq i \leq \frac{2n}{3} - 1 \right\} \cup \left\{ \alpha_{2i-1} \alpha_{\frac{2n}{3}+i} \cup \alpha_{2i} \alpha_{\frac{2n}{3}+i} \mid 1 \leq i \leq \frac{n}{3} \right\}.$$

Note that in the graph AT_n the total numbers of triangles will be $\frac{n}{3}$. Let us denote the total triangles as $p = \frac{n}{3}$

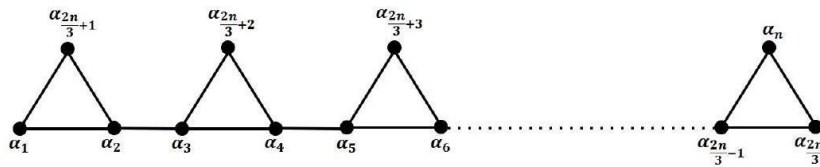


Figure 1
Alternate Triangular Snake Graph

Similarly, the alternate quadrilateral snake as shown in Figure 2 and is denoted as AQ_n whose vertex set is $V = \{\alpha_1, \alpha_2, \dots, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ and the edge set is defined as

$$E(AQ_n) = \{\alpha_{4k-3}\alpha_{4k-2} \cup \alpha_{4k-2}\alpha_{4k} \cup \alpha_{4k}\alpha_{4k-1} \cup \alpha_{4k-1}\alpha_{4k-3} \mid 1 \leq k \leq p\} \\ \cup \{\alpha_{4k}\alpha_{4k+2} \mid 1 \leq k \leq p-1\} \cup \{\alpha_{n-1}\alpha_n\}.$$

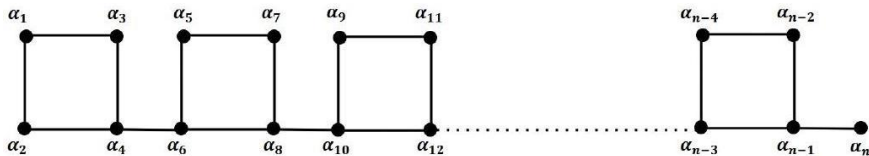


Figure 2
Alternate Quadrilateral Snake Graph

Note that in the graph AQ_n the total numbers of quadrilaterals will be $\frac{n}{4}$. Let us denote the total quadrilaterals as $p = \frac{n}{4}$.

And, the alternate pentagonal snake graph as shown in Figure 3 and is denoted as AP_n whose vertex set is $V = (\alpha_1, \alpha_2, \dots, \alpha_{2p}, \alpha_{2p+1}, \alpha_{2p+2}, \dots, \alpha_{4p}, \alpha_{4p+1}, \dots, \alpha_n)$ and the edge set is defined as

$$E(AP_n) = \{\alpha_i\alpha_{i+1} \mid 1 \leq i \leq 2p-1\} \\ \cup \{\alpha_{2k-1}\alpha_{2p+2k-1} \mid 1 \leq k \leq p\} \\ \cup \{\alpha_{2p+2k-1}\alpha_{4p+k} \mid 1 \leq k \leq p\} \\ \cup \{\alpha_{4p+k}\alpha_{2p+2k} \mid 1 \leq k \leq p\} \\ \cup \{\alpha_{2p+2k}\alpha_{2k} \mid 1 \leq k \leq p\}.$$

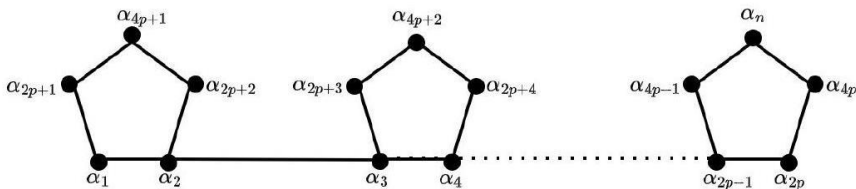


Figure 3
Alternate Pentagonal Snake Graph

Note that in the graph AP_n the total numbers of pentagon will be $\frac{n}{5}$. Let us denote the total pentagon as $p = \frac{n}{5}$. The k-polygon graph as shown in Figure 4 and is denoted as $AS_n(C_k)$, whose vertex set is $V(AS_n(C_k)) = \{\alpha_1, \alpha_2, \dots, \alpha_{2p}, \alpha_{2p+1}, \alpha_{2p+2}, \dots, \alpha_{4p}, \alpha_{4p+1}, \alpha_{4p+2}, \dots, \alpha_{6p}, \alpha_{6p+1}, \alpha_{6p+2}, \dots, \alpha_n\}$, and the edge set is defined as

$$\begin{aligned} E(AS_n(C_k)) = & \{\alpha_i \alpha_{i+1} \mid 1 \leq i \leq 2p-1\} \\ & \cup \{\alpha_{2i-1} \alpha_{2p+(k-2)(i-1)+1} \mid 1 \leq i \leq p\} \\ & \cup \{\alpha_{2p+(k-2)(i-1)+j} \alpha_{2p+(k-2)(i-1)+j+1} \\ & \mid 1 \leq i \leq p, 1 \leq j \leq k-3\} \\ & \cup \{\alpha_{2p+(k-2)i} \alpha_{2i} \mid 1 \leq i \leq p\}. \end{aligned}$$

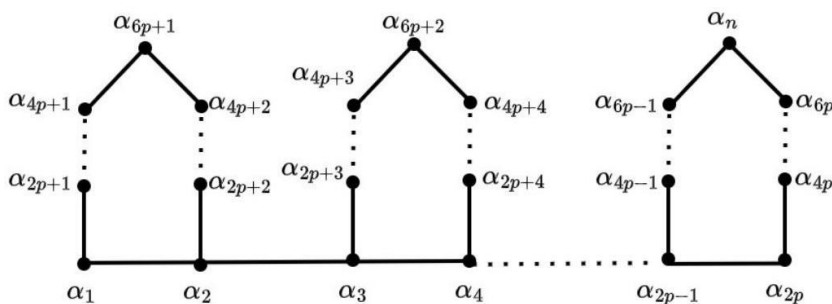


Figure 4
Alternate k-polygon Snake Graph

Note that in the graph $AS_n(C_n)$ the total numbers of polygon be $\frac{n}{k}$. Let us denote the total pentagon as $p = \frac{n}{k}$. In graph theory and combinatorics, snake graphs are widely used in the study of perfect matching, where the number of perfect matchings is closely related to continued fractions [4]. Their step-by-step construction helps in representing cluster variables in cluster algebras [20]. Another application of snake graphs can be widely seen in coding theory, where snake graphs have been used to model code words and to compute associated energy values directly from the graph structure and the use of LRS and SRS in such models improves error detection and localization capability and supports studies in Brauer configuration algebras [11].

3. Local and Strong Metric Dimension

In this section, we calculate the LMD and SMD of the alternate families of snake graphs.

Theorem 3: For $p \geq 2$, the LMD of alternate triangular snake graph AT_n is 2.

Proof: The alternating triangular snake graph AT_n is shown in Figure 1. From Figure 1, we see that the adjacent vertices of AT_n for $1 \leq i \leq \frac{2n}{3} - 1$ are $\{\alpha_i, \alpha_{i+1}\}$, and for $1 \leq i \leq \frac{n}{3}$ the adjacent vertices are $\{\alpha_{2i-1}, \alpha_{\frac{2n}{3}+i}\}$, and $\{\alpha_{2i}, \alpha_{\frac{2n}{3}+i}\}$. To prove the LMD of AT_n is 2, we divide the proof into following three cases depends upon n .

Case I: When $n \equiv 0(\text{mod}3)$. Then $n = 3p$, for some $p > 1$. Let $\mathcal{R} = \{\alpha_1, \alpha_{2p}\}$ be an ordered subset of $V(AT_n)$. We will show that $\mathcal{R}$ is a LRS for AT_n . In order to establish the required result, it suffices to determine the metric representations of adjacent vertices, with respect to $\mathcal{R}$ which are given below:

For the adjacent vertices α_i and α_{i+1} , we have

$$r(\alpha_i \mid \mathcal{R}) = (i - 1, 2p - i)$$

$$r(\alpha_{i+1} \mid \mathcal{R}) = (i, 2p - i - 1), \forall 1 \leq i \leq \frac{2n}{3} - 1.$$

For the adjacent vertices α_{2i-1} and $\alpha_{\frac{2n}{3}+i}$, we have

$$r(\alpha_{2i-1} \mid \mathcal{R}) = (2i - 2, 2p - 2i + 1),$$

$$r\left(\alpha_{\frac{2n}{3}+i} \mid \mathcal{R}\right) = (2i - 1, 2p - 2i + 1), \forall 1 \leq i \leq \frac{n}{3}.$$

Also, for the adjacent vertices α_{2i} and $\alpha_{\frac{2n}{3}+i}$, we have

$$r(\alpha_{2i} \mid \mathcal{R}) = (2i - 1, 2p - 2i),$$

$$r\left(\alpha_{\frac{2n}{3}+i} \mid \mathcal{R}\right) = (2i - 1, 2p - 2i + 1), \forall 1 \leq i \leq \frac{n}{3}.$$

Thus, the above vector representations of each neighbouring vertices is unique with respect to $\mathcal{R}$. Hence, $\mathcal{R}$ is a LRS for AT_n .

Case II: When $n \equiv 1(\text{mod}3)$. Then $n = 3p + 1$, for some $p > 1$. Here we consider one more triangle than the previous case and we choose $\mathcal{R} = \{\alpha_1, \alpha_{2p}\}$ as an ordered subset of $V(AT_n)$. So, the representation of vector for each adjacent vertex of $AT_n \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices α_i and α_{i+1} , where we have

$$r(\alpha_i \mid \mathcal{R}) = (i - 1, 2p - i)$$

$$r(\alpha_{i+1} \mid \mathcal{R}) = (i, 2p - i - 1), \forall 1 \leq i \leq \frac{2n}{3} - 1.$$

For the adjacent vertices α_{2i-1} and $\alpha_{\frac{2n}{3}+i}$, we have

$$r(\alpha_{2i-1} \mid \mathcal{R}) = (2i - 2, 2p - 2i + 1)$$

$$r\left(\alpha_{\frac{2n}{3}+i} \mid \mathcal{R}\right) = (2i - 1, 2p - 2i + 1), \forall 1 \leq i \leq \frac{n}{3} - 1.$$

Also, for the adjacent vertices α_{2i} and $\alpha_{\frac{2n}{3}+i}$, we have

$$r(\alpha_{2i} \mid \mathcal{R}) = (2i - 1, 2p - 2i)$$

$$r\left(\alpha_{\frac{2n}{3}+i} \mid \mathcal{R}\right) = (2i - 1, 2p - 2i + 1), \forall 1 \leq i \leq \frac{n}{3} - 1.$$

Thus, from the above vector representation, we can see that the neighbouring vertices of $AT_n \setminus \mathcal{R}$ has unique representation of vectors with respect to $\mathcal{R}$. Therefore, $\mathcal{R}$ is a LRS for AT_n .

Case III: When $n \equiv 2(\text{mod}3)$. Then $n = 3p + 2$, for some $p > 1$. Here we consider one more triangle than the previous case and we choose $\mathcal{R} = \{\alpha_1, \alpha_{2p+1}\}$ as an ordered subset of $V(AT_n)$. So, the representation of vector for each adjacent vertex of $AT_n \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices α_i and α_{i+1} , we have

$$r(\alpha_i \mid \mathcal{R}) = (i - 1, 2p + 1 - i)$$

$$r(\alpha_{i+1} \mid \mathcal{R}) = (i, 2p - i), \forall 1 \leq i \leq \frac{2n}{3} - 1.$$

For the adjacent vertices α_{2i-1} and $\alpha_{\frac{2n}{3}+i}$, we have

$$r(\alpha_{2i-1} \mid \mathcal{R}) = (2i - 2, 2p - 2i + 2)$$

$$r\left(\alpha_{\frac{2n}{3}+i} \mid \mathcal{R}\right) = (2i - 1, 2p - 2i + 2), \forall 1 \leq i \leq \frac{n}{3} - 1.$$

Also, for the adjacent vertices α_{2i} and $\alpha_{\frac{2n}{3}+i}$, we have

$$r(\alpha_{2i} \mid \mathcal{R}) = (2i - 1, 2p - 2i + 1),$$

$$r\left(\alpha_{\frac{2n}{3}+i} \mid \mathcal{R}\right) = (2i - 1, 2p - 2i + 2), \forall 1 \leq i \leq \frac{n}{3} - 1.$$

Since, every neighbouring pair of $AT_n \setminus \mathcal{R}$ have the unique vector representation with regard to $\mathcal{R}$, the vector representation above demonstrates this. Therefore, $\mathcal{R}$ is an LRS for AT_n .

In all the cases, we find that $\mathcal{R}$ is a LRS for AT_n with the cardinality 2. Hence, $\dim_\ell(AT_n) \leq 2$. As, the graph AT_n is not a bipartite graph, so the $\dim_\ell(AT_n) > 1$ (Using Theorem 1). Hence, $\dim_\ell(AT_n) = 2$. $\square$

Theorem 4: For $p \geq 2$, the LMD of alternate quadrilateral graph is 2.

Proof: Let AQ_n be an alternating quadrilateral snake graph as shown in Figure 2. From Figure 2 we see that for the adjacent vertices of AQ_n for $1 \leq k \leq p$ are $\{\alpha_{4k-3}, \alpha_{4k-2}\}, \{\alpha_{4k-2}, \alpha_{4k}\}, \{\alpha_{4k}, \alpha_{4k-1}\}, \{\alpha_{4k-1}, \alpha_{4k-3}\}$, and $\{\alpha_{n-1}, \alpha_n\}$. To prove the LMD of AQ_n is 2, we divide the proof into three cases depending up on n .

Case I: When $n \equiv 0(\text{mod}4)$, Then $n = 4p$, for some $p \geq 2$. Let us suppose that $\mathcal{R} = \{\alpha_2, \alpha_{n-3}\}$ be an ordered subset of $V(AQ_n)$.

Now, to show that $\mathcal{R}$ is a LRS for AQ_n we give the representation of vectors for each neighbouring vertices with respect to $\mathcal{R}$ which are given below:

For the adjacent vertices α_{4k-3} and α_{4k-2} , we have

$$\begin{aligned} r(\alpha_{4k-3} \mid \mathcal{R}) &= (4k-4, n-4k-1) \\ r(\alpha_{4k-2} \mid \mathcal{R}) &= (4k-5, n-4k), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k-2} and α_{4k} , we have

$$\begin{aligned} r(\alpha_{4k-2} \mid \mathcal{R}) &= (4k-5, n-4k) \\ r(\alpha_{4k} \mid \mathcal{R}) &= (4k-3, n-4k-2), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k} and α_{4k-1} , we have

$$\begin{aligned} r(\alpha_{4k} \mid \mathcal{R}) &= (4k-3, n-4k-2), \\ r(\alpha_{4k-1} \mid \mathcal{R}) &= (4k-4, n-4k-3), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k-1} and α_{4k-3} , we have

$$\begin{aligned} r(\alpha_{4k-1} \mid \mathcal{R}) &= (4k-4, n-4k-3), \\ r(\alpha_{4k-3} \mid \mathcal{R}) &= (4k-4, n-4k-1), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{n-1} and α_n , we have

$$\begin{aligned} r(\alpha_{n-1} \mid \mathcal{R}) &= (n-3, 2), \text{ and} \\ r(\alpha_n \mid \mathcal{R}) &= (n-2, 3) \end{aligned}$$

Hence, from the above vector representation, we can see that each neighbouring vertex of $AQ_n \setminus \mathcal{R}$ have the unique vector representation with respect $\mathcal{R}$. Thus, $\mathcal{R}$ is a LRS for AQ_n .

Case II: When $n \equiv 1(\text{mod}4)$, Then $n = 4p + 1$, for some $p \geq 2$. Here we consider one more quadrilateral than the previous case and we choose $\mathcal{R} = \{\alpha_2, \alpha_{n-4}\}$.

Now, we will show that $\mathcal{R}$ is a LRS for AQ_n . So, the representation of vector for each neighbouring vertex with respect to $\mathcal{R}$ is given below:

For the adjacent vertices α_{4k-3} and α_{4k-2} , where $1 \leq k \leq p$ we have

$$\begin{aligned} r(\alpha_{4k-3} \mid \mathcal{R}) &= (4k-5, n-4k-1), \\ r(\alpha_{4k-2} \mid \mathcal{R}) &= (4k-4, n-4k-2), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k-2} and α_{4k} , we have

$$\begin{aligned} r(\alpha_{4k-2} \mid \mathcal{R}) &= (4k-4, n-4k-2) \\ r(\alpha_{4k} \mid \mathcal{R}) &= (4k-2, n-4k-4), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k} and α_{4k-1} , we have

$$\begin{aligned} r(\alpha_{4k} \mid \mathcal{R}) &= (4k-2, n-4k-4), \\ r(\alpha_{4k-1} \mid \mathcal{R}) &= (4k-3, n-4k-3), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k-1} and α_{4k-3} , we have

$$\begin{aligned} r(\alpha_{4k-1} \mid \mathcal{R}) &= (4k-3, n-4k-3), \\ r(\alpha_{4k-3} \mid \mathcal{R}) &= (4k-5, n-4k-1), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{n-1} and α_n , we have

$$\begin{aligned} r(\alpha_{n-1} \mid \mathcal{R}) &= (n-3, 3), \text{ and} \\ r(\alpha_n \mid \mathcal{R}) &= (n-2, 4) \end{aligned}$$

Thus, from the above vector representation, we can observe that each neighbouring vertices of $AQ_n \setminus \mathcal{R}$ have the unique vector representation with respect $\mathcal{R}$. Thus, $\mathcal{R}$ is a LRS for AQ_n .

Case III: When $n \equiv 2(\text{mod}4)$, Then $n = 4p + 2$, for some $p \geq 2$. Here we consider one more quadrilateral than the previous case and we choose $\mathcal{R} = \{\alpha_2, \alpha_{n-4}\}$.

So, the representation of vector for each neighbouring vertex of $AQ_n \setminus \mathcal{R}$ with respect to $\mathcal{R}$ is given below:

For the adjacent vertices α_{4k-3} and α_{4k-2} , we have

$$\begin{aligned} r(\alpha_{4k-3} \mid \mathcal{R}) &= (4k - 5, n - 4k), \\ r(\alpha_{4k-2} \mid \mathcal{R}) &= (4k - 4, n - 4k - 1), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k-2} and α_{4k} , we have

$$\begin{aligned} r(\alpha_{4k-2} \mid \mathcal{R}) &= (4k - 4, n - 4k - 1) \\ r(\alpha_{4k} \mid \mathcal{R}) &= (4k - 2, n - 4k - 3), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k} and α_{4k-1} , we have

$$\begin{aligned} r(\alpha_{4k} \mid \mathcal{R}) &= (4k - 2, n - 4k - 3), \\ r(\alpha_{4k-1} \mid \mathcal{R}) &= (4k - 3, n - 4k - 2), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{4k-1} and α_{4k-3} , we have

$$\begin{aligned} r(\alpha_{4k-1} \mid \mathcal{R}) &= (4k - 3, n - 4k - 2), \\ r(\alpha_{4k-3} \mid \mathcal{R}) &= (4k - 5, n - 4k), \forall 1 \leq k \leq p. \end{aligned}$$

For the adjacent vertices α_{n-1} and α_n , we have

$$\begin{aligned} r(\alpha_{n-1} \mid \mathcal{R}) &= (n - 3, 3), \text{ and} \\ r(\alpha_n \mid \mathcal{R}) &= (n - 2, 4). \end{aligned}$$

Since, no neighbouring vertex of $AQ_n \setminus \mathcal{R}$ have the same vector representation with regard to $\mathcal{R}$. Therefore, $\mathcal{R}$ is an LRS for AQ_n .

From all of the above cases, we found that $\mathcal{R}$ is a LRS for AQ_n with cardinality of 2. Hence, $\dim_{\ell}(AQ_n) \leq 2$.

As, the graph AQ_n is not a bipartite graph, so the $\dim_{\ell}(AQ_n) > 1$ (Using Theorem 1). Hence, $\dim_{\ell}(AQ_n) = 2$. $\square$

Theorem 5: For $p \geq 2$, the LMD of alternate pentagonal snake graph is 2. Proof. Let AP_n be an alternating pentagonal snake graph as shown in Figure 3. From Figure 3 we see that for the adjacent vertices of AP_n for $1 \leq k \leq p$ are $\{\alpha_{2k-1}\alpha_{2p+2k-1}\}, \{\alpha_{2p+2k-1}\alpha_{4p+k}\}, \{\alpha_{4p+k}\alpha_{2p+2k}\}, \{\alpha_{2p+2k}\alpha_{2k}\}$, and for $1 \leq i \leq 2p - 1$ the adjacent vertex are $\{\alpha_i, \alpha_{i+1}\}$. To prove the LMD of AP_n is 2, we divide the proof into following three cases based on n .

Case I: When $n \equiv 0(\text{mod}5)$, Then $n = 5p$, for some $p \geq 2$. Let us suppose that $\mathcal{R} = \{\alpha_1, \alpha_{2p}\}$ be an ordered subset of $V(AP_n)$. We will show that $\mathcal{R}$ is a LRS for AP_n . For this, we give the representation of vector for each adjacent vertices of $AP_n \setminus \mathcal{R}$ with respect to the $\mathcal{R}$.

The vector representation of each adjacent vertex of $AP_n \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices $\alpha_{2k-1}\alpha_{2k}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{2k-1} \mid \mathcal{R}) &= (2k-2, 2(p-k)+1), \\ r(\alpha_{2k} \mid \mathcal{R}) &= (2k-1, 2(p-k)). \end{aligned}$$

For the adjacent vertices $\alpha_{2k}\alpha_{2k+1}$, where $1 \leq k \leq p-1$,

$$\begin{aligned} r(\alpha_{2k} \mid \mathcal{R}) &= (2k-1, 2(p-k)), \\ r(\alpha_{2k+1} \mid \mathcal{R}) &= (2k, 2(p-k)-1). \end{aligned}$$

For the adjacent vertices $\alpha_{2k-1}\alpha_{2(p+k)-1}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{2k-1} \mid \mathcal{R}) &= (2k-2, 2(p-k)+1), \\ r(\alpha_{2(p+k)-1} \mid \mathcal{R}) &= (2k-1, 2(p-k)+2). \end{aligned}$$

For the adjacent vertices $\alpha_{2(p+k)-1}\alpha_{4p+k}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{2(p+k)-1} \mid \mathcal{R}) &= (2k-1, 2(p-k)+2) \\ r(\alpha_{4p+k} \mid \mathcal{R}) &= (2k, 2(p-k)+2) \end{aligned}$$

For the adjacent vertices $\alpha_{4p+k}\alpha_{2p+2k}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{4p+k} \mid \mathcal{R}) &= (2k, 2(p-k)+2), \\ r(\alpha_{2(p+k)} \mid \mathcal{R}) &= (2k, 2(p-k)+1). \end{aligned}$$

For the adjacent vertices $\alpha_{2p+2k}\alpha_{2k}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{2(p+k)} \mid \mathcal{R}) &= (2k, 2(p-k)+1), \\ r(\alpha_{2k} \mid \mathcal{R}) &= (2k-1, 2(p-k)). \end{aligned}$$

Hence, from the above vector representation, we can see that no two adjacent vertices of $AP_n \setminus \mathcal{R}$ have the same vector representation with respect $\mathcal{R}$. Thus, $\mathcal{R}$ is a LRS for AP_n .

Case II: When $n \equiv 1(\text{mod } 5)$, Then $n = 5p + 1$, for some $p \geq 2$. Here we consider one more pentagon than the previous case and we choose $\mathcal{R} = \{\alpha_1, \alpha_n\}$ be an ordered subset of $V(AP_n)$. We will show that $\mathcal{R}$ is a LRS for AP_n . So, the representation of vector for each adjacent vertex of $AP_n \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices $\alpha_{2k-1}\alpha_{2k}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{2k-1} \mid \mathcal{R}) &= (2k-2, 2p-2k+2), \\ r(\alpha_{2k} \mid \mathcal{R}) &= (2k-1, 2p-2k+1). \end{aligned}$$

For the adjacent vertices $\alpha_{2k}\alpha_{2k+1}$, where $1 \leq k \leq p-1$,

$$\begin{aligned} r(\alpha_{2k} \mid \mathcal{R}) &= (2k-1, 2p-2k+1), \\ r(\alpha_{2k+1} \mid \mathcal{R}) &= (2k, 2p-2k). \end{aligned}$$

For the adjacent vertices $\alpha_{2k-1}\alpha_{2(p+k)-1}$, where $1 \leq k \leq p$,

$$\begin{aligned} r(\alpha_{2k-1} \mid \mathcal{R}) &= (2k-2, 2(p-k)+2), \\ r(\alpha_{2(p+k)-1} \mid \mathcal{R}) &= (2k-1, 2(p-k)+2). \end{aligned}$$

For the adjacent vertices $\alpha_{2(p+k)-1}\alpha_{4p+k}$, where $1 \leq k \leq p$,

$$r(\alpha_{2(k+p)-1} \mid \mathcal{R}) = (2k - 1, 2(p - k) + 2)$$

$$r(\alpha_{4p+k} \mid \mathcal{R}) = (2k, 2p - 2k + 3)$$

For the adjacent vertices $\alpha_{4p+k}\alpha_{2p+2k}$, where $1 \leq k \leq p$,

$$r(\alpha_{4p+k} \mid \mathcal{R}) = (2k, 2p - 2k + 3),$$

$$r(\alpha_{2(p+k)} \mid \mathcal{R}) = (2k, 2(p - k) + 1).$$

For the adjacent vertices $\alpha_{2p+2k}\alpha_{2k}$, where $1 \leq k \leq p$,

$$r(\alpha_{2(k+p)} \mid \mathcal{R}) = (2k, 2(p - k) + 1),$$

$$r(\alpha_{2k} \mid \mathcal{R}) = (2k - 1, 2(p - k) + 1).$$

Thus, from the above vector representation, we can observe that no two adjacent vertices of $AP_n \setminus \mathcal{R}$ have the same vector representation with respect $\mathcal{R}$. Therefore, $\mathcal{R}$ is a LRS for AP_n .

Case III: When $n \equiv 2(\text{mod}5)$, Then $n = 5p + 2$, for some $p \geq 2$. Here we consider one more pentagon than the previous case and we choose $\mathcal{R} = \{\alpha_1, \alpha_{2p+2}\}$ be an ordered subset of $V(AP_n)$. We will show that $\mathcal{R}$ is a LRS for AP_n . So, the representation of vector for each adjacent vertex of $AP_n \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices $\alpha_{2k-1}\alpha_{2k}$, where $1 \leq k \leq p$,

$$r(\alpha_{2k-1} \mid \mathcal{R}) = (2k - 2, 2p - 2k + 3)$$

$$r(\alpha_{2k} \mid \mathcal{R}) = (2k - 1, 2(p - k) + 2)$$

For the adjacent vertices $\alpha_{2k}\alpha_{2k+1}$, where $1 \leq k \leq p - 1$,

$$r(\alpha_{2k} \mid \mathcal{R}) = (2k - 1, 2p - 2k + 2),$$

$$r(\alpha_{2k+1} \mid \mathcal{R}) = (2k, 2p - 2k + 1).$$

For the adjacent vertices $\alpha_{2k-1}\alpha_{2(p+k)-1}$, where $1 \leq k \leq p$,

$$r(\alpha_{2k-1} \mid \mathcal{R}) = (2k - 2, 2p - 2k + 3),$$

$$r(\alpha_{2(k+p)-1} \mid \mathcal{R}) = (2k - 1, 2p - 2k + 3).$$

For the adjacent vertices $\alpha_{2(p+k)-1}\alpha_{4p+k}$, where $1 \leq k \leq p$,

$$r(\alpha_{2(k+p)-1} \mid \mathcal{R}) = (2k - 1, 2p - 2k + 3)$$

$$r(\alpha_{4p+k} \mid \mathcal{R}) = (2k, 2(p - k) + 4)$$

For the adjacent vertices $\alpha_{4p+k}\alpha_{2p+2k}$, where $1 \leq k \leq p$,

$$r(\alpha_{4p+k} \mid \mathcal{R}) = (2k, 2(p - k) + 4)$$

$$r(\alpha_{2(p+k)} \mid \mathcal{R}) = (2k, 2(p - k) + 2)$$

For the adjacent vertices $\alpha_{2p+2k}\alpha_{2k}$, where $1 \leq k \leq p$,

$$r(\alpha_{2(k+p)} \mid \mathcal{R}) = (2k, 2(p - k) + 2),$$

$$r(\alpha_{2k} \mid \mathcal{R}) = (2k - 1, 2p - 2k + 2).$$

Since, the neighbouring vertices have unique representation of vector with regard to $\mathcal{R}$. Therefore, $\mathcal{R}$ is an LRS for AP_n .

In all the above cases, we find that $\mathcal{R}$ is a LRS for AP_n with cardinality of 2. Hence, $\dim_\ell(AP_n) \leq 2$. As, the graph AP_n is not a bipartite graph, so the $\dim_\ell(AP_n) > 1$ (Using Theorem 1). Hence, $\dim_\ell(AP_n) = 2$. $\square$

Theorem 6: For $k \geq 4$, the LMD of alternate k -polygon snake graph is 2. *Proof.* Let $AS_n(C_k)$ be an alternating k -polygonal snake graph as shown in Figure 4. Let us suppose that $\mathcal{R} = \{\alpha_1, \alpha_{2p}\}$, then for every neighbouring vertex the vector representation of $AS_n(C_k) \setminus \mathcal{R}$ with respect to the $\mathcal{R}$ is given as:

Case I: When k is even.

Let us suppose that the ordered set $\mathcal{R} = \{\alpha_1, \alpha_{2p}\}$. Then the representation of vector for every neighbouring vertex of $AS_n(C_k) \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices α_{2i-1} , and $\alpha_{2p+(k-2)(i-1)+1}$, where $1 \leq k \leq p$,

$$r(\alpha_{2i-1} \mid \mathcal{R}) = (2i - 2, 2p - 2i + 1),$$

$$r(\alpha_{2p+(k-2)(i-1)+1} \mid \mathcal{R}) = (2i - 1, \min\{2p - 2i + 2, 2p - 2i + k - 1\}).$$

For adjacent vertices $\alpha_{2p+(k-2)(i-1)+j}$, and $\alpha_{2p+(k-2)(i-1)+j+1}$, where $1 \leq i \leq p, 1 \leq j \leq k - 3$.

$$r(\alpha_{2p+(k-2)(i-1)+j} \mid \mathcal{R}) = (\min\{2i + j - 2, 2i + k - j - 2\}, \\ \min\{2p - 2i + j + 1, 2p - 2i + k - j\}),$$

$$r(\alpha_{2p+(k-2)(i-1)+j+1} \mid \mathcal{R}) = (\min\{2i + j - 1, 2i + k - j - 3\}, \\ \min\{2p - 2i + j + 2, 2p - 2i + k - j - 1\}).$$

For adjacent vertices $\alpha_{2p+(k-2)i}$, and α_{2i} where $1 \leq i \leq p$

$$r(\alpha_{2p+(k-2)i} \mid \mathcal{R}) = (\min\{2i + k - 4, 2i\}, \min\{2p - 2i + k - 1, 2p - 2i\}), \\ r(\alpha_{2i} \mid \mathcal{R}) = (2i - 1, 2p - 2i).$$

For adjacent vertices α_{2i} , and α_{2i+1} where $1 \leq i \leq p$

$$r(\alpha_{2i} \mid \mathcal{R}) = (2i - 1, 2p - 2i), \\ r(\alpha_{2i+1} \mid \mathcal{R}) = (2i, 2p - 2i - 1)$$

Case II: When k is odd.

Let us suppose that the ordered set $\mathcal{R} = \{\alpha_1, \alpha_{2p+(k-2)}\}$. Then the representation of vector for every neighbouring vertex of $AS_n(C_k) \setminus \mathcal{R}$ in relation with $\mathcal{R}$ has been shown below:

For the adjacent vertices α_{2i-1} , and $\alpha_{2p+(k-2)(i-1)+1}$, where $1 \leq k \leq p$,

$$r(\alpha_{2i-1} \mid \mathcal{R}) = (2i - 2, 2i - 1),$$

$$r(\alpha_{2p+(k-2)(i-1)+1} \mid \mathcal{R}) = (2i - 1, 2i - 2).$$

For adjacent vertices $\alpha_{2p+(k-2)(i-1)+j}$, and $\alpha_{2p+(k-2)(i-1)+j+1}$, where $1 \leq i \leq p, 1 \leq j \leq k - 3$.

$$r(\alpha_{2p+(k-2)(i-1)+j} \mid \mathcal{R}) = (\min\{2i+j-2, 2i+k-j-2\}, \\ \min\{2i+j-3, 2i+k-j-3\}),$$

$$r(\alpha_{2p+(k-2)(i-1)+j+1} \mid \mathcal{R}) = (\min\{2i+j-1, 2i+k-j-3\}, \\ \min\{2i+j-2, 2i+k-j-4\}).$$

For the adjacent vertices $\alpha_{2p+(k-2)}$, and α_2 , we have

$$r(\alpha_{2p+(k-2)} \mid \mathcal{R}) = (2, 0) \\ r(\alpha_2 \mid \mathcal{R}) = (1, 1)$$

For adjacent vertices $\alpha_{2p+(k-2)i}$, and α_{2i} , where $2 \leq i \leq p$

$$r(\alpha_{2p+(k-2)i} \mid \mathcal{R}) = (\min\{2i+k-4, 2i\}, \min\{2i+k-5, 2i-1\}), \\ r(\alpha_{2i} \mid \mathcal{R}) = (2i-1, 2i).$$

For adjacent vertices α_{2i} , and α_{2i+1} where $1 \leq i \leq p$

$$r(\alpha_{2i} \mid \mathcal{R}) = (2i-1, 2i), \\ r(\alpha_{2i+1} \mid \mathcal{R}) = (2i, 2i+1)$$

Since no two neighbouring vertices of $AS_n(C_k) \setminus \mathcal{R}$ have the same representation of vector with regard to $\mathcal{R}$. Therefore, $\mathcal{R}$ is an LRS for $AS_n(C_k)$. In all the above cases, we find that $\mathcal{R}$ is a LRS for $AS_n(C_k)$ with cardinality of 2. Hence, $\dim_\ell(AS_n(C_k)) \leq 2$.

As, the graph $AS_n(C_k)$ is not a bipartite graph, so the $\dim_\ell(AS_n(C_k)) > 1$ (Using Theorem 1). Hence, $\dim_\ell(AS_n(C_k)) = 2$.

In the next theorem, we compute the SMD of alternate families of snake graphs. Here, to prove the results we will use the strong resolving graphs and boundary framework given by Oellermann and Peters-Fransen [24]. Note that all the families of graphs considered in this paper have geodesic spine structure and for each $1 \leq i \leq n$, we have $d(\alpha_1, \alpha_i) = i-1$, and $(\alpha_n, \alpha_i) = n-i$. (1)

To compute the SMD we need to find the boundary of graph, then we need to build the strong resolving graph and at the end we need to identify the vertex cover of graph from strong resolving graph.

Theorem 7: For $AT_n, n \geq 3$ the strong metric dimension is 2.

Proof: Let us consider the vertex α_1 and α_n of AT_n . So, the distance of the vertex α_1 with the end vertex α_n is $d(\alpha_1, \alpha_n) = n-1$ using Equation (1). Also, let us suppose $u \in N(\alpha_1)$. So, every neighbour say u of α_1 is closer to α_n i.e., $d(u, \alpha_n) < d(\alpha_1, \alpha_n)$. Therefore, α_1 is maximally distant from α_n .

Similarly, if $t \in N(\alpha_n)$ then $d(w, \alpha_1) < d(\alpha_n, \alpha_1)$. Therefore, α_n is maximally distant from α_1 . Hence, α_1 and α_n are MMD.

Next, if we choose any vertex say $v = \alpha_i \in V(G) \setminus \{\alpha_1, \alpha_n\}$, where $2 \leq i \leq n-1$. Then v is different from the two extreme vertices of AT_n , and by equation no. (1), $d(\alpha_{i+1}, \alpha_1) = i > d(\alpha_i, \alpha_1) = i-1$. So, the vertex v has a

neighbour $u \in N(v)$ that lies far from either α_1 or α_n . Consequently, $d(u, \alpha_1) > d(v, \alpha_1)$ or $d(u, \alpha_n) > d(v, \alpha_n)$.

Thus, the vertex v is not maximally distant from any vertex of AT_n . Therefore, the pair $\{\alpha_1, \alpha_n\}$ is the unique MMD pair in AT_n . Hence, $\partial(AT_n) = \{\alpha_1, \alpha_n\}$.

Now, the only MMD vertices in AT_n are α_1 and α_n . Hence in the strong resolving graph $(AT_n)_{SR}$ the vertices α_1 and α_n are adjacent, while no other pair of vertices is adjacent. So, the vertex cover set is equal to the boundary of AT_n which is $\partial(AT_n) = \{\alpha_1, \alpha_n\}$, with the cardinality of 2.

Thus, $\beta(AT_n)_{SR} = 2$, and by using theorem 2, $\text{sdim}(AT_n) = 2$. Similarly, we can find the SMD of the other families of alternate snake graphs.

Theorem 8: For AQ_n , $n \geq 4$ the strong metric dimension is 2.

Proof: Similar to Theorem 7.

Theorem 9: For AP_n , $n \geq 5$ the strong metric dimension is 2.

Proof: Similar to Theorem 7.

4. Conclusion

In this article we have computed the local and the strong metric dimensions of the alternate families of the snake graph. As the graphs are infinite and despite being increasing each time, new blocks, the LMD as well as SMD remains constant. This study shows the role of linear backbone structures, as they can be generalized to larger classes of edge-linked polygonal chains, and emphasizes the stabilizing function of linear backbone structures in distance-based graph parameters. In future one can also explore other types of resolving parameters, such as the fault-tolerant metric dimension and fractional metric dimension, for these graph families. Studying these could help in understanding how different distance-based parameters are connected with the structural properties of such networks.

Competing Interest

The authors claim they have no conflicting interests.

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Author Contributions

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