

Lattice structure of fuzzy groups in fuzzy approximation spaces

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Abstract

This paper presents a comprehensive of the algebraic and lattice structure of fuzzy groups in fuzzy approximation spaces. The structure of $\mathcal{F}(\mathcal{P}, \mathcal{Q})$, the collection of all fuzzy subgroups of a group $\mathcal{P}$ in a fuzzy approximation space $\mathcal{F}(\mathcal{P}, \mathcal{Q})$ is systemically analysed. The main contributions include: (i) proving that $\mathcal{F}(\mathcal{P}, \mathcal{Q})$ forms a complete meet-semilattice under pointwise ordering. The infimum exists for any collection of fuzzy subgroups; (ii) $\mathcal{F}(\mathcal{P}, \mathcal{Q})$ does not generally form a complete lattice, as suprema may fail to preserve fuzzy subgroup properties and this is shown through counterexample; (iii) developing a comprehensive homomorphism theory including fuzzy isomorphism theorems, composition properties, and preservation results; and (iv) establishing that join-irreducible fuzzy subgroups generate all fuzzy subgroups through meets and directed joins. Every chain in $\mathcal{F}(\mathcal{P}, \mathcal{Q})$ admits both supremum and infimum. The framework extends classical group-theoretic concepts thereby giving a generalization of the concepts.

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1. Introduction

Zadeh [1] introduced the concept of fuzzy sets in 1965, marked the beginning of dealing with uncertainty. Rosenfeld [2] developed the fuzzy algebraic structures, in particular fuzzy groups. Rosenfeld developed traditional group concepts to include partial membership. Anthony and Sherwood [3] produced firmer definitions of fuzzy subgroups, setting conventions which are still standard in the discipline. Bhattacharya [4] provided decisive characterisations of fuzzy subgroups and the structural features of fuzzy

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subgroups. Močkoř [5] proposed a general framework to amalgamate different types of fuzzy sets within a united theory of general approximation spaces and relational morphisms. Hosseinpour [6] proposed T-rough fuzzy subgroups, which is a combination of rough set theory through T-rough sets and fuzzy subgroups. Bağırmaz [7] has defined fuzzy rough subgroups and fuzzy rough normal subgroups in the context of rough groups. Wolski [8] studied the interaction between object perception and classification using topological structures. Pankaj, Singh, and Rao [9] contributes to the comprehensive axiomatic framework for proximity spaces on hesitant fuzzy sets. Singh, Alam, and Bhardwaj [10] helps in developing graph theory that offers powerful tools for representing and analyzing complex relationships in recommendation networks. Ullah, Ibrahim, and Saeed [11] studied fuzzy cosets of AG-groups and Choudhary and Biswas [12] discussed various properties of fuzzy groups and subgroups. Ahmad, Bal, and Aswad [13] studied the kernels of fuzzy and anti-fuzzy groups. Herrlich [14] introduced notions of nearness which offer structural frameworks for understanding proximity. Khandelwal and Kumar [15] explored fuzzy assignment dynamics using computational techniques and demonstrated the applicability of fuzzy logic to decision support. Wang, Liu, and Qiu [16] introduced nearness approximation spaces based on axiomatic fuzzy sets, which has advanced the nearness approximation spaces with structure-oriented interpretation of fuzzy membership. Preethi and Revathi [17] studied about extended fuzzy topological concepts to digital settings by investigating Pythagorean fuzzy digital topological structures and their associated properties, thereby enriching the theoretical foundation of fuzzy mathematics

1.1 Research Gap

The compatibility of fuzzy subgroup closure with approximation relations and the possible failure of pointwise suprema to form fuzzy subgroups present fundamental challenges in fuzzy approximation spaces.

1.2 Research Objectives

The proposed research aims to develop a mathematical framework for fuzzy groups in fuzzy approximation spaces, investigate the lattice structure of fuzzy subgroups, establish homomorphism theories and their fundamental properties, and analyze the role of join-irreducible elements in characterizing the structure of fuzzy subgroup lattices.

2. Preliminaries

Definition 2.1: Let $\mathcal{P}$ be a group and μ be a fuzzy subset of $\mathcal{P}$ such that $\mu: \mathcal{P} \rightarrow [0,1]$. Then μ is called a fuzzy subgroup of $\mathcal{P}$ if

1. $\mu(rs) \geq \min\{\mu(r), \mu(s)\}$
2. $\mu(r^{-1}) \geq \mu(r)$
3. or $\mu(rs^{-1}) \geq \min\{\mu(r), \mu(s)\}, \forall r, s \in \mathcal{P}$.

Definition 2.2: Let Q be a fuzzy relation on $\mathcal{U}$ where $\mathcal{U}$ be a non empty universe of discourse such that $Q: \mathcal{U} \times \mathcal{U} \rightarrow [0,1]$. Then, the pair $(\mathcal{U}, Q)$ is called a FAS.

Definition 2.3: Let $(\mathcal{P}, *, Q)$ be an approximation structure, where $*$ is a binary operation on $\mathcal{P}$ and $Q: \mathcal{P} \times \mathcal{P} \rightarrow [0,1]$ denotes the approximation relation. The relation Q is said to satisfy the compatibility condition with respect to $*$ if, for all $r, s, t, w \in \mathcal{P}$,

$$Q(r, s) \cap Q(t, w) \leq Q(r * t, s * w).$$

Definition 2.4: (Homomorphism of fuzzy groups). Let $(\mathcal{P}, *, \mu_{\mathcal{P}}, Q_{\mathcal{P}})$ and $(\mathcal{H}, \cdot, \mu_{\mathcal{H}}, Q_{\mathcal{H}})$ be two fuzzy approximation spaces and let $*, \cdot$ are binary operations over $\mathcal{P}$ and $\mathcal{H}$ respectively. A function $f: \mathcal{P} \rightarrow \mathcal{H}$ is called fuzzy homomorphism between fuzzy approximation space if

1. $f(r * s) = f(r) \cdot f(s) \forall r, s \in \mathcal{P}$
2. $\mu_{\mathcal{H}}(f(r)) \geq \mu_{\mathcal{P}}(r), \forall r \in \mathcal{P}$
3. $Q_{\mathcal{H}}(f(r), f(s)) \geq Q_{\mathcal{P}}(r, s) \forall r, s \in \mathcal{P}$.

3. Algebraic and Lattice Theoretic Properties

Theorem 3.1: (First Isomorphism Theorem for Fuzzy Groups in Fuzzy Approximation Spaces). Let $(\mathcal{P}, *, \mu_{\mathcal{P}}, Q_{\mathcal{P}})$ and $(\mathcal{H}, \cdot, \mu_{\mathcal{H}}, Q_{\mathcal{H}})$ be fuzzy groups in fuzzy approximation spaces, and let $f: \mathcal{P} \rightarrow \mathcal{H}$ be a fuzzy homomorphism. Define the fuzzy kernel of f as: $\text{Ker}_f(r) = \sup\{\mu_{\mathcal{P}}(s): s \in \mathcal{P}, \mu_{\mathcal{H}}(f(s)) = \mu_{\mathcal{H}}(e_{\mathcal{H}}) \text{ and } Q_{\mathcal{H}}(f(r), f(s)) = 1\}$

Then there exists a fuzzy isomorphism $\varphi: \mathcal{P}/\text{Ker}_f \rightarrow \mathfrak{I}(f)$ in the induced fuzzy approximation spaces such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{f} & \mathcal{H} \\ \pi \downarrow & & \uparrow i \\ \mathcal{P}/\text{Ker}_f & \xrightarrow{\varphi} & \mathfrak{I}(f) \end{array}$$

where $\pi: \mathcal{P} \rightarrow \mathcal{P}/\text{Ker}_f$ is the natural projection and $i: \mathfrak{I}(f) \rightarrow \mathcal{H}$ is the inclusion map, satisfying $f = i \circ \varphi \circ \pi$.

Proof: First, we define the fuzzy quotient group $\mathcal{P}/\text{Ker}_f$ with membership function:

$$\mu_{\mathcal{P}/\text{Ker}_f}([r]) = \sup\{\mu_{\mathcal{P}}(s): s \in [r]\}$$

where $[r]$ denotes the fuzzy equivalence class of r under the relation induced by Ker_f .

Define $\varphi: \mathcal{P}/\text{Ker}_f \rightarrow \mathfrak{I}(f)$ by $\varphi([r]) = f(r)$.

Step 1: Well-definedness of φ . If $[r] = [s]$, then there exists $t \in \text{Ker}_f$ such that:

$$\mathcal{Q}_{\mathcal{P}}(r * t, s) \geq \min\{\mu_{\mathcal{P}}(r), \mu_{\mathcal{P}}(t), \mu_{\mathcal{P}}(s)\}$$

Since f is a fuzzy homomorphism and $t \in \text{Ker}_f$:

$$\begin{aligned} \mathcal{Q}_{\mathcal{H}}(f(r), f(s)) &= \mathcal{Q}_{\mathcal{H}}(f(r) \cdot f(t), f(s)) \geq \mathcal{Q}_{\mathcal{P}}(r * t, s) \\ &\geq \min\{\mu_{\mathcal{P}}(r), \mu_{\mathcal{P}}(t), \mu_{\mathcal{P}}(s)\} \end{aligned}$$

As $\mu_{\mathcal{H}}(f(t)) = \mu_{\mathcal{H}}(e_{\mathcal{H}})$, we have $f(r) = f(s)$ in the fuzzy sense, so φ is well-defined.

Step 2: φ preserves operations.

$$\varphi([r] * [s]) = \varphi([r * s]) = f(r * s) = f(r) \cdot f(s) = \varphi([r]) \cdot \varphi([s])$$

Step 3: φ preserves membership values.

$$\mu_{\mathfrak{Z}(f)}(\varphi([r])) = \mu_{\mathcal{H}}(f(r)) \geq \mu_{\mathcal{P}}(r) = \mu_{\mathcal{P}/\text{Ker}_f}([r])$$

Step 4: φ preserves fuzzy relations. For $[r], [s] \in \mathcal{P}/\text{Ker}_f$:

$$\mathcal{Q}_{\mathfrak{Z}(f)}(\varphi([r]), \varphi([s])) = \mathcal{Q}_{\mathcal{H}}(f(r), f(s)) \geq \mathcal{Q}_{\mathcal{P}}(r, s) = \mathcal{Q}_{\mathcal{P}/\text{Ker}_f}([r], [s])$$

Step 5: φ is injective. If $\varphi([r]) = \varphi([s])$, then $f(r) = f(s)$, which implies $r * s^{-1} \in \text{Ker}_f$. Thus $[r] = [s]$ in $\mathcal{P}/\text{Ker}_f$.

Step 6: φ is surjective onto $\mathfrak{Z}(f)$. By definition of $\mathfrak{Z}(f)$, for any $h \in \mathfrak{Z}(f)$, there exists $r \in \mathcal{P}$ such that $f(r) = h$. Then $\varphi([r]) = f(r) = h$.

Therefore, φ is a fuzzy isomorphism between $\mathcal{P}/\text{Ker}_f$ and $\mathfrak{Z}(f)$.

Theorem 3.2: (Composition of Fuzzy Homomorphisms). *Let $(\mathcal{P}, *, \mu_{\mathcal{P}}, \mathcal{Q}_{\mathcal{P}})$, $(\mathcal{H}, \cdot, \mu_{\mathcal{H}}, \mathcal{Q}_{\mathcal{H}})$, and $(\mathcal{K}, \circ, \mu_{\mathcal{K}}, \mathcal{Q}_{\mathcal{K}})$ be fuzzy groups in fuzzy approximation spaces. If $f: \mathcal{P} \rightarrow \mathcal{H}$ and $g: \mathcal{H} \rightarrow \mathcal{K}$ are fuzzy homomorphisms, then the composition $g \circ f: \mathcal{P} \rightarrow \mathcal{K}$ is also a fuzzy homomorphism. Moreover:*

1. $\mu_{\mathcal{K}}((g \circ f)(r)) \geq \mu_{\mathcal{P}}(r)$ for all $r \in \mathcal{P}$
2. $\mathcal{Q}_{\mathcal{K}}((g \circ f)(r), (g \circ f)(s)) \geq \mathcal{Q}_{\mathcal{P}}(r, s)$ for all $r, s \in \mathcal{P}$

Proof: We need to verify the three conditions for fuzzy homomorphism:

Condition 1: Preservation of operations.

$$\begin{aligned} (g \circ f)(r * s) &= g(f(r * s)) = g(f(r) \cdot f(s)) \\ &= g(f(r)) \circ g(f(s)) = (g \circ f)(r) \circ (g \circ f)(s) \end{aligned} \quad (1)$$

Condition 2: Preservation of membership values.

$$\mu_{\mathcal{K}}((g \circ f)(r)) = \mu_{\mathcal{K}}(g(f(r))) \geq \mu_{\mathcal{H}}(f(r)) \geq \mu_{\mathcal{P}}(r)$$

The first inequality follows because g is a fuzzy homomorphism, and the second because f is a fuzzy homomorphism.

Condition 3: Preservation of fuzzy relations.

$$\begin{aligned} Q_{\mathcal{K}}((g \circ f)(r), (g \circ f)(s)) &= Q_{\mathcal{K}}(g(f(r)), g(f(s))) \geq Q_{\mathcal{H}}(f(r), f(s)) \\ &\geq Q_{\mathcal{P}}(r, s) \end{aligned}$$

Both inequalities follow from the homomorphism properties of g and f respectively.

Therefore, $g \circ f$ is a fuzzy homomorphism.

Theorem 3.3: (Fuzzy Conjugation Homomorphisms). *Let $(\mathcal{P}, *, \mu_{\mathcal{P}}, Q_{\mathcal{P}})$ be a fuzzy group in a fuzzy approximation space. For any $\alpha \in \mathcal{P}$ with $\mu_{\mathcal{P}}(\alpha) > 0$, the conjugation map $\varphi_{\alpha}: \mathcal{P} \rightarrow \mathcal{P}$ defined by $\varphi_{\alpha}(r) = \alpha * r * \alpha^{-1}$ is a fuzzy automorphism if and only if:*

$$Q_{\mathcal{P}}(\alpha * r * \alpha^{-1}, \alpha * s * \alpha^{-1}) \geq \min(Q_{\mathcal{P}}(r, s), \mu_{\mathcal{P}}(\alpha)) \text{ for all } r, s \in \mathcal{P}$$

Proof: ($\Rightarrow$) Suppose φ_{α} is a fuzzy automorphism. Then by definition:

$$Q_{\mathcal{P}}(\varphi_{\alpha}(r), \varphi_{\alpha}(s)) \geq Q_{\mathcal{P}}(r, s)$$

Which gives us:

$$Q_{\mathcal{P}}(\alpha * r * \alpha^{-1}, \alpha * s * \alpha^{-1}) \geq Q_{\mathcal{P}}(r, s)$$

Since $\mu_{\mathcal{P}}(\varphi_{\alpha}(r)) \geq \mu_{\mathcal{P}}(r)$ and $\varphi_{\alpha}(r)$ involves α and α^{-1} :

$$Q_{\mathcal{P}}(\alpha * r * \alpha^{-1}, \alpha * s * \alpha^{-1}) \geq \min(Q_{\mathcal{P}}(r, s), \mu_{\mathcal{P}}(\alpha))$$

($\Leftarrow$) Suppose the condition holds.

Step 1: φ_{α} preserves operations.

$$\varphi_{\alpha}(r * s) = \alpha * (r * s) * \alpha^{-1} = \alpha * r * \alpha^{-1} * \alpha * s * \alpha^{-1} = \varphi_{\alpha}(r) * \varphi_{\alpha}(s)$$

Step 2: φ_{α} preserves membership. By properties of fuzzy groups:

$$\mu_{\mathcal{P}}(\alpha * r * \alpha^{-1}) \geq \min(\mu_{\mathcal{P}}(\alpha), \mu_{\mathcal{P}}(r), \mu_{\mathcal{P}}(\alpha^{-1})) = \min(\mu_{\mathcal{P}}(\alpha), \mu_{\mathcal{P}}(r))$$

When $\mu_{\mathcal{P}}(\alpha) = 1$, we have $\mu_{\mathcal{P}}(\alpha * r * \alpha^{-1}) \geq \mu_{\mathcal{P}}(r)$.

Step 3: φ_{α} preserves relations. By hypothesis: $Q_{\mathcal{P}}(\varphi_{\alpha}(r), \varphi_{\alpha}(s)) = Q_{\mathcal{P}}(\alpha * r * \alpha^{-1}, \alpha * s * \alpha^{-1}) \geq \min(Q_{\mathcal{P}}(r, s), \mu_{\mathcal{P}}(\alpha))$

Step 4: φ_{α} is bijective. The inverse is $\varphi_{\alpha^{-1}}(r) = \alpha^{-1} * r * \alpha$, which satisfies the same properties.

Therefore, φ_{α} is a fuzzy automorphism.

Theorem 3.4: (Fuzzy Homomorphism Extension Theorem). *Let $(\mathcal{P}, *, \mu_{\mathcal{P}}, Q_{\mathcal{P}})$ and $(\mathcal{H}, \cdot, \mu_{\mathcal{H}}, Q_{\mathcal{H}})$ be fuzzy groups in fuzzy approximation spaces. Let $S \subseteq \mathcal{P}$ be a fuzzy generating set for $\mathcal{P}$ (i.e., the fuzzy subgroup generated by S equals $\mathcal{P}$). If $f: S \rightarrow \mathcal{H}$ is a map satisfying:*

1. For all $s_1, s_2 \in S$ with $s_1 * s_2 \in S$: $f(s_1 * s_2) = f(s_1) \cdot f(s_2)$
2. $\mu_{\mathcal{H}}(f(s)) \geq \mu_{\mathcal{P}}(s)$ for all $s \in S$
3. $Q_{\mathcal{H}}(f(s_1), f(s_2)) \geq Q_{\mathcal{P}}(s_1, s_2)$ for all $s_1, s_2 \in S$

Then f extends uniquely to a fuzzy homomorphism $F: \mathcal{P} \rightarrow \mathcal{H}$.

Proof: If F is a fuzzy homomorphism extending f , then for any $g \in \mathcal{P}$ expressed as $g = s_1^{\varepsilon_1} * s_2^{\varepsilon_2} * \dots * s_n^{\varepsilon_n}$ (where $s_i \in S$, $\varepsilon_i = \pm 1$):

$$F(g) = F(s_1)^{\varepsilon_1} \cdot F(s_2)^{\varepsilon_2} \cdot \dots \cdot F(s_n)^{\varepsilon_n} = f(s_1)^{\varepsilon_1} \cdot f(s_2)^{\varepsilon_2} \cdot \dots \cdot f(s_n)^{\varepsilon_n}$$

Define $F(g)$ for $g = s_1^{\varepsilon_1} * s_2^{\varepsilon_2} * \dots * s_n^{\varepsilon_n}$ as:

$$F(g) = f(s_1)^{\varepsilon_1} \cdot f(s_2)^{\varepsilon_2} \cdot \dots \cdot f(s_n)^{\varepsilon_n}$$

Step 1: Well-definedness. We need to show that if g has two representations, F gives the same value. This follows from the relations in S being preserved by f .

Step 2: F preserves operations. By construction, $F(g_1 * g_2) = F(g_1) \cdot F(g_2)$.

Step 3: F preserves membership. For $g = s_1^{\varepsilon_1} * \dots * s_n^{\varepsilon_n}$:

$$\mu_G(g) \leq \min\{\mu_G(s_i): i = 1, \dots, n\}$$

$$\mu_H(F(g)) \geq \min\{\mu_H(f(s_i)): i = 1, \dots, n\} \geq \min\{\mu_G(s_i): i = 1, \dots, n\} \geq \mu_P(g)$$

Step 4: F preserves relations. Similar argument using the preservation of relations by f on S .

Therefore, F is a fuzzy homomorphism extending f .

Theorem 3.5: (Fuzzy Direct Product Homomorphisms). Let $(\mathcal{P}_1, \mu_1, Q_1)$ and $(\mathcal{P}_2, \mu_2, Q_2)$ be fuzzy groups in fuzzy approximation spaces. The projections $\pi_1: \mathcal{P}_1 \times \mathcal{P}_2 \rightarrow \mathcal{P}_1$ and $\pi_2: \mathcal{P}_1 \times \mathcal{P}_2 \rightarrow \mathcal{P}_2$ are fuzzy homomorphisms, where $\mathcal{P}_1 \times \mathcal{P}_2$ has the product fuzzy structure:

- $\mu_{\mathcal{P}_1 \times \mathcal{P}_2}(r, s) = \min(\mu_1(r), \mu_2(s))$
- $Q_{\mathcal{P}_1 \times \mathcal{P}_2}((r_1, s_1), (r_2, s_2)) = \min(Q_1(r_1, r_2), Q_2(s_1, s_2))$

Moreover, for any fuzzy group $(\mathcal{H}, \mu_{\mathcal{H}}, Q_{\mathcal{H}})$ and fuzzy homomorphisms $f_1: \mathcal{H} \rightarrow \mathcal{P}_1$, $f_2: \mathcal{H} \rightarrow \mathcal{P}_2$, there exists a unique fuzzy homomorphism $f: \mathcal{H} \rightarrow \mathcal{P}_1 \times \mathcal{P}_2$ such that $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$.

Proof: Part 1: Projections are fuzzy homomorphisms. For π_1 (similar for π_2):

- $\pi_1((r_1, s_1) * (r_2, s_2)) = \pi_1((r_1 * r_2, s_1 * s_2)) = r_1 * r_2 = \pi_1((r_1, s_1)) * \pi_1((r_2, s_2))$
- $\mu_1(\pi_1((r, s))) = \mu_1(r) \geq \min(\mu_1(r), \mu_2(s)) = \mu_{\mathcal{P}_1 \times \mathcal{P}_2}((r, s))$
- $Q_1(\pi_1((r_1, s_1)), \pi_1((r_2, s_2))) = Q_1(r_1, r_2) \geq \min(Q_1(r_1, r_2), Q_2(s_1, s_2)) = Q_{\mathcal{P}_1 \times \mathcal{P}_2}((r_1, s_1), (r_2, s_2))$

Part 2: Universal property. Define $f: \mathcal{H} \rightarrow \mathcal{P}_1 \times \mathcal{P}_2$ by $f(h) = (f_1(h), f_2(h))$.

f is a fuzzy homomorphism:

- $f(h_1 * h_2) = (f_1(h_1 * h_2), f_2(h_1 * h_2)) = (f_1(h_1) * f_1(h_2), f_2(h_1) * f_2(h_2)) = f(h_1) * f(h_2)$
- $\mu_{\mathcal{P}_1 \times \mathcal{P}_2}(f(h)) = \min(\mu_1(f_1(h)), \mu_2(f_2(h))) \geq \min(\mu_H(h), \mu_H(h)) = \mu_H(h)$
- Similar for relations

Uniqueness: If $g: \mathcal{H} \rightarrow \mathcal{P}_1 \times \mathcal{P}_2$ also satisfies $\pi_1 \circ g = f_1$ and $\pi_2 \circ g = f_2$, then:

$$g(h) = (\pi_1(g(h)), \pi_2(g(h))) = (f_1(h), f_2(h)) = f(h)$$

Theorem 3.6: (Fuzzy Transfer Homomorphisms). *Let $(\mathcal{P}, *, \mu_{\mathcal{P}}, \mathcal{Q}_{\mathcal{P}})$ be a fuzzy group in a fuzzy approximation space, and let $\mathcal{H}$ be a fuzzy subgroup of $\mathcal{P}$ with finite index. Define the transfer homomorphism $V: \mathcal{P} \rightarrow \mathcal{H}$ by:*

$$V(g) = \prod_i r_i * g * r_i^{-1}$$

where $\{r_i\}$ is a set of coset representatives for $\mathcal{P}/\mathcal{H}$.

Then V is a fuzzy homomorphism satisfying:

1. $V|_{\mathcal{H}} = \text{id}^{[\mathcal{P}:\mathcal{H}]}$ (the $[\mathcal{P}:\mathcal{H}]$ -th power map on $\mathcal{H}$)
2. $\mu_{\mathcal{H}}(V(g)) \geq \mu_{\mathcal{P}}(g)^{[\mathcal{P}:\mathcal{H}]}$ for all $g \in \mathcal{P}$
3. $\mathcal{Q}_{\mathcal{H}}(V(g_1), V(g_2)) \geq \mathcal{Q}_{\mathcal{P}}(g_1, g_2)$ for all $g_1, g_2 \in \mathcal{P}$

Proof:

Step 1: V preserves operations. For $g_1, g_2 \in \mathcal{P}$:

$$V(g_1 * g_2) = \prod_i r_i * (g_1 * g_2) * r_i^{-1} = \prod_i (r_i * g_1 * r_i^{-1}) * (r_i * g_2 * r_i^{-1})$$

By rearranging using the fact that conjugates of $\mathcal{H}$ -elements remain in $\mathcal{H}$:

$$V(g_1 * g_2) = V(g_1) * V(g_2)$$

Step 2: Membership preservation. Each conjugate $r_i * g * r_i^{-1}$ has membership at least $\mu_{\mathcal{P}}(g)$ in the fuzzy sense. Since $\mathcal{H}$ is a fuzzy subgroup:

$$\mu_{\mathcal{H}}(V(g)) = \mu_{\mathcal{H}}(\prod_i r_i * g * r_i^{-1}) \geq \min\{\mu_{\mathcal{P}}(r_i * g * r_i^{-1}) : i\} \geq \mu_{\mathcal{P}}(g)$$

Step 3: Relation preservation. For the fuzzy relation:

$$\mathcal{Q}_{\mathcal{H}}(V(g_1), V(g_2)) \geq \min\{\mathcal{Q}_{\mathcal{P}}(r_i * g_1 * r_i^{-1}, r_i * g_2 * r_i^{-1}) : i\} \geq \mathcal{Q}_{\mathcal{P}}(g_1, g_2)$$

Step 4: Restriction to $\mathcal{H}$. For $h \in \mathcal{H}$, in each coset $r_i \mathcal{H}$, the element $r_i * h * r_i^{-1}$ lies in the same coset as r_i . After reordering: $V(h) = h^{[\mathcal{P}:\mathcal{H}]}$.

3.1 Lattices of Fuzzy Groups in Fuzzy Approximation Spaces

This subsection studies the lattice structure of fuzzy groups in fuzzy approximation spaces, investigating whether the collection of all fuzzy subgroups forms a lattice under natural ordering. In the context of fuzzy groups operating

within approximation spaces, this investigation reveals both the possibilities and fundamental limitations of extending classical lattice-theoretic results to uncertain algebraic environments.

3.1.1 Partial Order Structure and Basic Framework

Let $(\mathcal{P}, *, Q)$ be a fuzzy approximation space where $\mathcal{P}$ is a group with operation $*$ and $Q: \mathcal{P} \times \mathcal{P} \rightarrow [0, 1]$ is a fuzzy relation satisfying reflexivity and symmetry. We denote by $\mathcal{F}(\mathcal{P}, Q)$ the collection of all fuzzy subgroups of $\mathcal{P}$ in the fuzzy approximation space $(\mathcal{P}, Q)$.

Definition 3.7: For fuzzy subgroups $\mu_1, \mu_2 \in \mathcal{F}(\mathcal{P}, Q)$, we define a partial order by:

$$\mu_1 \leq \mu_2 \text{ if and only if } \mu_1(r) \leq \mu_2(r) \text{ for all } r \in \mathcal{P}$$

Lemma 3.8: The relation $\leq$ defines a partial order on $\mathcal{F}(\mathcal{G}, \mathcal{R})$.

Proof: We verify the three properties:

- **Reflexivity:** For any $\mu \in \mathcal{F}(\mathcal{P}, Q)$, we have $\mu(r) \leq \mu(r)$ for all $r \in \mathcal{P}$, hence $\mu \leq \mu$.
- **Antisymmetry:** If $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_1$, then $\mu_1(r) \leq \mu_2(r)$ and $\mu_2(r) \leq \mu_1(r)$ for all $r \in \mathcal{P}$, implying $\mu_1(r) = \mu_2(r)$ for all r , hence $\mu_1 = \mu_2$.
- **Transitivity:** If $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_3$, then $\mu_1(r) \leq \mu_2(r) \leq \mu_3(r)$ for all $r \in \mathcal{P}$, hence $\mu_1 \leq \mu_3$.

3.1.2 Existence of Greatest Lower Bounds

The fundamental structure theorem for meets in fuzzy approximation spaces establishes that intersections of fuzzy subgroups remain fuzzy subgroups.

Theorem 3.9: (Existence of Meet). Let $(\mathcal{P}, *, Q)$ be a fuzzy approximation space and let $\mu_1, \mu_2 \in \mathcal{F}(\mathcal{P}, Q)$. Define $\mu: \mathcal{P} \rightarrow [0, 1]$ by

$$\mu(r) = \min\{\mu_1(r), \mu_2(r)\}$$

for all $r \in \mathcal{P}$. Then:

1. $\mu \in \mathcal{F}(\mathcal{P}, Q)$
2. $\mu = \inf\{\mu_1, \mu_2\}$ in the partial order $(\mathcal{F}(\mathcal{P}, Q), \leq)$

Proof: We must show that μ satisfies the fuzzy subgroup condition in the approximation space. From Theorem 4.3, we need to show that for all $r, s \in \mathcal{P}$, there exists $t \in \mathcal{P}$ such that:

$$Q(r * s^{-1}, t) \geq \min\{\mu(r), \mu(s), \mu(t)\}$$

Since $\mu_1, \mu_2 \in \mathcal{F}(\mathcal{P}, Q)$, there exist $t_1, t_2 \in \mathcal{P}$ such that:

$$Q(r * s^{-1}, t_1) \geq \min\{\mu_1(r), \mu_1(s), \mu_1(t_1)\} \quad (3)$$

$$Q(r * s^{-1}, t_2) \geq \min\{\mu_2(r), \mu_2(s), \mu_2(t_2)\} \quad (4)$$

Choose t optimally by:

$$t = \begin{cases} t_1 & \text{if } \mu(t_1) \geq \mu(t_2) \\ t_2 & \text{otherwise} \end{cases}$$

Then $\mu(t) = \min\{\mu_1(t), \mu_2(t)\} \geq \min\{\mu(t_1), \mu(t_2)\}$, and the required inequality follows.

For the infimum property, clearly $\mu \leq \mu_1$ and $\mu \leq \mu_2$. If $v \in \mathcal{F}(\mathcal{P}, Q)$ with $v \leq \mu_1$ and $v \leq \mu_2$, then $v(r) \leq \min\{\mu_1(r), \mu_2(r)\} = \mu(r)$ for all $r \in \mathcal{P}$, hence $v \leq \mu$.

Example 3.10: Let $\mathcal{P} = S_3$ be the symmetric group on 3 elements with the fuzzy relation:

$$Q((g_1, g_2)) = \begin{cases} 1 & \text{if } g_1 = g_2 \\ 0.8 & \text{if } g_1 g_2^{-1} \in A_3 \setminus \{e\} \\ 0.4 & \text{otherwise} \end{cases}$$

Define two fuzzy subgroups:

$$\mu_1: \mu_1(e) = 1, \mu_1((123)) = 0.9, \mu_1((132)) = 0.9 \quad (5)$$

$$\mu_1((12)) = 0, \mu_1((13)) = 0, \mu_1((23)) = 0 \quad (6)$$

$$\mu_2: \mu_2(e) = 1, \mu_2((12)) = 0.7 \quad (7)$$

$$\mu_2(g) = 0 \text{ for other } g \in S_3 \quad (8)$$

Their intersection $\mu = \mu_1 \wedge \mu_2$ is:

$$\mu(r) = \begin{cases} 1 & \text{if } r = e \\ 0 & \text{otherwise} \end{cases}$$

This is the trivial fuzzy subgroup, which clearly satisfies the approximation conditions.

Corollary 3.11: *The collection $\mathcal{F}(\mathcal{P}, Q)$ forms a complete meet-semilattice. That is, every non-empty subset $\{\mu_i\}_{i \in I} \subseteq \mathcal{F}(\mathcal{P}, Q)$ has an infimum given by*

$$\mu(r) = \inf_{i \in I} \{\mu_i(r)\}$$

3.1.3 Non-existence of Least Upper Bounds

The critical limitation in the lattice structure arises from the failure of supremum operations to preserve the fuzzy subgroup property within approximation spaces.

Theorem 3.12: (Non-existence of Lattice Structure). *The collection $\mathcal{F}(\mathcal{P}, Q)$ does not necessarily form a lattice. Specifically, there exist fuzzy approximation spaces $(\mathcal{P}, *, Q)$ and fuzzy subgroups $\mu_1, \mu_2 \in \mathcal{F}(\mathcal{P}, Q)$ such that their supremum does not exist in $\mathcal{F}(\mathcal{P}, Q)$.*

Proof: We construct an explicit counter example using the Klein four-group:

Let $\mathcal{P} = \{e, \alpha, \beta, \gamma\}$ with the group structure:

- e is the identity
- $\alpha^2 = \beta^2 = \gamma^2 = e$
- $\alpha\beta = \beta\alpha = \gamma, \alpha\gamma = \gamma\alpha = \beta, \beta\gamma = \gamma\beta = \alpha$

Define the fuzzy relation $Q: \mathcal{P} \times \mathcal{P} \rightarrow [0,1]$ by:

$$Q(r, s) = \begin{cases} 1 & \text{if } r = s \\ 0.3 & \text{if } r \neq s \end{cases}$$

Define fuzzy subgroups:

$$\mu_1 : \mu_1(e) = 1, \mu_1(k^2) = 0.7, \text{ others} = 0 \quad (9)$$

$$\mu_2 : \mu_2(e) = 1, \mu_2(n) = 0.7, \text{ others} = 0 \quad (10)$$

The pointwise maximum gives non-zero values to both k^2 and n , but their product nk^2 would need membership at least 0.7 for closure. However, the approximation constraints with $\mathcal{R}(nk^2, t) = 0.6$ or 0.2 for most t create inconsistencies preventing the existence of a supremum.

4. Conclusion

The study highlights the non-classical lattice behavior of fuzzy subgroup structures, preserves essential meet-semilattice properties, and establishes homomorphism results that support further algebraic and categorical developments in fuzzy approximation spaces.

Conflict of interest: The authors declare that they have no conflict of interest.

References

- [1] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353 (1965), doi: 10.1016/S0019-9958(65)90241-X.
- [2] A. Rosenfeld, "Fuzzy groups," *J. Math. Anal. Appl.*, vol. 35, no. 3, pp. 512–517 (1971), doi: 10.1016/0022-247X(71)90199-5.
- [3] J. M. Anthony and H. Sherwood, "Fuzzy subgroups redefined," *J. Math. Anal. Appl.*, vol. 69, no. 1, pp. 124–130 (1979), doi: 10.1016/0022-247X(79)90182-3.
- [4] P. Bhattacharya, "Fuzzy subgroups: Some characterization," *J. Math. Anal. Appl.*, vol. 128, no. 1, pp. 241–252 (1987), doi: 10.1016/0022-247X(87)90228-9.
- [5] J. Močkoř, "Unification of methods for new types of fuzzy sets: General approximation spaces with relational morphisms," *Soft Comput.*, vol. 29, no. 2, pp. 521–542 (2025), doi: 10.1007/s00500-025-10440-2.

- [6] E. Hosseinpour, "T-rough fuzzy subgroups of groups," *J. Math. Comput. Sci.*, vol. 12, no. 3, pp. 186–195 (2014), doi: 10.22436/jmcs.012.03.02.
- [7] N. Bağirmaz, "Fuzzy rough subgroups on approximation space," *J. Adv. Appl. Comput. Math.*, vol. 10, pp. 65–70 (2023), doi: 10.15377/2409-5761.2023.10.6.
- [8] M. Wolski, "Perception and classification: A note on near sets and rough sets," *Fundam. Inform.*, vol. 101, nos. 1–2, pp. 143–155 (2010), doi: 10.3233/FI-2010-281.
- [9] Pankaj, R. Singh, and R. V. N. Srinivas Rao, "Proximity structures on hesitant fuzzy sets and its application," *Sci. Rep.*, vol. 16, Art. no. 16363 (2026), doi: 10.1038/s41598-026-44045-y.
- [10] R. Singh, K. Alam, and N. Bhardwaj, "Graph-based recommendation algorithm for personalized suggestions," *J. Discrete Math. Sci. Cryptogr.*, vol. 28, no. 3, pp. 931–944 (2025), doi: 10.47974/JDMSC-2204.
- [11] A. Ullah, M. Ibrahim, and T. Saeed, "Fuzzy cosets in AG-groups," *AIMS Math.*, vol. 7, no. 3, pp. 3321–3344 (2022), doi: 10.3934/math.2022185.
- [12] M. Choudhary and S. Biswas, "Fuzzy group and fuzzy subgroup," *Int. J. Eng. Sci. Math.*, vol. 8, pp. 35–41 (2019).
- [13] K. D. Ahmad, M. Bal, and M. Aswad, "The kernel of fuzzy and anti-fuzzy groups," *J. Neutrosophic Fuzzy Syst.*, vol. 1, no. 1, pp. 48–54 (2021), doi: 10.54216/JNFS.010105.
- [14] H. Herrlich, "A concept of nearness," *Gen. Topol. Appl.*, vol. 4, no. 3, pp. 191–212 (1974), doi: 10.1016/0016-660X(74)90021-X.
- [15] A. Khandelwal and A. Kumar, "Exploring fuzzy assignment dynamics: A computational journey with 'R'," *J. Inf. Optim. Sci.*, vol. 46, no. 3, pp. 815–829 (2025), doi: 10.47974/JIOS-1835.
- [16] L. Wang, X. Liu, and W. Qiu, "Nearness approximation space based on axiomatic fuzzy sets," *Int. J. Approx. Reason.*, vol. 53, no. 2, pp. 200–211 (2012), doi: 10.1016/j.ijar.2011.10.001.
- [17] N. Preethi and G. K. Revathi, "A view on Pythagorean fuzzy digital feeble topological spaces, nets and prefilters," *J. Interdiscip. Math.*, vol. 28, no. 4, pp. 1607–1622 (2025), doi: 10.47974/JIM-2155.

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