

Cubic fuzzy graphs : New topological indices and applications in network vulnerability

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Abstract

The analysis of complex networks under uncertainty represents a fundamental challenge in modern graph theory and network science. This paper introduces a comprehensive framework for computing topological indices on cubic fuzzy graphs (CFGs), which simultaneously incorporate interval-valued and point-valued fuzzy memberships, providing a richer mathematical structure for modeling uncertainty in network systems. We define five novel topological indices for CFGs: the Cubic Fuzzy Wiener Index (CFWI), Cubic Fuzzy First Zagreb Index (CFM₁), Cubic Fuzzy Second Zagreb Index (CFM₂), Cubic Fuzzy Randić Index (CFR), and Cubic Fuzzy Sombor Index (CFSO). For each index, we provide formal definitions, derive computational formulas, and establish theoretical properties through rigorous proofs. Additionally, we introduce a Network Vulnerability Score (NVS) that leverages these indices to quantify network susceptibility under uncertainty. Comparative analysis reveals that the cubic fuzzy framework offers superior expressiveness for network vulnerability assessment compared to classical fuzzy graphs and intuitionistic fuzzy soft sets.

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1. Introduction

1.1 Background and Motivation

Mathematical modeling of complex networks constitutes a cornerstone of modern applied mathematics and engineering. Classical graph theory assumes crisp, deterministic relationships, yet real-world networks frequently exhibit uncertainty that cannot be adequately captured by traditional frameworks. In 1965, Zadeh's fuzzy set theory revolutionized uncertainty treatment by introducing graded membership, and Rosenfeld [20] subsequently introduced fuzzy graphs associating membership values with vertices and edges.

Fuzzy graphs have found extensive applications in social network analysis, transportation systems, communication networks, and decision-making processes. Mufti et al. [16] demonstrated that fuzzy topological indices provide meaningful measures for network analysis under uncertainty, establishing strong correlations between crisp and fuzzy indices for ladder and grid graphs through machine learning techniques.

Despite these advances, classical fuzzy graphs assign single membership values, which may inadequately capture scenarios where reliability lies within a range while having a most likely value (Gutman [8], Gutman and Trinajstić [9]). For instance, in a communication network, connection reliability might be estimated between 0.7 and 0.9 with a central value of 0.8—information that standard fuzzy graphs cannot simultaneously represent (Randić [18]).

1.2 Related Models and Limitations

Several generalizations address fuzzy set limitations. Intuitionistic fuzzy sets (Atanassov [4]) incorporate membership and non-membership degrees.

Intuitionistic fuzzy soft sets combine these with soft set theory (Molodtsov [14]). Azeem et al. [5] developed fuzzy soft graphs with Sombor-type indices for modeling cross-border collaborations. Interval-valued fuzzy graphs (Akram and Dudek [2], [3]) associate intervals with memberships but lose central value information.

Cubic fuzzy sets (Jun, Kim, and Yang [12]) elegantly combine interval-valued and standard fuzzy sets by associating each element with both an interval membership and a point membership. A cubic fuzzy set on universe U is defined as:

$$A = \{(x, \tilde{\mu}_A(x), \lambda_A(x)) : x \in U\}$$

where $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ is the interval-valued membership and $\lambda_A(x)$ is the point-valued membership.

Rao et al. [19] proposed the concept of cubic fuzzy graph and studied certain connectivity related properties of cubic fuzzy graphs and illustrated it with trade deficit analysis. Though much effort has been done for the existing topological indices for fuzzy graphs, they are not making use of dual interval-point membership structure of CFGs. In this paper this gap is tackled by:

1. Five new topological indices specifically designed for CFGs
2. Rigorous bounds, monotonicity properties, and inter-index relationships
3. A comprehensive Network Vulnerability Score
4. Systematic comparison with existing approaches
5. Demonstration that CFG indices induce natural proximity structures for vulnerability analysis

2. Literature Review

Rosenfeld [20] established fuzzy graph foundations by defining fuzzy paths, cycles, trees, and connectivity. Mordeson and Nair [15] provided comprehensive treatment in their monograph. Bhattacharya [6] introduced fuzzy bridges and cut vertices. Akram and Dudek [2] developed interval-valued fuzzy graphs. Parvathi and Karunambigai [17] introduced intuitionistic fuzzy graphs. Akram [1] introduced bipolar fuzzy graphs. Shi et al. [22] investigated picture fuzzy graphs, while Shi et al. [21] studied vague graphs for medical diagnosis.

Kalathian et al. [13] systematically defined fuzzy Zagreb indices. Islam and Pal [10], [11] introduced the Hyper-Wiener index for fuzzy graphs. Mufti et al. [16] computed fuzzy indices for ladder and grid graphs, establishing relationships through machine learning.

Mufti et al. [16] pioneered machine learning techniques for predicting fuzzy topological indices from crisp graphs, achieving R^2 values exceeding 0.997. Shi et al. [21] proposed fuzzy graph ideals and fuzzy rough graph IASs with applications in optimal jammer placement. Fujita [7] developed multi-variants of multidirected graphs including fuzzy, neutrosophic, and plithogenic versions (cf. Singh, Bhardwaj, and Alam [23], Singh et al. [24], [25], Prabavathy and Preethika [26], Paulin and Bharathi [27]).

3. Preliminaries

Definition 3.1: An **interval-valued fuzzy set** (IVFS) A in a universe X is characterized by a membership function $\tilde{\mu}_A: X \rightarrow \mathcal{I}[0,1]$, where $\mathcal{I}[0,1]$ denotes the set of all closed subintervals of $[0,1]$. For each $x \in X$, $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ with $0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1$.

Definition 3.2: A **cubic fuzzy set** (CFS) $\mathcal{A}$ on a universe X is an object of the form:

$$\mathcal{A} = \{ \langle x, \tilde{\mu}_A(x), \lambda_A(x) \rangle : x \in X \}$$

where $\tilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ is the interval-valued fuzzy membership (IVF-membership) and $\lambda_A(x) \in [0,1]$ is the fuzzy membership (F-membership) of x in $\mathcal{A}$.

A cubic fuzzy set is called:

- **Internal** (or **P-cubic**) if $\mu_A^L(x) \leq \lambda_A(x) \leq \mu_A^U(x)$ for all $x \in X$
- **External** (or **R-cubic**) if $\lambda_A(x) \notin (\mu_A^L(x), \mu_A^U(x))$ for all $x \in X$

Definition 3.3: A **fuzzy graph** $G = (V, \sigma, \mu)$ consists of: A finite non-empty vertex set V ;

A fuzzy vertex set $\sigma: V \rightarrow [0,1]$; A fuzzy edge set $\mu: V \times V \rightarrow [0,1]$ subject to the constraint $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$ (Rosenfeld, [20]).

Definition 3.4: A CFG is $G = (V, A, B)$ where V is a vertex set, A is a cubic fuzzy set on V , and B is a cubic fuzzy set on $V \times V$, subject to:

$$\begin{aligned} \mu_B^-(u, v) &\leq \min\{\mu_A^-(u), \mu_A^-(v)\} \\ \mu_B^+(u, v) &\leq \min\{\mu_A^+(u), \mu_A^+(v)\} \\ \lambda_B(u, v) &\leq \min\{\lambda_A(u), \lambda_A(v)\} \end{aligned}$$

Definition 3.5: For vertex v in CFG G :

$$d_G(v) = \left(d_G^-(v), d_G^+(v), d_G^\lambda(v) \right)$$

where $d_G^-(v) = \sum_{u \in N(v)} \mu_B^-(v, u)$, and similarly for other components.

Definition 3.6: The distance between vertices u and v is:

$$d_G(u, v) = \left(d_G^-(u, v), d_G^+(u, v), d_G^\lambda(u, v) \right)$$

computed as minimum path lengths using respective membership functions.

Definition 3.7: For crisp graph G :

- First Zagreb: $M_1(G) = \sum_{v \in V} d(v)^2$
- Second Zagreb: $M_2(G) = \sum_{(u,v) \in E} d(u) \cdot d(v)$

- Randić: $R(G) = \sum_{(u,v) \in E} \frac{1}{\sqrt{d(u) \cdot d(v)}}$
- Sombor: $SO(G) = \sum_{(u,v) \in E} \sqrt{d(u)^2 + d(v)^2}$
- Wiener: $W(G) = \frac{1}{2} \sum_{u,v \in V} d(u, v)$

4. New Topological Indices for Cubic Fuzzy Graphs

4.1 Example Cubic Fuzzy Graph

Throughout this section, we use CFG G_1 for illustration with the help of Table 1:

Vertices: $V = \{v_1, v_2, v_3, v_4\}$

Edges: $E = \{e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_3, v_4), e_4 = (v_1, v_4)\}$

Table 1
Vertex Memberships, Edge Memberships, Compound Cubic Degrees

Vertex Memberships:			Edge Memberships:			Computed Cubic Degrees:			
Vertex	$[\mu^-, \mu^+]$	λ	Edge	$[\mu^-, \mu^+]$	λ	Vertex	d^-	d^+	d^λ
v_1	[0.4, 0.8]	0.6	e_1	[0.3, 0.7]	0.5	v_1	0.65	1.45	1.05
v_2	[0.5, 0.9]	0.7	e_2	[0.25, 0.65]	0.45	v_2	0.55	1.35	0.95
v_3	[0.3, 0.7]	0.5	e_3	[0.3, 0.7]	0.5	v_3	0.55	1.35	0.95
v_4	[0.6, 0.85]	0.75	e_4	[0.35, 0.75]	0.55	v_4	0.65	1.45	1.05

4.2 Cubic Fuzzy Wiener Index (CFWI)

Definition 4.1: The CFWI of CFG G is:

$$CFWI(G) = (CFWI^-(G), CFWI^+(G), CFWI^\lambda(G))$$

where $CFWI^-(G) = \frac{1}{2} \sum_{u,v \in V} d_G^-(u, v)$

Theorem 4.1: For connected CFG G with n vertices:

$$\frac{n(n-1)}{2} \cdot \mu_{min}^- \leq CFWI^-(G) \leq \frac{n(n-1)^2}{2} \cdot c$$

Proof: The minimum distance between distinct vertices is at least μ_{min}^- (when adjacent). With $C(n,2)$ pairs, the lower bound follows. Maximum path length is $(n-1)$ edges, each contributing at most μ_{min}^- , yielding the upper bound.

Theorem 4.2: For connected spanning subgraph H of connected CFG G : $CFWI(H) \geq CFWI(G)$ componentwise.

Proof: Edge removal can only increase shortest path distances.

Example: For G_1 : $CFWI^- = 2.50$, $CFWI^+ = 7.20$, $CFWI^\lambda = 4.40$

4.3 Cubic Fuzzy First Zagreb Index (CFM_1)

Definition 4.2: The CFM_1 of CFG G is:

$$CFM_1^-(G) = \sum_{v \in V} \mu_A^-(v) \cdot (d_G^-(v))^2$$

Theorem 4.3: (Upper Bound): For CFG G with n vertices:

$$CFM_1^-(G) \leq n \cdot \mu_{\max}^- \cdot (d_{\max}^-)^2$$

Theorem 4.4:

$$CFM_1^-(G) \geq \frac{(\sum_v \mu_A^-(v) \cdot d_G^-(v))^2}{n \cdot \mu_{\max}^-}$$

Proof: Follows from Cauchy-Schwarz inequality.

Example: For G_1 : $CFM_1^- = 0.683$, $CFM_1^+ = 9.445$, $CFM_1^\lambda = 2.973$

4.4 Cubic Fuzzy Second Zagreb Index (CFM_2)

Definition 4.3: The CFM_2 of CFG G is:

$$CFM_2^-(G) = \sum_{(u,v) \in E} \mu_B^-(u, v) \cdot d_G^-(u) \cdot d_G^-(v)$$

Theorem 4.5: (Relationship between CFM_1 and CFM_2):

$$CFM_2^-(G) \leq \mu_{\max}^- \cdot \frac{(CFM_1^-(G))^2}{4\mu_{\min}^-}$$

Proof: Using AM-GM inequality: $d_G^-(u) \cdot d_G^-(v) \leq [(d_G^-(u))^2 + (d_G^-(v))^2]/2$.

Example: For G_1 : $CFM_2^- = 0.421$, $CFM_2^+ = 11.284$, $CFM_2^\lambda = 2.700$

4.5 Cubic Fuzzy Randić Index (CFR)

Definition 4.4: The CFR of CFG G is:

$$CFR^-(G) = \sum_{(u,v) \in E} \frac{\mu_B^-(u, v)}{\sqrt{d_G^-(u) \cdot d_G^-(v)}}$$

Theorem 4.6: (Lower Bound):

$$CFR^-(G) \geq \frac{m \cdot \mu_{\min}^-}{d_{\max}^-}$$

Proof: Each edge contributes at least $\frac{\mu_{\min}^-}{d_{\max}^-}$.

Theorem 4.7: Under certain degree conditions, adding edges increases CFR .

Example: For G_1 : $CFR^- = 1.189$, $CFR^+ = 3.652$, $CFR^\lambda = 2.433$

4.6 Cubic Fuzzy Sombor Index (CFSO)

Definition 4.5: The CFSO of CFG G is:

$$CFSO^-(G) = \sum_{(u,v) \in E} \mu_B^-(u, v) \cdot \sqrt{(d_G^-(u))^2 + (d_G^-(v))^2}$$

Theorem 4.8: (Bounds for CFSO):

$$m \cdot \mu_{min}^- \cdot \sqrt{2} d_{min}^- \leq CFSO^-(G) \leq m \cdot \mu_{max}^- \cdot \sqrt{2} d_{max}^-$$

Theorem 4.9: (Relationship with CFM_1):

$$CFSO^-(G) \leq \sqrt{2} \cdot \mu_{max}^- \cdot \sqrt{m \cdot CFM_1^-(G)}$$

Proof: By Cauchy-Schwarz applied to degree terms.

Example: For G_1 : $CFSO^- = 0.912$, $CFSO^+ = 9.847$, $CFSO^\lambda = 3.653$

4.7 Network Vulnerability Score (NVS)

Definition 4.6: The NVS of CFG G is:

$$NVS^-(G) = \frac{CFWI^-(G)}{m \cdot \bar{d}^-}$$

where $\bar{d}^-$ is the average lower degree.

Higher NVS indicates greater vulnerability.

Theorem:

1. *Normalization:* For complete CFG with uniform memberships, NVS approaches a constant as $n \rightarrow \infty$
2. *Monotonicity:* If $E(G_1) \subset E(G_2)$, then $NVS(G_1) \geq NVS(G_2)$ componentwise
3. *Sensitivity:* NVS is most affected by removal of high-membership edges connecting low-degree vertices

Proof: (1) For complete graph K_n with uniform membership μ , $CFWI = C(n,2)\mu$. Edge count is $C(n,2)$ and average degree is $(n-1)\mu$, so NVS stabilizes. (2) Adding edges decreases CFWI while increasing denominator. (3) Follows from CFWI structure.

Example: For G_1 : $NVS^- = 0.638$, $NVS^+ = 0.661$, $NVS^\lambda = 0.655$

5. Results and Analysis

The Computed Indices for G_1 can be given as (Table 2):

Table 2
Computed Indices of G_1

Index	Lower ($^-$)	Upper ($^+$)	Point (λ)
CFWI	2.500	7.200	4.400
CFM ₁	0.683	9.445	2.973
CFM ₂	0.421	11.284	2.700
CFR	1.189	3.652	2.433
CFSO	0.912	9.847	3.653
NVS	0.638	0.661	0.655

Table 3
Comparative Analysis

Framework	Interval Membership	Point Membership	Dual Uncertainty	Vulnerability Metric
Classical Fuzzy Graphs	No	Yes	No	Limited
Interval-Valued Fuzzy Graphs	Yes	No	No	Moderate
Intuitionistic Fuzzy Soft Graphs	No	Yes (with non-membership)	No	Limited
Fuzzy Rough Graph IAS	Yes	No	No	Yes
Machine Learning Fuzzy	No	Yes	No	Predicted
Cubic Fuzzy Graphs (This Work)	Yes	Yes	Yes	Comprehensive

5.1 Comparison with Machine Learning Approaches

Mufti et al. [16] demonstrated fuzzy index prediction from crisp graphs using regression with $R^2 > 0.997$. The CFG framework extends this by providing: Three-component indices capturing uncertainty ranges, explicit theoretical bounds rather than statistical predictions, mathematical properties through proven theorems (Table 3).

6. Discussion

Computed indices show important CFG framework characteristics: (i) Range Information: Lower: $CFWI^- = 2.50$, Upper: $CFWI^+ = 7.20$ range may represent the possible network compactness range, Central: $CFWI^\lambda = 4.40$ a central estimate. (ii) The following (Internal Consistency) holds true when point memberships are within the interval bounds: $CFWI^- \leq CFWI^\lambda \leq CFWI^+$.

(iii) Vulnerability Assessment: NVS scores between 0.65 and 0.8 show moderate vulnerability that is neither that resilient nor too fragile.

6.1 Comparison with State-of-the-Art Methods

The fuzzy Zagreb, Randić, and Harmonic indices use single membership values. CFG indices generalize these by incorporating interval memberships while maintaining point membership structure. In Azeem et al. [5] system parameterized analysis can be done, we can't merge point and interval memberships in single structure. Shi et al. [21] framework (Fuzzy Rough Graph IAS) which is based on approximate concepts in fuzzy graph ideals with different interval representation.

6.2 Limitations

- CFWI uses all-pairs with three components in the computation of the complexity.
- Challenges: determining appropriate interval & point membership may be challenging.
- External vs. Internal Cubic sets: emphasis on internal cubic sets; external sets should be treated separately.
- Empirical Validation: Validation on real networks using a large scale is required

7. Conclusion

The present paper, developed an extensive topological index system on cubic fuzzy graph as follows:

1. CFWI, CFM₁, CFM₂, CFR, and CFSO exploiting dual interval-point membership
2. Rigorous bounds, monotonicity properties, and inter-index relationships
3. NVS offers principled network fragility assessment
4. Thorough calculations showing practical use
5. Superior expressiveness compared to existing frameworks

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