

Alternative rings with potent and nilpotent elements

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Abstract

In this paper we take an alternative ring R with certain conditions on the nilpotent elements and potent elements of R . We prove if $[ab, ba]$ is potent $\forall a \in R \setminus N(R), b \in R \setminus N(R)$ then all idempotents of R are contained in the center C of R . Using this we derived if x in $R \setminus N(R)$ is of the form $x = n + p$, where n and p are nilpotent and potent, then R is commutative.

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Introduction

“The center C is defined as $C = \{c \in N/[c, R] = 0\}$, an element x is potent if there exists $k > 1$ such that $x^k = x$, an element x of a ring R is called nilpotent if there exists some positive integer n such that $x^n = 0$, the nucleus N of a ring R , the set of all elements n in R such that $(n, R, R) = (R, n, R) = (R, R, n) = 0$ and an alternative ring R is a ring having

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$(aa)b = a(ab), b(aa) = (ba)a, \forall a, b \text{ in } R$ are known as the left and right alternative laws”.

[1, 2] gives some results on semiprimality and nilpotency of nonassociative rings. Jacobson’s theorem [4] states a ring R taking the potent $\forall a \text{ in } R, \exists$ an integer potent $k(x) > 1$ such that $a^{k(x)} = a$ is commutative. Hasanali and Yacub [3] generalized this theorem by imposing some weaker conditions on an associative ring. In this paper we consider an alternative ring R with some conditions on the nilpotent elements and potent elements of R . We show that if $[ab, ba]$ is potent $\forall a \in R \setminus N(R), b \in R \setminus N(R)$, then all idempotents of R are having in center C of R . Using this we prove that if x in $R \setminus C$ is of nature $x = n + p$, where n is nilpotent and p is potent, then R is commutative. [5], Zorn [6], and Behn, Correa, and Hentzel [1] have explained about the alternative rings and its proprieties of nonassociative rings. [7, 8] states some results on commutative rings.

In this paper, R represents an alternative ring C as center and $N(R)$ is set of R ’s nilpotent members, and P is the set of its potent element’s. $P = \{x/x \in R, a^{k(x)} = a \text{ for some } k(x) > 1\}$

We first establish the below results:

Lemma 1.1: Consider R an alternative Ring holds

- (i) $\forall x \text{ in } R \setminus C$ can be expressed as $x = n + p$ for certain n in $N(R)$ and certain p in P and
- (ii) $[ab, ba]$ is potent $\forall a \in R \setminus N(R), b \in R \setminus N(R)$, implies R is a commutative, then
- (i) $C(R) \subseteq J \subseteq N(R) \cup C$ and s(ii) $N(R) \subseteq J$

Proof of (i): In [3], $C(R) \hat{=} J$. It is enough to prove

$$J \subseteq N \cup C \quad (1)$$

Consider $x \in J$ and $x \notin C$, by statement (i) of lemma,

$$x = n + p, n \in N, p^r = p \text{ for certain } r > 1 \quad (2)$$

$n \in N$ take $n^q = 0$, since $r \geq 2, x \in J$

we know

$$(n-a)^{(r-1)q+1} \hat{=} J. \quad (3)$$

Since $p^r = p$, by using again (2) and (3)

$$p = p^r = p^{(r-1)q+1} = (n-a)^{(r-1)q+1} \hat{=} J,$$

and hence p^{r-1} is an idempotent of J . Therefore $p^{r-1} = 0$, and thus $p = p^r = 0$.

Using (2), $x = n \hat{=} N(R)$, which proves (i) and completed the result (i).

Proof of (ii): Let $\bar{R} = R/C(R)$ and $\bar{n} = n + C(R)$. Consider $\bar{n} \hat{=} N(R)$.

implies $\bar{n}^k = 0$ for some +ve integer k . Take x in R .

Since $\bar{R} = R/C(R)$ is commutative, hence $\bar{n} \bar{x}$ is nilpotent in $\bar{R}$, say $(\bar{n}\bar{x})^r = 0$.

Therefore, $(nx)^r \in C(R) \cap N \subseteq C$ (by result (1)).

Thus, $(nx)^r \in N(R)$ or $(nx)^r \in C$. Now, if $(nx)^r \in C$, then

$((nx)^r)^k = (nx)^r (nx)^r \dots (nx)^r = n^k y$ for few y in R , and makes $(nx)^{rk} = 0$ (due to $n^k = 0$). Hence, in every case, $nx \in N(R)$, which means nx is right-quasi-regular $\forall x$ in R , and we have $n \in J$. Completes (ii).

Lemma 1.2: Let R is an alternative ring which satisfies (i) $\forall x$ in $R \setminus C$ expressed in $x = n + p$ for certain n in $N(R)$ and certain p in P and (ii) $[ab, ba]$ is potent $\forall a \in R \setminus N(R), b \in R \setminus N(R)$, then R commutes. Then,

- (1) R has an ideal $N(R)$
- (2) $\forall x \in R$, has $x \in C$ or $x^k - x \in N(R)$ for few integer $k > 1$;
- (3) If $f : R \rightarrow R^\circ$ is a homomorphism from R onto R° , then the set N° of nilpotents of R° is contained in $f(N(R)) \subseteq C^\circ$, where C° itself center of R°

Proof of (1): By Lemma 1.1, we have

$$N(R) \cap J \cap N(R) \subseteq C. \quad (4)$$

Take $d \in N(R)$, $e \in N(R)$. Then $d \in J$, $e \in J$, get

$d - e \in J \cap N(R) \subseteq C$. implies $d - e \in N(R)$ or $d - e \in C$.

Hence, if $d - e \in C$, later $de = ed$ implies $d - e \in N(R)$ (because $d \in N(R)$, $e \in N(R)$). Thus, in any case $d - e \in N(R)$. Next, suppose $d \in N(R)$, $x \in R$. Then $d \in J$, $x \in R$, and hence $dx \in J \subseteq N(R) \subseteq C$.

Hence $dx \in N(R)$ or $dx \in C$. Now, if $dx \in C$, then by induction $(dx)^k = d^k x^k$ $\forall$ non negative integers k , gives $dx \in N(R)$ (because $d \in N(R)$). Hence all cases, $dx \in N(R)$. Same as, $d \in N(R)$, $x \in R$ gives $xd \in N(R)$ thus the result proved.

Proof of (2): Let $x \in R$, $x \notin C$. Then, by statement of lemma (i),

$$x = n + p, n \in N(R), p^k = p, k > 1.$$

$$\text{Since, } x - n = p = p^k = (x - n)^k, n \in N(R), k > 1. \quad (5)$$

By part (1), $N(R)$ is an ideal, also $a \in N(R)$, hence by (5), we get $x^k - x \in N(R)$, $k > 1$.

Proof of (3): Suppose $d^\circ \in N^\circ$, $d^\circ \notin C^\circ$. Then $(d^\circ)^s = 0$ for few non negative integer s . Let $t \in R$ be such that $f(t) = t^\circ$. Since $t^\circ \in C^\circ$, therefore $t \in C$, and by part (2),

$$t - t^r \in N(R) \text{ for few integer } r > 1. \quad (6)$$

It is easy verified that

$$t - t^{s+1} t^{s(r-2)} = (t - t^s) + t^{s-1} (t - t^s) + \dots + (t^{r-1})^{s-1} (t - t^s). \quad (7)$$

Combining (6) and (7), we conclude that

$$t - t^{s+1} t^{s(r-2)} \in N(R), (r \geq 2),$$

and hence $f(t - t^{s+1} t^{s(r-2)}) \in f(N(R))$, which implies $t^\circ \in f(N(R))$, since $(t^\circ)^s = 0$. Thus $t^\circ \in f(N(R)) \subseteq C^\circ$ for all $t^\circ \in N^\circ$, which completes (3).

Lemma 1.3: *Let R an alternative ring holds the requirements $[ab, ba]$ is potent $\forall a \in R \setminus N(R), b \in R \setminus N(R)$, then R commutes. Then, every idempotents of R are exists within the center C of R .*

Proof: Consider $i^2 = i \hat{I} R, x \hat{I} R$, and let $c = ix - ixi$.

We assert that $c = 0$.

Suppose not, consider $g = i + c$. Then

$$g^2 = g, ig = g, gi = i. \quad (8)$$

Furthermore, given that $c \neq 0$ by assumption, it follows that $i \neq 0$ and $g \neq 0$, so by hypothesis,

$[gi, ig]$ is potent; so

$$[gi, ig]^r = [gi, ig] \text{ for some integer } r > 1. \quad (9)$$

It is easily verified that (8) and (9) gives $c = 0$ (because $r > 1$), a contradiction.

This contradiction shows that $c = 0$, and hence $ix = ixi$. Similarly $ix = ixi$.

From [1] we have the following lemma.

Lemma 1.4: *Let an alternative ring R having center C and $N(R)$ its nilpotent members. Consider that (i) $N(R)$ is commutative; (ii) $\forall n \hat{I} N(R)$ and $p \hat{I} R, np - pn$ commutes with p ; (iii) $\forall p \hat{I} R$, we have $p \hat{I} C$ or $p^n - p \hat{I} N(R)$ for some $k = k(p) > 1$. Then R is commutative.*

We are now proving the main theorem.

Theorem 1: *Assume that $N(R)$ is the set having nilpotent elements in R , which is an alternate ring with center C . Let $P = \{x/x \in R, a^{k(x)} = a \text{ for few } k(x) > 1\}$ represent a set having potent elements in R . Suppose that*

- (i) $\forall x$ in $R \setminus C$ can be expressed as $x = n + p$ for certain n in $N(R)$ and certain p in P .
- (ii) $[n, p] = np - pn$ is potent for all n in $N(R)$ and p in $N(R)$.
- (iii) $[ab, ba]$ is potent $\forall x \hat{I} R \setminus N(R), y \hat{I} R \setminus N(R)$, then R is commutative.

Proof: By Lemma 1.2 (1), R 's ideal is $N(R)$, and therefore

$$np - pn \hat{I} N(R) \text{ for } n \hat{I} N(R), p \hat{I} N(R). \quad (10)$$

Additionally, since $np = pn$ is potent according to hypothesis (ii), $np - pn = 0$ (as $np - pn$ is also nilpotent, by (10)).

Hence, all the elements $N(R)$ of nilpotent of R is commutative. (11)

As is widely known,

$R @$ A subdirect sum of rings that are subdirectly irreducible $R_t (t \hat{I} G)$. (12)

Two cases can be distinguished.

Case 1: R_t lacks an identity.

First, observe that R_i satisfies both hypothesis (i), (iii) of theorem 1 and the conclusions of Lemma 1.3. Let $x_i \in R_i$, $x_i \in C_i$ (center of R_i), and let $f(x) = x_i$, $x \in R$, where $f: R \rightarrow R_i$ is the natural homomorphism of R onto R_i (see 12).

Then $x \in C$, and thus by Lemma 1.2(2), $x^n - x \in N(R)$ for some integer $n > 1$. Hence $x^m = x^{m+1}g(x)$ for some positive integer m and some $g(1) \in Z[1]$. This implies, by re-iterating, that

$$x^m = x^m(xg(x))^m = x^m e; e = (xg(x))^m \text{ is idempotent.}$$

Combining this with Lemma 1.3, we conclude that

$$x^m = x^m e, e = e^2, e \text{ central.}$$

Letting $x_i = f(x)$, $e_i = f(e)$, we see that this implies, in R_i ,

$$x_i^m = x_i^m e_i, e_i = e_i^2, e_i \text{ central.} \quad (13)$$

Since C_i is central idempotent in the subdirectly irreducible ring R_i , and since R_i does not have an identity in the present case, therefore $e_i = 0$, and hence by (13), $x_i^m = 0$; that is, x_i is nilpotent in R_i .

We have thus shown that

$$R_i = N_i \oplus C_i; N_i = \{\text{nilpotents of } R_i\}; C_i = \text{center of } R_i. \quad (14)$$

Moreover, by Lemma 1.2(3), $N_i \subseteq f(N(R)) \subseteq C_i$. Also, by (11), $f(N(R))$ is commutative, and hence N_i is commutative. Combining this with (14), we conclude that R_i is commutative.

Case 2: R_i has an identity 1.

We recall that R_i satisfies hypothesis (i), (iii) of theorem 1 and all the conclusions of Lemma 1.2. We also recall that N_i and C_i denote the set of nilpotents and the center of R_i , respectively. Let $a_i \in N_i$, $b_i \in R_i \setminus N_i$. Then, $a_i \in N_i$, $b_i \notin N_i$, and hence by hypothesis (iii)

$$[(1_i + a_i)b_i, b_i(1_i + a_i)] \text{ is potent,}$$

which implies

$$[b_i, b_i a_i] + [a_i b_i, b_i] + [a_i b_i, b_i a_i] \text{ is potent.} \quad (15)$$

We saw in the proof in case 1, N_i is commutative, and N_i is an ideal of R_i , too

(Lemma 1.2 (1)). Therefore, the commutators $[a_i b_i, b_i a_i]$ in (15) is zero, and hence (15) now reduces to $[b_i, b_i a_i] + [a_i b_i, b_i]$ is potent, which is equivalent to

$$[a_i b_i - b_i a_i, b_i] \text{ is potent, } (a_i \in N_i, b_i \notin N_i). \quad (16)$$

Since N_i is commutative, (16) is trivially satisfied if $b_i \in N_i$, and hence

$$[a_i b_i - b_i a_i, b_i] \text{ is potent for all } a_i \in N_i, b_i \in R_i. \quad (17)$$

Note that, since N_i is an ideal of R_i , therefore the potent commutator in (17) is also nilpotent, and hence it must be zero. Thus,

$$a_i b_i - b_i a_i \text{ commutes with } b_i \text{ for all } a_i \in N_i, b_i \in R_i. \quad (18)$$

Combining our results, we see that R_i satisfies all the hypothesis of Lemma 1.4 (see (18), Lemma 1.2(2), and the above proof that N_i is commutative) and

hence by Lemma 1.4, R_i is commutative. This completes the proof of Theorem 1.

As said in the beginning, Theorem 1 generalizes Jacobson $x^{n(x)} = x^n$ theorem.

We end with the following:

Corollary 1.1: R be an alternative ring such that $\forall x \in R \setminus C$, $x = n + p$, where $n \in N(R)$ and $p \in P$.

Assume also that $0 \nabla a, b$ in $[ab, ba] = 0, \forall a, b$ in R and the nilpotent of R commute. Hence R is commutative.

Conclusion

We take an alternative ring R by imposing some certain conditions on the nilpotent elements and potent elements of R . We proved if $[ab, ba]$ is potent $\forall a \in R \setminus N(R), \in R \setminus N(R)$ then all idempotents of R are contained in the center C of R . Using this we derived if x in $R \setminus N(R)$ is of the form $x = n + p$, where n and p are nilpotent and potent, then R is commutative.

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