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Some explorations on odd-even graceful labeling of graphs

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Abstract

Among the vast branches of mathematics, graph theory plays a vital role in applied mathematics and scientific computing. Graph labeling is one of the most prominent areas of graph theory. There exist various types of graphs labeling in the available literature, such as graceful labeling, even graceful labeling, Odd graceful labeling, Odd-Even graceful labeling, and Even-Even graceful labeling etc. The present work deals with a unique kind of graph labeling known as Odd-Even graceful labeling. This paper signifies Odd-Even graceful labeling of some classes of graphs, like coconut tree, comb graph, quadrilateral snake graphs, tristar, and 4-star.

Subject Classification: *05C38, 05C78.*

Keywords: *Odd-even graceful labeling, Coconut tree, Comb graph, Quadrilateral snake, Star graph.*

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1. Introduction

Graph theory deals with networks of connected nodes and edges. It provides a framework for modelling relationships and structures in various fields, including computer science, engineering, social sciences, and biology. It provides powerful tools for solving problems involving networks, paths, and relationships. Graceful labeling links captivating concepts within the realm of combinatorics and graph theory, offering a unique perspective on assigning labels to the vertices of a graph in a harmonious and aesthetically pleasing manner. In this mathematical discipline, the primary objective is to label the vertices of a graph in such a way that the resulting set of edge labels forms a sequence of distinct integers. One of the fundamental aspects of graceful labeling is the idea that the labels assigned to the vertices, when extended to the edges, form a sequence in which the absolute differences of the labels on the edges are unique. It has applications in various fields, including coding theory, communication networks, the design of certain types of puzzles, optimal circuit designs, graph decomposition tasks, the ruler-compass problem, and radio astronomy.

Rosa [8] introduced a focus on graceful graphs and methods for graceful labeling. Earlier, he named graceful labeling as β-valuation. Gallian [4] provided a comprehensive overview and resource on graph labeling research, acknowledging the vast number of papers and the challenge of keeping up with discoveries. Basher [1] demonstrated that the splitting graphs of several standard graphs, including paths, stars, bi-stars, complete bipartite graphs, and cycles are indeed Odd-Even graceful. Nurvazly, Chasanah, and Wiranto [7] explored the concepts of graceful, $\hat{\rho}$ (rho hat), and Odd-Even graceful labeling in graph theory, focusing on specific subgraphs of the millipede graph. Boonklurb, Ruamkaew, and Singhun [3] examined the directed edge-graceful labeling of a specific digraph structure: a graph formed by joining 'c' cycles of the same size 'a' at a single vertex, denoted as C(c × a). Mishra, Behera, and Nayak [6] focused on graceful labeling of a specific type of graph called three distant trees, utilizing a technique known as the component moving operation. Behera, Mishra, and Nayak [2] discussed new classes of graphs that possess Even-Even and Odd-Even graceful labeling. Lau et al. [5] specifically investigated these labeling for graphs with extremal maximum vertex degrees. SenthilKumar, Anbarasan, and Pushparaj [9] proved edge odd graceful labeling for some graphs. Senthilkumar and Venkatesan [10] proved several classes of graphs to prove they are odd vertex even edge root square mean graphs. Sumathi and Tamilselvi [11] determined the modular chromatic number for inflated graphs derived from specific tree-related structures. The present paper explores Odd-Even graceful labeling of various types of graphs like tristar, 4-star, Coconut tree, comb graph and a class of hairy cycle graphs with cycle as C_3.

Definition 1.1: Odd-Even graceful labeling of graph G having q edges assigns unique odd integer $f: V(G) \to \{1, 3, 5, ..., 2q+1\}$, resulting absolute difference between labels of adjacent vertices (forming edges) yields a unique even integer from $\{2, 4, 6, ..., 2q\}$. A graph that possesses an Odd-Even graceful labeling is an Odd-Even graceful graph.

Definition 1.2: A path is an alternate sequence of vertices and edges in which no vertices and no edges are repeated. But for a closed path, the initial and terminal vertices are repeated. P_n represents a path having n vertices.

Definition 1.3: Cycle is a closed path with three or more vertices. The notation C_n refers to a cycle with n vertices.

Definition 1.4: Coconut tree $CT(m, n)$ is formed by joining with n edges with one end of a path P_m called central path.

Definition 1.5: A comb graph $P_n \odot K_1$ is formed by attaching n pendant edges to every vertex of the path P_n.

Definition 1.6: The quadrilateral snake $C_4{}^n \oplus K_{1,n}$ is a type of snake graph which is formed by the adjacent joining of n number of cycles of length 4 with n edges attached at one end of the terminal C_4.

Definition 1.7: The quadrilateral snake $K_{1,n} \oplus C_4{}^n \oplus K_{1,n}$ is a type of snake graph which is formed by the adjacent joining of n number of cycles of length 4 attached to n edges at both ends of the initial and the terminal C_4.

Definition 1.8: The tristar $S_n \oplus S_{n+2} \oplus S_n$ is formed by adjacent joining of three star graphs with n, $n+2$ and n edges respectively, such that every two consecutive star graphs have one edge in common.

Definition 1.9: The 4-star $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ is formed by joining four star graphs having n, $n+2$, $n+2$ and n edges respectively such that every two consecutive star graphs have one edge in common.

Definition 1.10: $C_3 \odot 3K_{1,n}$ is a hairy cycle graph that is created by connecting n edges to every vertex of C_3.

2. Main Results

Theorem 2.1: *Coconut tree $CT(m,n)$ exhibits Odd-Even graceful labeling.*

Proof: Suppose $CT(m,n)$ be coconut tree, having $m+n$ vertices and $m+n-1(=q)$ edges, generated by joining n pendant edges attached to one end of the central path of the coconut tree with m vertices. Let $v_i, i = 1,2,3,...,n$ be pendant vertices joined to the upper end of the central path of the coconut tree, and $u_i, i = 1,2,...,m$ be the vertices on the central path of the coconut tree, starting from top end.

The vertex labeling is

$$f(u_i) = \begin{cases} i \ , & i \text{ is odd} \\ 2m-i+1, & i \text{ is even} \end{cases}, \quad f(v_i) = 2m+2i-1,\ i = 1,2,3,...,n$$

Edge labeling is

$$|f(u_{i+1}) - f(u_i)| = 2m-2i, i = 1,2,...,m-1$$
$$|f(u_1) - f(v_i)| = 2m+2i-2, i = 1,2,...,n$$

Since edge labeling of $CT(m,n)$ ranges from 2 to $2q$, where the labelled set of vertices is $\{1,3,5,...,2q+1\}$. Hence, the graph is Odd-Even graceful.

Illustration 2.1: Theorem 2.1 is explained in Figure 1 as follows for $m = 5$, $n = 7$.

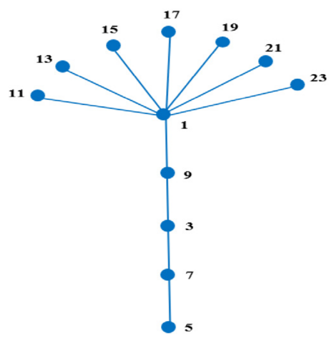

Figure 1

CT **(5, 7) with Odd-Even graceful labeling.**

Theorem 2.2: *Every Comb graph* $P_n \odot K_1$ *is Odd-Even graceful.*

Proof: Suppose $P_n \odot K_1$ be a comb graph having $2n$ vertices and $q = 2n-1$ edges, where n pendant vertices $v_i, i = 1,2,3,\ldots,n$ are attached with each vertex of the path P_n. And $u_i, i = 1,2,3,\ldots,n$ are vertices on path P_n.

The vertex labeling is

$$f(u_i) = \begin{cases} 2i-1, & i \text{ is odd} \\ 4n-2i+1, & i \text{ is even} \end{cases}, \quad f(v_i) = \begin{cases} 4n-2i+1, & i \text{ is odd} \\ 2i-1, & i \text{ is even} \end{cases}$$

And the edge labeling is

$$| f(u_i) - f(u_{i+1}) | = 4n-4i,\ i = 1,2,3,\ldots,n-1$$

$$| f(u_i) - f(v_i) | = 4n-4i+2,\ i = 1,2,3,\ldots,n.$$

Putting all the values of i in the above edge labeling formula, edge labels of comb graph range from 2 to $2q$, where the labelled set of vertices is $\{1,3,5,\ldots,2q+1\}$. Hence, graph becomes Odd-Even graceful.

Illustration 2.2: Theorem 2.2 is explained in Figure 2 as follows for n = 8.

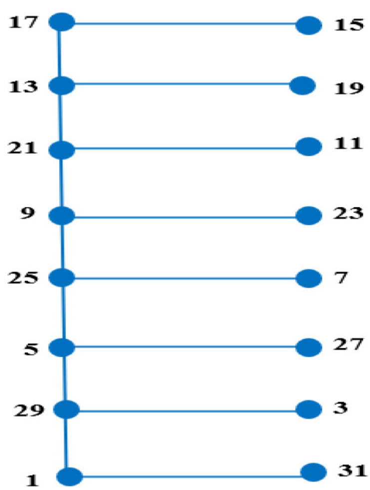

Figure 2

$P_8 \odot K_1$ **with Odd-Even graceful labeling.**

Theorem 2.3: *Quadrilateral snake* $C_4^{\,n} \oplus K_{1,n}$ *becomes Odd-Even graceful for* $n \geq 1$.

Proof: Suppose $C_4^{\,n} \oplus K_{1,n}$ represents a quadrilateral snake formed by joining n number of C_4 side by side with the tail end attached to $K_{1,n}$. Vertices and edges of $C_4^{\,n} \oplus K_{1,n}$ are $4n+1$, $5n(=q)$ respectively. Suppose u_i, v_i, and $w_i, i=1,2,3,\ldots,n$ be vertices at top, middle, and bottom of n adjacent C_4 respectively and $x_i, i=1,2,3,\ldots,n$ be pendant vertices attached at right end cycle of the quadrilateral snake.

Now the vertex labeling is

$$f(u_i)=10n-4i+5, f(w_i)=10n-4i+3,$$
$$f(x_i)=5n+2i-3,\ i=1,2,3,\ldots,n$$
$$f(v_i)=4i-3, i=1,2,3,\ldots,n+1$$

And the edge labeling is

$$f(u_iv_i)=10n-8i+8, i=1,2,3,\ldots,n, f(u_{i-1}v_i)=10n-8i+12, i=2,3,\ldots,n+1,$$
$$f(v_iw_i)=10n-8i+6, i=1,2,3,\ldots,n, f(v_iw_{i-1})=10n-8i+10, i=2,3,\ldots,n+1$$
$$f(v_{n+1}x_i)=2i, i=1,2,\ldots,n$$

By expanding edge labeling, edge labelled set is a set of even integers from 2 to $2q$, where the labelled set of vertices is $\{1,3,5,\ldots,2q+1\}$. Hence, graph is Odd-Even graceful.

Illustration 2.3: Theorem 2.3 is explained in Figure 3 as follows for $n = 4$.

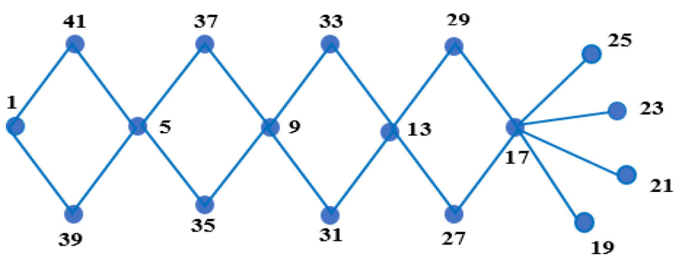

Figure 3

$C_4^{\,4} \oplus K_{1,4}$ **with Odd-Even graceful labeling.**

Theorem 2.4: *The quadrilateral snake* $K_{1,n} \oplus C_4^{\,n} \oplus K_{1,n}$ *becomes Odd-Even graceful for* $n \geq 1$.

Proof: Suppose $K_{1,n} \oplus C_4^{\,n} \oplus K_{1,n}$ be a quadrilateral snake with initial and terminal ends of cycle joined with $K_{1,n}$ and having $5n+1$ vertices and $6n(=q)$ edges. Suppose u_i, v_i, and $w_i, i = 1,2,3,\ldots,n$ be vertices at top, middle, and bottom of n adjacent C_4 respectively. $x_i, y_i, i = 1,2,3,\ldots,n$ be pendant vertices attached at right end cycle and the left end cycle of the quadrilateral snake respectively.

The vertex labeling is

$$f(u_{i+1}) = 10n - 4i + 1, i = 0,1,2,\ldots,n-1,$$
$$f(w_i) = 10n - 4i + 3, i = 1,2,\ldots,n$$
$$f(v_i) = 4i - 3, i = 1,2,3,\ldots,n+1$$
$$f(x_{i+1}) = 4n + 2i + 3, i = 0,1,2,\ldots,n-1,$$
$$f(y_i) = 10n + 2i + 1, i = 1,2,\ldots,n$$

The edge labeling is

$$f(u_i v_i) = 10n - 8i + 8,\ f(u_i v_{i+1}) = 10n - 8i + 4,\ f(v_{i+1} w_i) = 10n - 8i + 2,$$
$$f(v_i w_i) = 10n - 8i + 6, f(v_{n+1} x_i) = 2i, f(v_1 y_i) = 10n + 2i, i = 1,2,\ldots,n$$

After expansion the edge labeling ranges from 2 to $2q$, where the labelled set of vertices is $\{1,3,5,\ldots,2q+1\}$. Hence, graph becomes Odd-Even graceful.

Illustration 2.4: Theorem 2.4 is explained in Figure 4 as follows for $n = 4$.

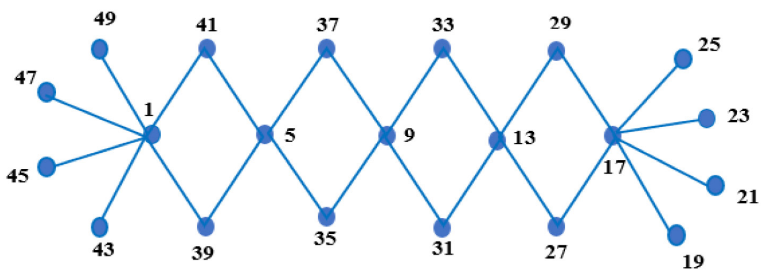

Figure 4

$K_{1,4} \oplus C_4^{\,4} \oplus K_{1,4}$ **with Odd-Even graceful labeling.**

Theorem 2.5: *The tristar* $S_n \oplus S_{n+2} \oplus S_n$ *is Odd-Even graceful for* $n \geq 2$.

Proof: Suppose $S_n \oplus S_{n+2} \oplus S_n$ be a tristar with $3n+3$ vertices and $3n+2(=q)$ edges. Let $u_i, i = 1,2,3$ be the vertices on the central path P_3

of the tristar. Suppose x_i, y_i and $z_i, i = 1, 2, 3, \ldots, n$ be vertices joined with u_1, u_2 and u_3 respectively.

The vertex labeling is

$$f(x_i) = 6n - 2i + 7, f(y_i) = 2n - 2i + 3, f(z_i) = 4n - 2i + 5, i = 1, 2, 3, \ldots, n,$$
$$f(u_1) = 1, f(u_2) = 4n + 5, f(u_3) = 2n + 3$$

And the edge labeling is

$$f(u_1x_i) = 6n - 2i + 6, f(u_2y_i) = 2n + 2i + 2, f(u_3z_i) = 2n - 2i + 2, i = 1, 2, 3, \ldots, n,$$
$$f(u_1u_2) = 4n + 4, f(u_2u_3) = 2n + 2$$

So, the edge labeling of the above graph contains a set of even integers ranging from 2 to $2q$, where the labelled set of vertices is $\{1, 3, 5, \ldots, 2q + 1\}$. Hence, the graph becomes Odd-Even graceful.

Illustration 2.5: Theorem 2.5 is explained in Figure 5 as follows for $n = 6$.

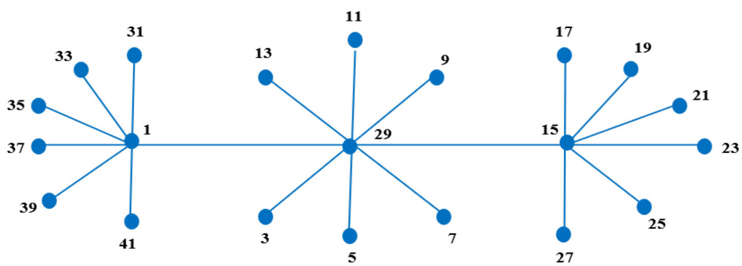

Figure 5

$S_6 \oplus S_8 \oplus S_6$ **with Odd-Even graceful labeling.**

Theorem 2.6: *The 4-star* $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ *is Odd-Even graceful for* $n \geq 2$.

Proof: Suppose $S_n \oplus S_{n+2} \oplus S_{n+2} \oplus S_n$ be a 4-star with central path P_4 such that both ends u_1, u_4 of the central path are joined to n vertices, while the internal vertices u_2, u_3 are joined to $n + 2$ vertices. Total number of vertices and edges are $4n + 4$ and $4n + 3 (= q)$ respectively. Suppose x_i, y_i, z_i and $v_i, i = 1, 2, 3, \ldots, n$ be the vertices joined with vertices u_1, u_2, u_3 and u_4 respectively.

The vertex labeling is

$$f(x_i) = 8n - 2i + 9, f(y_i) = 2n - 2i + 3, f(z_i) = 6n - 2i + 7,$$
$$f(v_i) = 4n - 2i + 5, i = 1, 2, 3, \ldots, n, f(u_1) = 1, f(u_2) = 6n + 7,$$
$$f(u_3) = 2n + 3, f(u_4) = 4n + 5$$

And the edge labeling is

$$f(u_1x_i) = 8n - 2i + 8, f(u_2y_i) = 4n + 2i + 4, f(u_3z_i) = 4n - 2i + 4,$$
$$f(u_4v_i) = 2i, i = 1, 2, 3, \ldots, n, f(u_1u_2) = 6n + 6, f(u_2u_3) = 4n + 4,$$
$$f(u_3u_4) = 2n + 2$$

So, the edge labeling of the above 4-star contains a set of even integers ranging from 2 to $2q$, where the labelled set of vertices is $\{1, 3, 5, \ldots, 2q+1\}$. Hence, the graph is Odd-Even graceful.

Illustration 2.6: Theorem 2.6 is explained in Figure 6 as follows for n = 6.

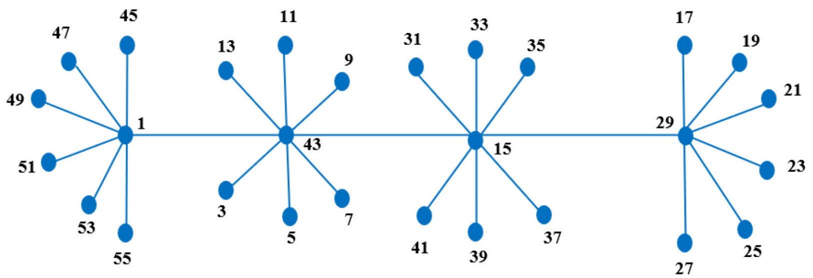

Figure 6

$S_6 \oplus S_8 \oplus S_8 \oplus S_6$ **with Odd-Even graceful labeling.**

Theorem 2.7: $C_3 \odot 3K_{1,n}$ *becomes Odd-Even graceful for* $n \geq 2$.

Proof: Suppose a graph $C_3 \odot 3K_{1,n}$ which is formed by joining n pendant edges to every vertex u_1, u_2, u_3 of C_3. It has $3n+3$ vertices and edges. Suppose x_i, y_i and $z_i, i = 1, 2, 3, \ldots, n$ be vertices are joined with u_1, u_2 and u_3 respectively.

The vertex labeling is

$$f(x_i) = 6n - 2i + 9, f(y_i) = 4n - 2i + 7, f(z_i) = 2n - 2i + 3, i = 1, 2, 3, \ldots, n,$$
$$f(u_1) = 1, f(u_2) = 2n + 5, f(u_3) = 4n + 7$$

And the edge labeling is

$$f(u_1x_i) = 6n - 2i + 8, f(u_2y_i) = 2n - 2i + 2, f(u_3z_i) = 2n + 2i + 4, i = 1, 2, 3, \ldots, n,$$
$$f(u_1u_2) = 2n + 4, f(u_2u_3) = 2n + 2, f(u_1u_3) = 4n + 6$$

Putting all the values of i in the above edge labeling formula, edge labels of $C_3 \odot 3K_{1,n}$ form a set of even integers from 2 to $2q$, where the labelled set of vertices is $\{1,3,5,...,2q+1\}$. Hence, the graph becomes Odd-Even graceful.

Illustration 2.7: Theorem 2.7 is explained in Figure 7 as follows for $n = 6$.

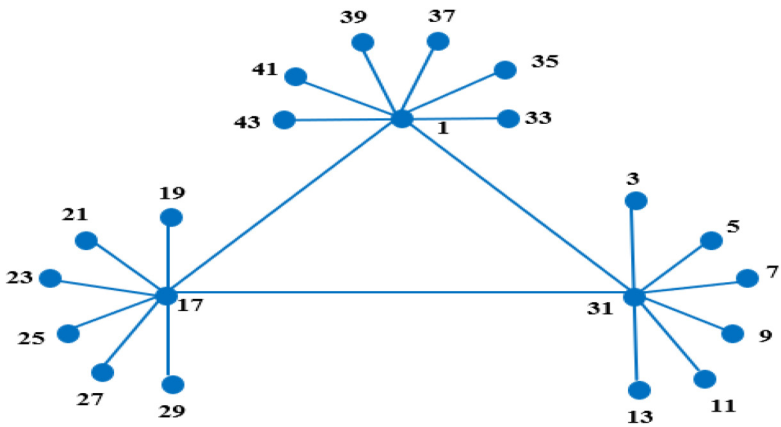

Figure 7

$C_3 \odot 3K_{1,6}$ **with Odd-Even graceful labeling.**

3. Discussion and Conclusion

The present paper emphasizes on Odd-Even graceful labeling of graphs like coconut tree, comb graph, a class of quadrilateral snakes, tristar, 4-star, and a class of hairy cycle graph. The results are found to be more interesting and inspiring for strengthening more research on odd graceful graphs. The problems are still open for *n-star* graphs $(n \geq 5)$. One can also try for n-gonal snakes with $(n \geq 5)$. Furthermore, hairy cycle graphs with cycle of length $n \geq 3$ may be shown as Odd-Even graceful.

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