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Different classes of graceful coronas and their second kind

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Abstract

A graph labeling assigns integer to the components of a graph (vertices, edges, or both), adhering to specific conditions. Let G be a graph with q edges. Then the graceful labeling is defined as a one-to-one mapping from the vertex set of G to $\{0, 1, ..., q\}$ so that each edge label arises due to the absolute difference of vertex labels of its corresponding vertices, making all edge labels dissimilar. This paper demonstrates that a class of coronas is graceful and of the second kind under specific conditions. The corona graph considered here is a cycle graph with each or every other vertex attached to two pendant edges.

Subject Classification: *05C10, 05C38, 05C78.*

Keywords: *Graceful labeling, 2nd kind graceful labeling, Cycle, Corona, Pendant edge.*

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1. Introduction

Rosa [11] developed an elegant labeling methodology, the most widely used graph labeling approach. A graceful graph is defined as an arrangement of integers from $\{0,1,...,q\}$ to its vertices so that each edge labeling, which is generated by taking the absolute difference of its corresponding vertex labeling, is a unique integer ranging from $\{1,2,...,q\}$ where q is the cardinality of the graph's edge set. Gallian [6] presented a theory of elegant labeling, according to which all trees display graceful labeling. Two popular approaches are highlighted in the survey: harmonious and graceful labeling. [9] Conducted a survey exploring whether this labelling exists for graphs with the highest conceivable maximum vertex degrees. [7] Studied the possibility of edge-odd gracefulness in the Cartesian product of two pathways. [2] Demonstrates how certain derived graphs (splitting graphs) from conventional graphs, such as pathways, cycles, and stars, can be labelled with odd integers for vertices, yielding even edge labels when their differences are taken. Within the field of graph labeling theory, this trait is known as odd-even elegant. [4] Propose a technique for orienting edges by $C(c\times a)$ creating edge-graceful labeling with direction under specific circumstances. [3] Demonstrated that under some circumstances, some coronas are graceful of the second kind. Pendant graph-related graceful labeling has several applications and consequences. Murugesan and Uma [10] discussed a study on super vertices using graceful graphs. Mathematicians and graph theorists have continued to study different kinds of graphs and the circumstances in which graceful labeling is possible over time. [8] Showed in 2021 that some elegant hairy cycles accept an alpha label. [12] Proposed an arithmetic sequential graceful graph and presented the idea of arithmetic sequential graceful labeling in 2024. [5] Investigated graceful labeling in graph theory in 2024, with a particular emphasis on pendant graphs. [1] looked into the elegant labeling of a certain multiple-fan graph variation, namely a multiple-fan graph with k pendants, in 2021. The current study is unique and stands apart from all previous research in that it limits the elegant labeling of coronas together with the second kind under varied conditions. Cycles with two pendant edges connected to each vertex are demonstrated to display either elegant labeling or gracious labeling of the second sort under specific circumstances as specified in the corresponding theorems. To demonstrate that the corona graphs are elegant or graceful of the second sort under the criteria outlined in the theorems of this paper, the pendant edges are also linked at alternating points of a cycle.

2. Preliminaries

Definition 2.1: A closed path with more than three vertices is called a cycle. This represents a cycle with vertices C_m.

Definition 2.2: A vertex with a degree of one is known as a pendant vertex. similarly, an edge that has one end vertex of degree 1 is called a pendant edge.

Definition 2.3: A corona graph means a cyclic graph by attaching n pendant edges at some or all of its vertices. Usually, it is denoted as $C_m \odot nK_1$.

Definitions: 2.4: Let $G = (V, E)$ denotes a graph where $V(G)$ represents vertex set and $E(G)$ denotes edge set, with total number of edges being $|E(G)| = q$. A graceful labeling refers to vertex labeling defined by injective function $f : V \to [0, 1, ...q]$, producing edge labeling $\phi(xy) = |f(x) - f(y)|$ $= \{1, 2, ..., q\}$, where $x, y \in V$. The graph satisfies the graceful labeling known as a graceful graph.

Definition 2.5: Graceful labeling of a corona is of 2nd kind, if a pendant vertex is assigned with $2m$.

The Figure 1 provides examples of graceful labeling of 2nd kind.

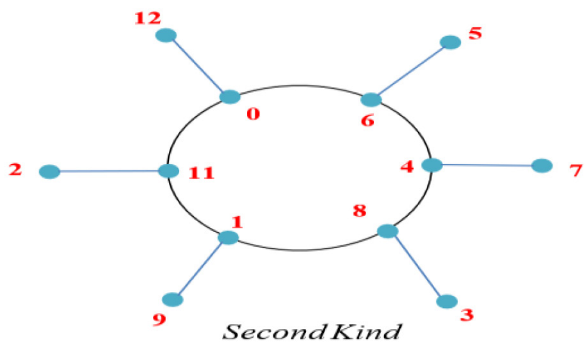

Figure 1

Graceful labeling of 2nd Kind.

3. Results And Discussion

Theorem 3.1: *Every alternate corona* $C_m \odot 2K_1$ *has a graceful labeling of the* 2^{nd} *kind, when* $m \equiv 6 \pmod 8$; $m > 3$.

Proof: Let us consider w_i, $i = 1, 2, \ldots, m$ be the vertices of a cycle C_m and $C_m \odot 2K_1$ has $2m$ edges. Let (x_i, y_i), $i = 1, 2, \ldots, m$ be the pair of alternate pendant vertices for the cycle C_m. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 6 (mod 8)$; $m > 3$.

$$f(w_i) = \begin{cases} 0, & i = m-1 \\ 2m-2, & i = m \\ \frac{m}{2}, & i = m-2 \\ \frac{i+1}{2}, & i = 1, 3, 5, \ldots\ldots\ldots, m-3 \\ 2m-2-\frac{3i}{2}, & i = 2, 4, 6, \ldots\ldots\ldots, \frac{3m-2}{4}-2 \\ 2m-3-\frac{3i}{2}, & i = \frac{3m-2}{4}, \frac{3m-2}{4}+2, \frac{3m-2}{4}+4, \ldots\ldots\ldots, m-4 \end{cases}$$

$$f(x_i) = \begin{cases} 2m, & i = m-1 \\ 2m-1, & i = m \\ 2m-3(\frac{i+1}{2}), & i = 1, 3, 5, \ldots\ldots\ldots\ldots, \frac{3m-2}{4}-1 \\ 2m-3(\frac{i+1}{2})-1, & i = \frac{3m-2}{4}+1, \frac{3m-2}{4}+3, \ldots\ldots. m-3 \\ 2m-\frac{3i}{2}-1, & i = 2, 4, 6, \ldots\ldots\ldots, \frac{3m-2}{4}-2 \\ 2m-\frac{3i}{2}-2, & i = \frac{3m-2}{4}, \frac{3m-2}{4}+2, \ldots\ldots\ldots, m-2 \end{cases}$$

Its corresponding edge labeling is given by

$$| f(w_i) - f(w_1) | = 2m-3, i = m$$

$$|f(w_i)-f(w_{i+1})| = \begin{cases} 2m-2, & i=m-1 \\ \dfrac{m}{2}, & i=m-2 \\ 2m-2i-4, & i=1,3,5,\ldots\ldots\ldots\ldots,\dfrac{3m-2}{4}-3 \\ 2m-2i-5, & i=\dfrac{3m-2}{4}-1,\dfrac{3m-2}{4}+1,\ldots\ldots.m-3 \\ 2m-2i-3, & i=2,4,6,\ldots\ldots\ldots,\dfrac{3m-2}{4}-2 \\ 2m-2i-4, & i=\dfrac{3m-2}{4},\dfrac{3m-2}{4}+2,\ldots\ldots\ldots,m-4 \end{cases}$$

$$|f(w_i)-f(x_{i+1})| = \begin{cases} 2m-2i-3, & i=1,3,5,\ldots\ldots\ldots\ldots,\dfrac{3m-2}{4}-3 \\ 2m-2i-4, & i=\dfrac{3m-2}{4}-1,\dfrac{3m-2}{4}+1,\ldots\ldots.m-3 \\ 2m-1, & i=m-1 \end{cases}$$

$$|f(w_i)-f(x_i)| = \begin{cases} 2m-2i-2, & i=1,3,5,\ldots\ldots\ldots\ldots,\dfrac{3m-2}{4}-1 \\ 2m-2i-3, & i=\dfrac{3m-2}{4}+1,\dfrac{3m-2}{4}+3,\ldots\ldots.m-3 \\ 2m, & i=m-1 \end{cases}$$

Since labelled edges belong to $\{1,2,3,\ldots,2m\}$ with pendant vertex assigned the label $2m$. Hence the graph satisfies graceful labeling of the 2nd kind.

Illustration 3.1: If $m \equiv 6(mod\,8)$; $m > 3$, let $m = 14$. Then $C_{14} \odot 2K_1$ has a graceful labeling of the second kind, which is explained in Figure 1 as follows.

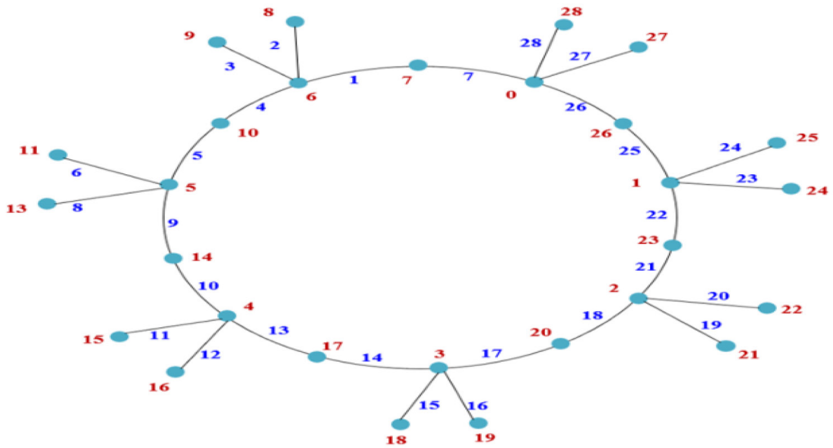

Figure 2

Alternate $C_{14} \odot 2K_1$ with graceful labeling of the 2nd kind.

From Figure 2, it is clear that the above corona $C_{14} \odot 2K_1$ is graceful of the 2nd kind, since each edge is labeled with dissimilar integers belonging to $\{1,2,3,\ldots,28\}$ and 28 is assigned to the pendant vertex.

Theorem 3.2: *Every 1, 4-alternate corona $C_m \odot 2K_1$ has a graceful labeling, if $m \equiv 0(mod\,12); m > 5$.*

Proof: Let us consider $w_i, i = 1,2,\ldots,m$ be the vertices of a cycle C_m and $C_m \odot 2K_1$ has $\frac{5m}{3}$ edges. Let (x_i, y_i), $i = 1,2,\ldots,m$ be the pair of 1, 4 alternate pendant vertices for the cycle. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 0(mod\,12); m > 5$.

$$f(w_i)=\begin{cases} 0, & i=m \\ \frac{5m}{3}-\frac{5}{6}(i-1), & i=1,7,13,\dots\dots\dots,\frac{m-2}{2}-4 \\ \frac{5m}{3}-\frac{5}{6}(i-1)-1, & i=\frac{m-2}{2}+2,\frac{m-2}{2}+8,\dots\dots\dots,m-5 \\ \frac{5m}{3}-\frac{5}{6}(i-3)-1, & i=3,9,15,\dots\dots\dots,\frac{m-2}{2}-2 \\ \frac{5m}{3}-\frac{5}{6}(i-3)-2, & i=\frac{m-2}{2}+4,\frac{m-2}{2}+10,\dots\dots\dots,m-3 \\ \frac{5m}{3}-\frac{5}{6}(i-5)-4, & i=5,11,17,\dots\dots\dots,\frac{m-2}{2} \\ \frac{5m}{3}-\frac{5}{6}(i-5)-5, & i=\frac{m-2}{2}+6,\frac{m-2}{2}+12,\dots\dots\dots,m-1 \\ \frac{5}{6}(i-2)+3, & i=2,8,14,\dots\dots\dots\dots,m-4 \\ \frac{5}{6}(i-4)+4, & i=4,10,16,\dots\dots\dots\dots,m-2 \\ \frac{5i}{6}, & i=6,12,18,\dots\dots\dots\dots\dots,m-6 \end{cases}$$

$$f(x_i)=\begin{cases} \frac{5}{4}(i-1)+1, & i=1,5,9,\dots\dots\dots\dots,\frac{2m}{3}-3 \\ \frac{5m}{3}-\frac{5}{4}(i-3)-2, & i=3,7,11,\dots\dots\dots,\frac{m}{3}-1 \\ \frac{5m}{3}-\frac{5}{4}(i-3)-3, & i=\frac{m}{3}+3,\frac{m}{3}+7,\frac{m}{3}+11,\dots\dots\dots,\frac{2m}{3}-1 \\ \frac{5}{4}(i-2)+2, & i=2,6,10,\dots\dots\dots\dots,\frac{2m}{3}-2 \\ \frac{5m}{4}-\frac{5i}{4}+2, & i=4,8,12,\dots\dots\dots\dots,\frac{m}{3} \\ \frac{5m}{4}-\frac{5i}{4}+1, & i=\frac{m}{3}+4,\frac{m}{3}+8,\frac{m}{3}+12,\dots\dots\dots,\frac{2m}{3} \end{cases}$$

Its corresponding edge labeling is given by

$$|f(w_i)-f(w_1)|=\frac{5m}{3}, i=m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases}\frac{5m}{3}-\frac{5}{3}(i-1)-3, & i=1,4,7,.........,\frac{m}{2}-2\\ \frac{5m}{3}-\frac{5}{3}(i-1)-4, & i=\frac{m}{2}+1,\frac{m}{2}+4,.........,m-2\\ \frac{5m}{3}-\frac{5}{3}(i-2)-4, & i=2,5,8,.........,\frac{m}{2}-1\\ \frac{5m}{3}-\frac{5}{3}(i-2)-5, & i=\frac{m}{2}+2,\frac{m}{2}+5,.........,m-4\\ \frac{5m}{3}-\frac{5}{3}(i-3)-5, & i=3,6,9,.........,\frac{m}{2}-3\\ \frac{5m}{3}-\frac{5}{3}(i-3)-6, & i=\frac{m}{2},\frac{m}{2}+3,.........,m-3\\ \frac{5m}{6}, & i=m-1\end{cases}$$

$$|f(w_i)-f(x_j)|=\begin{cases}\frac{5m}{3}-\frac{5}{3}(i-1)-1, & i=1,4,7,.........,\frac{m}{2}-2\\ \frac{5m}{3}-\frac{5}{3}(i-1)-2, & i=\frac{m}{2}+1,\frac{m}{2}+4,.........,m-2\end{cases}$$

$$|f(w_i)-f(x_{j+1})|=\begin{cases}\frac{5m}{3}-\frac{5}{3}(i-1)-2, & i=1,4,7,.........,\frac{m}{2}-2\\ \frac{5m}{3}-\frac{5}{3}(i-1)-3, & i=\frac{m}{2}+1,\frac{m}{2}+4,.........,m-2\end{cases}$$

where, $j=\frac{2}{3}(i-1)+1$

Since the edge labeling belongs to the set $\left\{1,2,3,...,\frac{5m}{3}\right\}$. Hence, the graph is graceful.

Illustration 3.2: If $m\equiv 0(mod 12)$; $m>5$, let $m=12$. Then $C_{12}\odot 2K_1$ has a graceful labeling, which is explained in Figure 2 as follows.

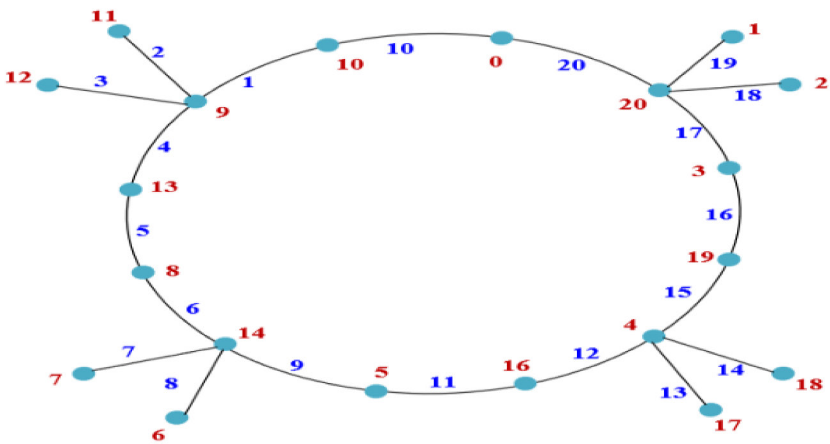

Figure 3

1,4-alternate $C_{12} \odot 2K_1$ **with graceful labeling**

From Figure 3, it is clear that the above corona $C_{12} \odot 2K_1$ is graceful, since each edge is labeled with dissimilar integers belonging to {1, 2, 3, ..., 20}.

Theorem 3.3: *Every 1, 4-alternate corona* $C_m \odot 2K_1$ *has a graceful labeling, if* $m \equiv 3(mod\,12)$; $m > 5$.

Proof: Let us consider $w_i, i = 1,2,...,m$ be vertices of cycle C_m and $C_m \odot 2K_1$ has $\frac{5m}{3}$ edges. Let (x_i, y_i), $i = 1,2,...,m$ be the pair of 1,4-alternate pendant vertices for the corona C_m. Then the following functions produce the vertex labeling of $C_m \odot 2K_1$ when $m \equiv 3(mod\,12); m > 5$

$$f(w_i)=\begin{cases} 0, & i=m \\ \frac{5m}{3}-\frac{5}{6}(i-1), & i=1,7,13,.........,m-2 \\ \frac{5}{6}(i-4)+5, & i=\frac{m-3}{2}+4,\frac{m-3}{2}+10,.........,m-5 \\ \frac{5m}{3}-\frac{5}{6}(i-3)-1, & i=3,9,15,.........,m-6 \\ \frac{5i}{6}+1, & i=\frac{m-3}{2}+6,\frac{m-3}{2}+12,.........,m-3 \\ \frac{5m}{3}-\frac{5}{6}(i-5)-4, & i=5,11,17,.........,m-4 \\ \frac{5}{6}(i-2)+4, & i=\frac{m-3}{2}+2,\frac{m-3}{2}+8,.........,m-1 \\ \frac{5}{6}(i-2)+3, & i=2,8,14,..........,\frac{m-3}{2}-4 \\ \frac{5}{6}(i-4)+4, & i=4,10,16,..........,\frac{m-3}{2}-2 \\ \frac{5i}{6}, & i=6,12,18,............,\frac{m-3}{2} \end{cases}$$

$$f(x_i)=\begin{cases} \frac{5}{4}(i-1)+1, & i=1,5,9,..........,\frac{m}{3}-4 \\ \frac{5m}{3}-\frac{5}{4}(i-3)-2, & i=3,7,11,.........,\frac{2m}{3}-3 \\ \frac{5}{4}(i-1)+2, & i=\frac{m}{3},\frac{m}{3}+4,\frac{m}{3}+8,.........,\frac{2m}{3}-1 \\ \frac{5}{4}(i-2)+2, & i=2,6,10,..........,\frac{m}{3}-3 \\ \frac{5m}{3}-\frac{5}{4}(i-4)-3, & i=4,8,12,..........,\frac{2m}{3}-2 \\ \frac{5}{4}(i-2)+3, & i=\frac{m}{3}+1,\frac{m}{3}+5,\frac{m}{3}+9,.........,\frac{2m}{3} \end{cases}$$

Its corresponding edge labeling is given by

$$|f(w_i)-f(w_1)|=\frac{5m}{3}, i=m$$

$$|f(w_i)-f(w_{i+1})|=\begin{cases}\frac{5m}{3}-\frac{5}{3}(i-1)-3, & i=1,4,7,.........,\frac{m-3}{2}-2\\ \frac{5m}{3}-\frac{5}{3}(i-1)-4, & i=\frac{m-3}{2}+1,\frac{m-3}{2}+4,.........,m-2\\ \frac{5m}{3}-\frac{5}{3}(i-2)-4, & i=2,5,8,.........,\frac{m-3}{2}-1\\ \frac{5m}{3}-\frac{5}{3}(i-2)-5, & i=\frac{m-3}{2}+2,\frac{m-3}{2}+5,.........,m-4\\ \frac{5m}{3}-\frac{5}{3}(i-3)-5, & i=3,6,9,.........,\frac{m-3}{2}\\ \frac{5m}{3}-\frac{5}{3}(i-3)-6, & i=\frac{m-3}{2}+3,\frac{m-3}{2}+6,.........,m-3\\ \frac{5m+9}{6}, & i=m-1\end{cases}$$

$$|f(w_i)-f(x_j)|=\begin{cases}\frac{5m}{3}-\frac{5}{3}(i-1)-1, & i=1,4,7,.........,\frac{m-3}{2}-2\\ \frac{5m}{3}-\frac{5}{3}(i-1)-2, & i=\frac{m-3}{2}+1,\frac{m-3}{2}+4,.........,m-2\end{cases}$$

$$|f(w_i)-f(x_{j+1})|=\begin{cases}\frac{5m}{3}-\frac{5}{3}(i-1)-2, & i=1,4,7,.........,\frac{m-3}{2}-2\\ \frac{5m}{3}-\frac{5}{3}(i-1)-3, & i=\frac{m-3}{2}+1,\frac{m-3}{2}+4,.........,m-2\end{cases}$$

Where, $j=\frac{2}{3}(i-1)+1$

Since the edge labeling belongs to the set $\{1,2,3,...,\frac{5m}{3}\}$.

Hence, the graph is graceful.

Illustration 3.3: Similar to Theorem 3.2.

4. Discussion and Conclusion

This paper awakens the graceful labeling of a class of coronas satisfying certain conditions, along with the graceful labeling of the second kind. In some cases, it has been proven that a class of coronas, which are formed by attaching two pendant edges of a cycle, are either graceful or have a graceful labeling of the 2^{nd} kind. Furthermore, coronas having alternate pendant edges or 1,4-alternate pendant edges either have a graceful labeling or a graceful labeling of the 2^{nd} kind. This work will help a researcher to extend the coronas either by attaching more than two pendant edges at each vertex or every possible alternate vertices of a cycle to prove it graceful or graceful labeling of the 1^{st},$2^{nd,}$ and 3^{rd} kind. Also, one can extend the pendant edges of a cycle more than once to verify the graceful labeling of all kinds.

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