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Unshackle games on $2m \times (2n + 1)$ Grid graphs, Ladder graphs and Ladder-like graphs

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Abstract

The Unshackle game is a combinatorial game for two players that starts with prisoners and shackles on a board such that each shackle has two ends, and each end is shackled to one prisoner. Two players alternately take turns by destroying a shackle on the board until all shackles have been destroyed. A prisoner is free when all shackles that are shackled to him are destroyed, and the player who frees the most prisoners wins and the other loses. Both players draw if neither of them can win. The prisoners and the shackles on the board can be considered as vertices and edges of a graph. This article determines the outcomes of the Unshackle games on $2m \times (2n + 1)$ grid graphs by constructing a winning strategy for the second player, and determines the outcomes of the Unshackle games on ladder graphs and on ladder-like graphs by using mathematical induction.

Subject Classification: *05C57.*

Keywords: *Unshackle game, Combinatorial game, Impartial game, Game on graph, 2-person game, Winning strategy, Grid graph, Ladder graph.*

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1. Introduction

In 2022, the Unshackle game was first introduced in [5]. This game is a modification of the Strings-and-Coins game introduced by Berlekamp, Conway, and Guy [3], while the two games differ slightly in their rules. The *Unshackle Game* is a game for two players, called *Player I* and *Player II*, and they are the *opponents* of each other. There are finitely many *states* in the game, each of which consists of a finite set of prisoners and a finite set of shackles, where each shackle has two ends, and each end is shackled to one prisoner. Every *play* in the game starts at a designated *initial state*. The players *move* alternately from one state to another by destroying one of the shackles. A prisoner is said to be *free* if all shackles that are shackled to that prisoner are destroyed. The play ends at a *terminal state*, namely a state in which all prisoners are free, i.e., all shackles are destroyed. Then, the player who frees the most prisoners *wins* and the other *loses*. Both players *draw* if neither of them can win. This game is a type of combinatorial game [1] and is impartial [2], since the rules that specify the moves of both players are the same.

In any Unshackle game, the relation that associates each shackle with two prisoners in any state is equivalent to a relation that associates each edge with two vertices in any graph. This suggests that the game can be considered a game on a graph in which vertices represent prisoners, and edges represent shackles. The following definition gives the graph version of the game.

Definition 1.1: ([5]). A (graph) Unshackle Game is a two-player game that starts with a graph. Two players move alternately by removing one of the edges in the graph. Each play ends when all vertices are isolated, i.e., all edges are removed. Then, the player who isolated the most vertices wins and the other loses. Both players draw if neither of them can win.

Any graph Unshackle game, hereafter called an Unshackle game, is a game played on a graph and a combinatorial game. Consequently, each graph generates a unique Unshackle game. Note that the notation $\mathbf{U}(G)$ denotes the Unshackle game on a graph G.

Throughout this article, standard graph theoretical terminology, knowledge and notation follow from a textbook written by Gross and Yellen [4]. Note that the notations $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of a graph G, respectively.

As in general combinatorial game, every move from the initial state in any Unshackle game are Player I's. Additionally, every state in any Unshackle game is determined by a graph which is a subgraph of the graph at the initial state. Note that the notation $G(s)$ denotes the graph at a state s.

For any play in an Unshackle game, the player's *score* is defined as the number of vertices isolated by the player in the play. Note that the notations $sc_1(P)$ and $sc_2(P)$ denote Player I's score and Player II's score of a play P, respectively.

Throughout this article, standard terminology and notation in combinatorial game theory follow from a textbook written by Beck [1]. In order to analyze the

Unshackle game, the necessary notions and tools from combinatorial game theory are provided.

Definition 1.2: The game-tree of a combinatorial game is a rooted-tree in which the vertices represent the states in the game, the root represents the initial state, and the children of each vertex represent its options, i.e., vertex u is the parent of vertex v if there is a legal move from state u to state v.

Remark that each leaf in the game-tree of a combinatorial game represents a terminal state in the game, each edge incident to a vertex at an even level and its child represents a Player I's move, and each edge incident to a vertex at an odd level and its child represents a Player II's move. Moreover, each path from the root to a leaf can represent a play. Note that the notation $\mu(P)$ denotes the number of moves in a play P in a combinatorial game, i.e., if P is considered a path from the root to a leaf in the game-tree, then $\mu(P) = |E(P)|$.

Every combinatorial game can be represented by a pair (T, F) in which T is the game-tree of the game, and F is a function from the set of plays to the set of possible outcomes, which consists of **a win for Player I**, **a win for Player II** and **a draw for both players**. Note that the notation $par(s)$ denotes the state that has s as one of its options, the notation $opt(s)$ denotes the set of all options of a state s in a combinatorial game, and the notation $child_T(v)$ denote the set of all children of a vertex v in a rooted-tree T, i.e., if T is the game-tree of the combinatorial game, then for each state s, $opt(s) = child_T(s)$.

A *strategy* in a combinatorial game is a plan constructed to determine moves of a player from some given states to specific options. Some states may be assigned a move to one or more options, while some states may not be assigned any moves by the strategy.

Definition 1.3: A strategy-tree for any player in a combinatorial game is a rooted-subtree S of the game-tree that satisfies the following conditions.

(1) The initial state is the root in S.

(2) For each non-terminal state s, if $s \in S$, then $child_S(s) \neq \emptyset$. Moreover, if the legal moves from s is the opponent's, then $child_S(s) = opt(s)$.

Every strategy in a combinatorial game can be represented by a strategy-tree. For unique representation, states that are not assigned any moves by the strategy will be considered to be assigned moves to all of their options.

Definition 1.4: A strategy in a combinatorial game is a winning strategy for a player if that player wins in all possible plays satisfying the strategy, and it is a drawing strategy for a player if that player draws in at least one play satisfying the strategy and wins in the other plays.

Theorem 1.5: ((Zermelo) [1]). *In any combinatorial game, either exactly one player has a winning strategy, or both players have drawing strategies.*

According to Theorem 1.5, the combinatorial games can be divided into three classes. The following is the definition of the outcomes of the combinatorial games based on the theorem.

Definition 1.6: The outcome of a combinatorial game is one of the following.

- Player I has a winning strategy.
- Player II has a winning strategy.
- Both players have drawing strategies.

Remark 1.7: Let (T, F) be an Unshackle game on a graph G containing no isolated vertices. Then, the following holds.

(1) For each play P, $\mu(P) = |E(G)|$.

(2) For each play P, $sc_1(P) + sc_2(P) = |V(G)|$.

(3) For each play P,

$$F(P) = \begin{cases} \text{a win for Player I} & \text{if } sc_1(P) > sc_2(P), \\ \text{a win for Player II} & \text{if } sc_1(P) < sc_2(P), \\ \text{a draw for both players} & \text{if } sc_1(P) = sc_2(P). \end{cases}$$

Example 1.8: Figure 1 shows the Unshackle game on the path with four vertices. The game consists of six plays, four of which result in a win for Player I, and two of which result in a draw for both players. Moreover, the outcome of this game is that Player I has a winning strategy.

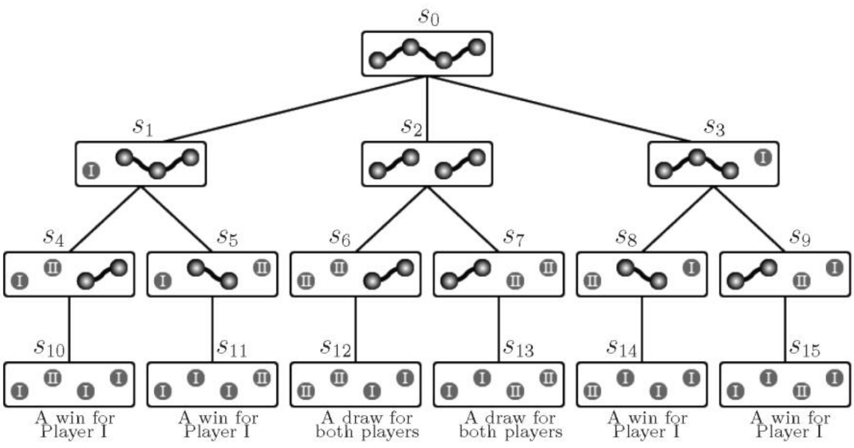

Figure 1
The Unshackle game on the path with four vertices

In classifying Unshackle games based on their outcomes, this article chooses to classify graphs instead because each graph generates a unique Unshackle game.

Definition 1.9: The notations $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{D}$ are the following.

- $\mathcal{W}_1 = \{G|\text{Player I has a winning strategy in } \mathbf{U}(G)\}$.
- $\mathcal{W}_2 = \{G|\text{Player II has a winning strategy in } \mathbf{U}(G)\}$.
- $\mathcal{D} = \{G|\text{both players have drawing strategies in } \mathbf{U}(G)\}$.

Any graph containing no edges can be classified into the class $\mathcal{D}$. Furthermore, any graph containing isolated vertices can be classified based on the induced subgraph on the set of its non-isolated vertices, because isolated vertices in the initial state will not be counted toward the players' scores. For convenience, this article denotes $G \approx H$ if $G - I_1 \cong H - I_2$ where I_1 and I_2 are the sets of all isolated vertices in G and H, respectively, i.e., if $G \approx H$, then G and H belong to the same class among $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{D}$.

Lemma 1.10: *If G is a graph in $\mathcal{W}_2$, then either adding or deleting one edge in G results in a graph in $\mathcal{W}_1$.*

Proof: Assume that $G \in \mathcal{W}_2$. Then, Player II has a winning strategy σ in $\mathbf{U}(G)$. Let e be any edge in G, let u and v be any vertices in G, and let e' be a new edge connecting two vertices u and v. Clearly, the strategy σ is a winning strategy for Player I in $\mathbf{U}(G-e)$. Moreover, the strategy in which Player I remove e' for their first move and play according to σ for all their other moves is a winning strategy for Player I in $\mathbf{U}(G+e')$. Therefore, $G - e \in \mathcal{W}_1$ and $G + e' \in \mathcal{W}_1$.

This article classifies certain cases of grid graphs, including ladder graphs, which are a special case of grid graphs, as well as ladder-like graphs, which are subgraphs of ladder graphs. The definition of a grid graph is the same as that of a mesh defined in [4]. For convenience, any grid graph in this article is considered a plane graph in the Euclidean plane $\mathbb{R}^2$.

Definition 1.11: A grid graph is a plane graph G in which $V(G) = \{0,1,2,\dots,m-1\} \times \{0,1,2,\dots,n-1\}$ where m, $n \in \mathbb{N}$, $E(G) = \{\overline{pq}|p,q \in V(G) \text{ and } d_u(p,q) = 1\}$. A grid graph with the vertex-set $\{0,1,2,\dots,m-1\} \times \{0,1,2,\dots,n-1\}$ is denoted by $G_{m,n}$.

Note that the notation $\overline{pq}$ denotes the line segment joining two points p and q, and the notation $d_u(p,q)$ denotes the Euclidean distance between two points p and q, i.e., if the coordinates of p and q are (x_1,y_1) and (x_2,y_2), respectively, then $d_u(p,q) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.

Definition 1.12: A ladder graph is a grid graph $G_{n,2}$ where $n \in \mathbb{N}$. A ladder graph with $2n$ vertices is denoted by L_n.

Example 1.13: Figure 2 shows the grid graph $G_{4,5}$ and the ladder graph L_6.

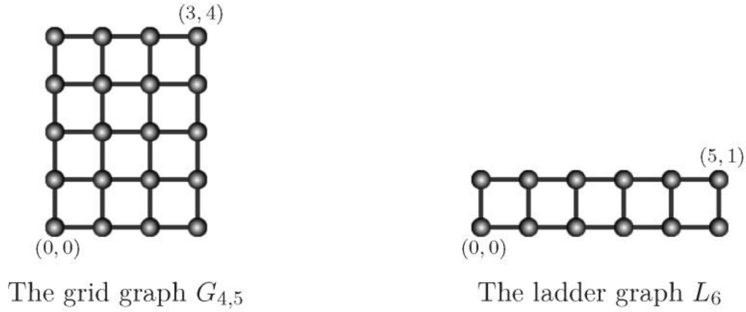

Figure 2
The grid graph $G_{4,5}$ and the ladder graph L_6

Definition 1.14: A ladder-like graph is one of the following.

- $\alpha_n = L_{n+1} - (0,0)$
- $\beta_n = L_{n+1} - \overline{(n,0)(n,1)}$
- $\gamma_n = L_{n+2} - \{(0,0), \overline{(n+1,0)(n+1,1)}\}$
- $\delta_n = L_{n+2} - \{(0,0), (n+1,0)\}$
- $\varepsilon_n = L_{n+2} - \{(0,0), (n+1,1)\}$
- $\zeta_n = L_{n+2} - \{\overline{(0,0)(0,1)}, \overline{(n+1,0)(n+1,1)}\}$

Example 1.15: Figure 3 shows the graphs α_n, β_n, γ_n, δ_n, ε_n and ζ_n.

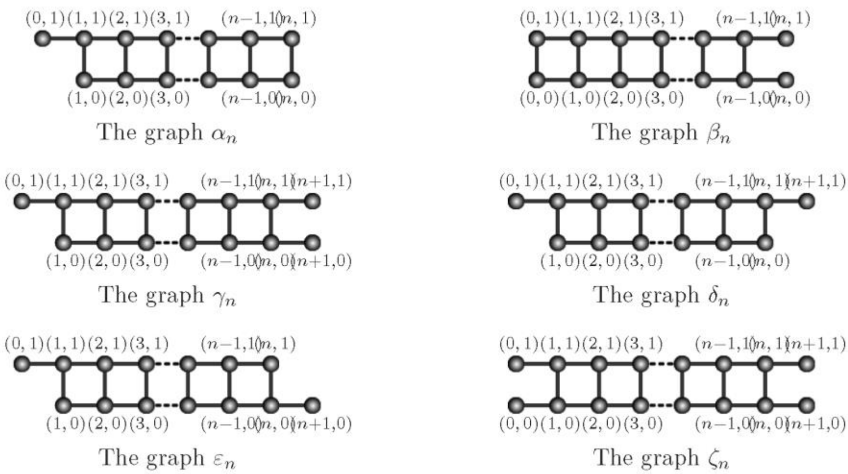

Figure 3
The graphs α_n, β_n, γ_n, δ_n, ε_n and ζ_n

Remark that $G_{m,n} \cong G_{n,m}$ for all positive integers m and n. It follows that $G_{m,n}$ and $G_{n,m}$ belong to the same class among $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{D}$. Leelathanakit, Boonklurb, and Singhun [5] classified $G_{m,n}$ in the cases where m and n are both odd, where $m = n$, where $m = 1$ and n is even, where $m = 2$ and n is odd, and where $(m, n) = (2,4)$.

Theorem 1.16: ([5]). *Let m and n be positive integers, and $(m, n) \neq (1,1)$.*

(1) *If m and n are both odd, then $G_{m,n} \in \mathcal{W}_2$.*

(2) *If $m = n$, then $G_{m,n} \in \mathcal{W}_2$.*

(3) *If n is even, then $G_{1,n} \in \mathcal{W}_1$.*

(4) *If n is odd, then $L_n \cong G_{2,n} \in \mathcal{W}_1$.*

(5) $L_4 \cong G_{2,4} \in \mathcal{W}_2$.

The following definition, strategies and lemmas were given in [5] and are all necessary tools for the main results of this article.

Definition 1.17: A plane graph G in $\mathbb{R}^2$ is line symmetric if there is a straight line l called a line of symmetry, that satisfies the following conditions.

(1) For each $v \in V(G)$, there is a vertex $v' \in V(G)$ such that $d_u(v, l) = d_u(v', l)$ and $d_u(v, v') = 2d_u(v, l)$, the vertex v' is called the shadow of v under the reflection across the line of symmetry l, denoted by $\partial_l v$.

(2) For each $e \in E(G)$, there is an edge $e' \in E(G)$ such that, for each point $p \in \mathbb{R}^2$ lying on e, there is a point $p' \in \mathbb{R}^2$ lying on e' such that $d_u(p, l) = d_u(p', l)$ and $d_u(p, p') = 2d_u(p, l)$, the edge e' is called the shadow of e under the reflection across the line of symmetry l, denoted by $\partial_l e$.

Note that the notation $d_u(p, l)$ denotes the Euclidean distance between a point p and a straight line l, i.e., $d_u(p, l) = \min\{d_u(p, q) | q \text{ lies on } l\}$.

Strategy 1.18: ([5]). For any given state s in the Unshackle game on a line symmetric plane graph with a line of symmetry l, Player II chooses a move according to the following.

If Player I's move from $par(s)$ to s is the removal of an edge e, and $G(s)$ contains $\partial_l e$, **then**

Player II chooses to remove $\partial_l e$ from $G(s)$.

Otherwise,

Player II chooses to remove any edge from $G(s)$.

Lemma 1.19: ([5]). Let G be a line symmetric plane graph with no isolated vertices, and let l be a line of symmetry of G. If G contains no edges passing across or lying on l, and contains at least one vertex lying on l, then Strategy 1.18 is a winning strategy for Player II in $\mathbf{U}(G)$.

Moreover, for all plays satisfying Strategy 1.18, all of the vertices lying on l and exactly half of the vertices not lying on l are isolated by Player II.

Strategy 1.20: ([5]). For any given state s in the Unshackle game $\mathbf{U}(G_{1,n})$ where n is even, Player I chooses a move according to the following.

If s is the initial state, **then**

Player I chooses to remove $\overline{(0,0)(0,1)}$ from $G(s)$.

Otherwise, **if** Player II's move from $par(s)$ to s is the removal of an edge e, and $G(s)$ contains $\partial_{y=\frac{n}{2}}e$, **then**

Player I chooses to remove $\partial_{y=\frac{n}{2}}e$ from $G(s)$.

Otherwise,

Player I chooses to remove any edge from $G(s)$.

Lemma 1.21: *Let n be a positive even integer. Strategy 1.20 is a winning strategy for Player I in $\mathbf{U}(G_{1,n})$.*

Moreover, for all plays satisfying Strategy 1.18, the vertices $(0,0)$, $\left(0,\frac{n}{2}\right)$ and exactly half of the remaining vertices are isolated by Player I.

2. Main Results

This article presents two main results. The first result is to classify the grid graph $G_{m,n}$ in the case where m and n have different parities, as shown in Theorem 2.2, and the second result is to classify the ladder graph L_n in the case where n is even, together with all ladder-like graphs serving as the auxiliary graphs for L_n, as shown in Theorem 2.7.

The following strategy is constructed for Player I in $\mathbf{U}(G_{m,n})$ with m and n of different parities. It is necessary for the proof of the first main result. Note that the notation $G[V]$ denote the subgraph of a graph G induced by $V \subseteq V(G)$.

Strategy 2.1: For any given state s in the Unshackle game $\mathbf{U}(G_{m,n})$ where $m \geq 3$ is odd and n is even, define $G_1 = G_{m,n}[\{(x,y) \in V(G_{m,n}) | x = \frac{m-1}{2}\}]$ and $G_2 = G_{m,n} - E(G_1)$, Player I chooses a move according to the following.

If s is the initial state, **or** Player II's move from $par(s)$ to s is the removal of an edge e in G_1, and $G(s)[V(G_1)]$ contains $\partial_{y=\frac{n}{2}}e$, **then**

Player I chooses a move from $G(s)[V(G_1)]$ according to Strategy 1.20, treating G_1 as $G_{1,n}$ up to the natural graph isomorphism that maps $(0,y)$ to $(\frac{m-1}{2}, y)$.

Otherwise, **if** Player II's move from $par(s)$ to s is the removal of an edge e in G_2, and $G(s)[V(G_2)]$ contains $\partial_{x=\frac{m-1}{2}}e$, **then**

Player I chooses a move from $G(s)[V(G_2)]$ according to Strategy 1.18.

Otherwise,

Player I chooses to remove any edge from $G(s)$.

Theorem 2.2: *Let m and n be positive integers such that m and n have different parities. Then, $G_{m,n} \in \mathcal{W}_1$.*

Proof: WLOG, assume that m is odd and n is even.

By Theorem 1.16 (3), $G_{1,n} \in \mathcal{W}_1$.

Assume that $m \geq 3$. Let G_1 and G_2 be two subgraphs of $G_{m,n}$ as defined in Strategy 2.1. Then, $G_{m,n} = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, $|E(G_1)|$ is odd, and $|E(G_2)|$ is even. Let P be any play in $\mathbf{U}(G_{m,n})$ such that all of Player I's moves follow Strategy 2.1. Then, there are plays P_1 in $\mathbf{U}(G_1)$ and P_2 in $\mathbf{U}(G_2)$ such that the sequences of moves corresponding to P_1 and P_2 are subsequences of the sequence of moves corresponding to P. Moreover, for each edge e in $G_{m,n}$, the player who removes e in P is the same as the one who removes e in P_1, but different from the one who removes e in P_2. Consequently, all of Player I's moves in P_1 follow Strategy 1.20, and all of Player I's moves in P_2 follow Strategy 1.18.

Consider the vertices in $V(G_1)$. Clearly, $|V(G_1)| = n$. By Lemma 1.21, in P_1, Player I isolates at least $\frac{n}{2} + 1$ vertices. By Lemma 1.19, in P_2, Player II isolates all n vertices. Then, for each vertex $v \in V(G_1)$, if Player I isolates v in P_1, then Player I also isolates v in P, but if Player II isolates v in P_1, then either Player I or Player II isolates v in P depending on whether the last edge incident to v that is removed in P belongs to $E(G_1)$ or $E(G_2)$. This implies that, in P, Player I isolates at least $\frac{n}{2} + 1$ vertices.

Consider the vertices in $V(G_{m,n}) \backslash V(G_1)$. Clearly, $\left|V(G_{m,n}) \backslash V(G_1)\right| = (m-1)n$. By Lemma 1.19, in P_2, Player II isolates exactly $\frac{(m-1)n}{2}$ vertices. This implies that, in P, Player I isolates exactly $\frac{(m-1)n}{2}$ vertices.

Hence, $sc_1(P) \geq \frac{(m-1)n}{2} + \frac{n}{2} + 1 = \frac{mn}{2} + 1 > \frac{mn}{2} = \frac{|V(G_{m,n})|}{2} = \frac{sc_1(P)+sc_2(P)}{2}$.

Therefore, Strategy 2.1 is a winning strategy for Player I in $\mathbf{U}(G_{m,n})$. Consequently, $G_{m,n} \in \mathcal{W}_1$.

The following four theorems, classify the unions of two graphs under various assumptions. These are necessary for the proof of the second main result.

Theorem 2.3: *Let G and H be two vertex-disjoint graphs in which $|E(G)|$ and $|E(H)|$ are both even. If $G \in \mathcal{W}_2$ and either $H \in \mathcal{W}_2$ or $H \in \mathcal{D}$, then $G \cup H \in \mathcal{W}_2$.*

Proof: Assume that $G \in \mathcal{W}_2$ and either $H \in \mathcal{W}_2$ or $H \in \mathcal{D}$. Therefore, Player II has a winning strategy σ_1 in $\mathbf{U}(G)$, and either a winning or a drawing strategy σ_2 in $\mathbf{U}(H)$.

First, to construct a strategy for Player II in $\mathbf{U}(G \cup H)$, let σ be a strategy for Player II in which, for any given state s,

(1) if Player I's move from $par(s)$ to s is the removal of an edge in G, then Player II's move from s follows σ_1,

(2) if Player I's move from $par(s)$ to s is the removal of an edge in H, then Player II's move from s follows σ_2.

Then, Player II can make a play according to σ for all their moves, because G and H are vertex-disjoint, and $|E(G)|$ and $|E(H)|$ are both even.

Next, to show that σ is a winning strategy for Player II in $\mathbf{U}(G \cup H)$, let P be any play in $\mathbf{U}(G \cup H)$ such that all of Player II's moves follow σ. Then, there are plays P_1 in $\mathbf{U}(G)$ and P_2 in $\mathbf{U}(H)$ such that the sequences of moves corresponding to P_1 and P_2 are subsequences of the sequence of moves corresponding to P. Moreover, for each edge e in $G \cup H$, the player who removes e in P is the same as the one who removes e in either P_1 or P_2. Consequently, all of Player II's moves in P_1 follow σ_1, and all of Player II's moves in P_2 follow σ_2.

Due to the disjointness of G and H, for each vertex v in $G \cup H$, the player who isolates v in P is the same as the one who isolates v in either P_1 or P_2. Then, $sc_1(P) = sc_1(P_1) + sc_1(P_2)$ and $sc_2(P) = sc_2(P_1) + sc_2(P_2)$. The strategies σ_1 and σ_2 ensure that $sc_2(P_1) \geq sc_1(P_1) + 1$ and $sc_2(P_2) \geq sc_1(P_2)$, respectively. Hence, $sc_2(P) \geq sc_1(P) + 1 > sc_1(P)$.

Therefore, σ is a winning strategy for Player II in $\mathbf{U}(G \cup H)$. Consequently, $G \cup H \in \mathcal{W}_2$.

Theorem 2.4: *Let G and H be two vertex-disjoint graphs in which $|E(G)|$ is odd and $|E(H)|$ is even. If $G \in \mathcal{W}_1$ and either $H \in \mathcal{W}_2$ or $H \in \mathcal{D}$, then $G \cup H \in \mathcal{W}_1$.*

Proof: The proof is similar to that of Theorem 2.6.

Theorem 2.5: *Let G and H be two graphs that satisfy the following conditions.*

(1) $E(G) \cap E(H) = \emptyset$.

(2) $|V(G) \cap V(H)| \leq 1$.

(3) $|E(G)|$ and $|E(H)|$ are both even.

(4) $G \in \mathcal{W}_2$ and $H \in \mathcal{W}_2$.

Then, $G \cup H \in \mathcal{W}_2$.

Proof: Let v denote the unique vertex in $V(G) \cap V(H)$ if $|V(G) \cap V(H)| = 1$. By Theorem 2.3, the statement holds when $|V(G) \cap V(H)| = 0$, and also when v is isolated in G or H. Assume that $|V(G) \cap V(H)| = 1$, and v is neither isolated in G nor in H.

Since $G \in \mathcal{W}_2$ and $H \in \mathcal{W}_2$, Player II has winning strategies σ_1 and σ_2 in $\mathbf{U}(G)$ and $\mathbf{U}(H)$, respectively.

Let σ be a strategy for Player II in $\mathbf{U}(G \cup H)$ which is the same as that defined in the proof of Theorem 2.3. Then, Player II can make a play according to σ for all their moves, because $E(G) \cap E(H) = \emptyset$, and $|E(G)|$ and $|E(H)|$ are both even.

Let P be any play in $\mathbf{U}(G \cup H)$ such that all of Player II's moves follow σ. Then, there are plays P_1 in $\mathbf{U}(G)$ and P_2 in $\mathbf{U}(H)$ such that the sequences of moves corresponding to P_1 and P_2 are subsequences of the sequence of moves corresponding to P. Moreover, for each edge e in $G \cup H$, the player who removes e in P is the same as the one who removes e in either P_1 or P_2. Consequently, all of Player II's moves in P_1 follow σ_1, and all of Player II's moves in P_2 follow σ_2.

Since v is neither isolated in G nor in H, there exists a unique edge e such that the removal of e in either P_1 or P_2 isolates v, but the removal of e in P does not isolate v. Then, $sc_1(P) = sc_1(P_1) + sc_1(P_2) - 1$ and $sc_2(P) = sc_2(P_1) + sc_2(P_2)$ if the removal of e in P is Player I's, and $sc_1(P) = sc_1(P_1) + sc_1(P_2)$ and $sc_2(P) = sc_2(P_1) + sc_2(P_2) - 1$ if the removal of e in P is Player II's. The strategies σ_1 and σ_2 ensure that $sc_2(P_1) \geq sc_1(P_1) + 1$ and $sc_2(P_2) \geq sc_1(P_2) + 1$, respectively. Hence, $sc_2(P) \geq sc_1(P) + 3$ if the removal of e in P is Player I's, and $sc_2(P) \geq sc_1(P) + 1$ if the removal of e in P is Player II's. It follows that $sc_2(P) > sc_1(P)$.

Therefore, σ is a winning strategy for Player II in $\mathbf{U}(G \cup H)$. Consequently, $G \cup H \in \mathcal{W}_2$.

Theorem 2.6: *Let G and H be two graphs that satisfy the following conditions.*
(1) $E(G) \cap E(H) = \emptyset$.
(2) $|V(G) \cap V(H)| \leq 1$.
(3) $|E(G)|$ *is odd and* $|E(H)|$ *is even.*
(4) $G \in \mathcal{W}_1$ and $H \in \mathcal{W}_2$.
Then, $G \cup H \in \mathcal{W}_1$.

Proof: The proof is similar to that of Theorem 2.5.

Theorem 2.7: *Let n be a positive even integer. Then, L_n, α_{n-1}, β_n, γ_{n-1}, δ_n, ε_n and ζ_n belong to $\mathcal{W}_2$.*

Proof: Prove by induction on i that L_{2i}, α_{2i-1}, β_{2i}, γ_{2i-1}, δ_{2i}, ε_{2i} and ζ_{2i} belong to $\mathcal{W}_2$ for all $i \geq 1$.

Basis step: $i = 1$. The graphs L_2, α_1, β_2, γ_1, δ_2, ε_2 and ζ_2 are shown in Figure 4.

(A1) By Theorem 1.16 (2), $L_2 = G_{2,2} \in \mathcal{W}_2$.

(B1) By Theorem 1.16 (1), $\alpha_1 \cong G_{1,3} \in \mathcal{W}_2$.

(C1) Consider $\mathbf{U}(\beta_2)$. There are six options for Player I's first move. For each option of Player I, there exists an option for Player II's first move such that the number of vertices isolated by Player II is greater than or equal to the number of vertices isolated by Player I in the first move. These are shown in Figure 5.

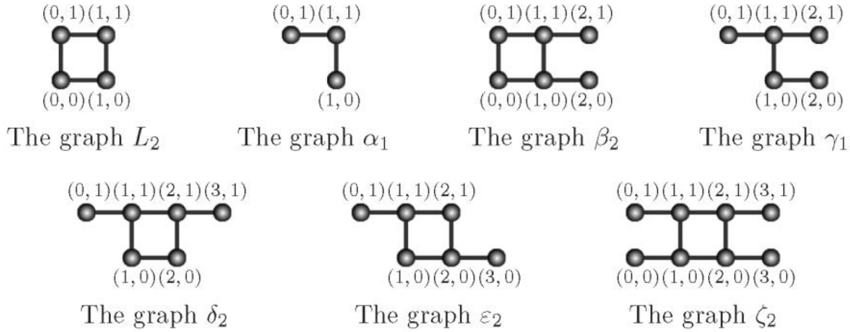

Figure 4
The graphs L_2, α_1, β_2, γ_1, δ_2, ε_2 and ζ_2

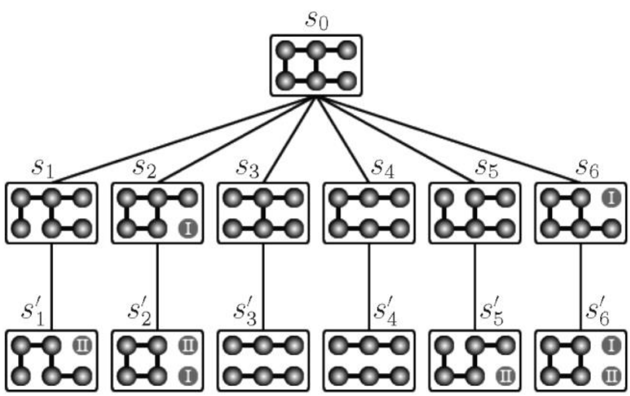

Figure 5
The first three levels of a strategy-tree for Player II in $\mathbf{U}(\beta_2)$

By Theorem 1.16 (1), $G(s'_1) \cong G(s'_5) \approx G_{1,5} \in \mathcal{W}_2$. By Theorems 1.16 (1) and 2.3, $G(s'_3) = G(s'_4) \in \mathcal{W}_2$. By **(A1)**, $G(s'_2) = G(s'_6) \approx G_{2,2} \in \mathcal{W}_2$. Hence, $\beta_2 \in \mathcal{W}_2$.

(D1) By Theorems 1.16 (1) and 2.5, because γ_1 is the union of two graphs isomorphic to $G_{1,3}$ sharing exactly one common vertex $(1,1)$, $\gamma_1 \in \mathcal{W}_2$.

(E1) Consider $\mathbf{U}(\delta_2)$. There are six options for Player I's first move. For each option of Player I, there exists an option for Player II's first move such that the number of vertices isolated by Player II is greater than or equal to the number of vertices isolated by Player I in the first move. These are shown in Figure 6.

By **(A1)**, $G(s'_1) = G(s'_3) \approx G_{2,2} \in \mathcal{W}_2$. By Theorems 1.16 (1) and 2.3, $G(s'_2) = G(s'_4) \in \mathcal{W}_2$. By Theorem 1.16 (1), $G(s'_5) \cong G(s'_6) \approx G_{1,5} \in \mathcal{W}_2$. Hence, $\delta_2 \in \mathcal{W}_2$.

(F1) By Lemma 1.19, because ε_2 has a planar drawing which is a line symmetric plane graph shown in Figure 7, $\varepsilon_2 \in \mathcal{W}_2$.

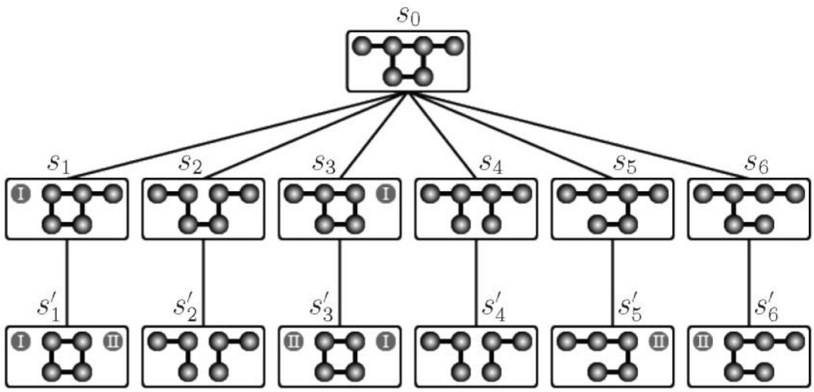

Figure 6
The first three levels of a strategy-tree for Player II in $\mathbf{U}(\delta_2)$

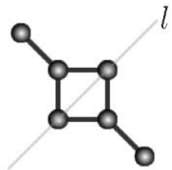

Figure 7
A line symmetric plane graph isomorphic to ε_2

(G1) Consider $\mathbf{U}(\zeta_2)$. There are eight options for Player I's first move. For each option of Player I, there exists an option for Player II's first move such that the number of vertices isolated by Player II is greater than or equal to the number of vertices isolated by Player I in the first move. These are shown in Figure 8.

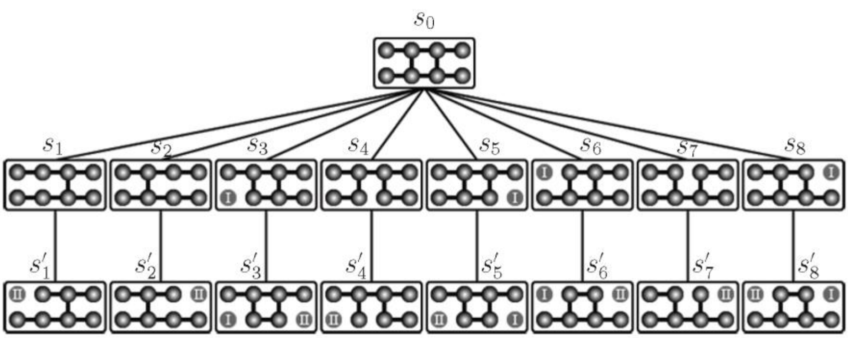

Figure 8
The first three levels of a strategy-tree for Player II in $\mathbf{U}(\zeta_2)$

By **(D1)** and Theorems 1.16 (1) and 2.5, $G(s'_1) = G(s'_2) \cong G(s'_4) = G(s'_7) \in \mathcal{W}_2$. By **(E1)**, $G(s'_3) = G(s'_5) = G(s'_6) = G(s'_8) \approx \delta_2 \in \mathcal{W}_2$. Hence, $\zeta_2 \in \mathcal{W}_2$.

Induction step: $i \geq 2$. Assume that L_{2j}, α_{2j-1}, β_{2j}, γ_{2j-1}, δ_{2j}, ε_{2j} and ζ_{2j} belong to $\mathcal{W}_2$ for all $j \in \{1,2,\ldots,i-1\}$. Let u_j and v_j denote the vertices $(j,0)$ and $(j,1)$ for all $j \in \{0,1,2,\ldots,2i+1\}$, respectively.

(A2) The graph L_{2i} is shown in Figure 9.

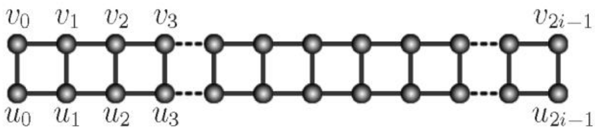

Figure 9
The ladder graph L_{2i}

Consider $\mathbf{U}(L_{2i})$. There are four cases, up to the symmetry of L_{2i}, for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{u_0v_0}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{2i-1}v_{2i-1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is ζ_{2i-2}, shown in Figure 10.

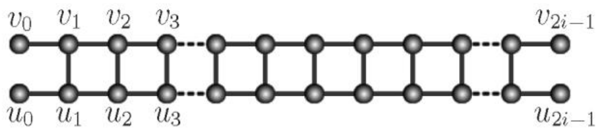

Figure 10
The graph $L_{2i} - \{\overline{u_0v_0}, \overline{u_{2i-1}v_{2i-1}}\}$

By the induction hypothesis, $L_{2i} - \{\overline{u_0v_0}, \overline{u_{2i-1}v_{2i-1}}\} = \zeta_{2i-2} \in \mathcal{W}_2$.

Case 2: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{1,3,5,\ldots,2i-3\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0-1}u_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of a graph isomorphic to α_{i_0} and a graph isomorphic to β_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 11.

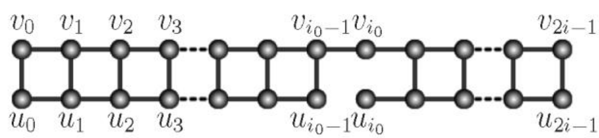

Figure 11
The graph $L_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\}$, $i_0 \in \{1, 3, 5, \ldots, 2i-3\}$

By the induction hypothesis, both α_{i_0} and β_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $L_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\} \in \mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{i_0-1}u_{i_0}}$, $i_0 \in \{1,3,5,\ldots,2i-3\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0}v_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of a graph isomorphic to α_{i_0} and a graph isomorphic to β_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 12.

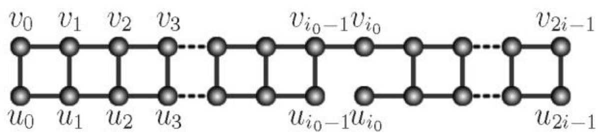

Figure 12
The graph $L_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{u_{i_0}v_{i_0}}\}$, $i_0 \in \{1,3,5,\ldots,2i-3\}$

By the induction hypothesis, both α_{i_0} and β_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $L_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{u_{i_0}v_{i_0}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of $\overline{u_{i_0-1}u_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_{i_0-1}v_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the disjoint union of L_{i_0} and a graph isomorphic to L_{2i-i_0}, shown in Figure 13.

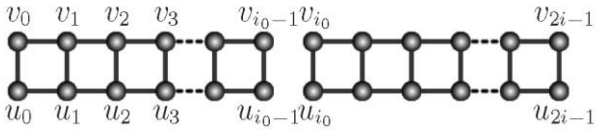

Figure 13
The graph $L_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_{i_0-1}v_{i_0}}\}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$

By the induction hypothesis, both L_{i_0} and L_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.3, $L_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_{i_0-1}v_{i_0}}\} \in \mathcal{W}_2$.

Consequently, $L_{2i} \in \mathcal{W}_2$.

(B2) The graph α_{2i-1} is shown in Figure 14.

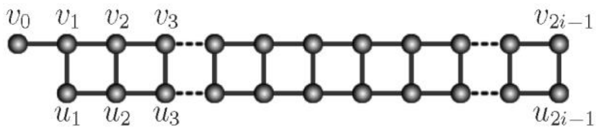

Figure 14
The graph α_{2i-1}

Consider $\mathbf{U}(\alpha_{2i-1})$. There are six cases for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{v_0v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_iv_i}$. Then, Player I's first move isolates v_0, Player II's first move isolates no vertices, and the resulting graph is the disjoint union of a trivial graph and a line symmetric plane graph containing exactly two vertices lying on a line of symmetry $x = i$, shown in Figure 15.

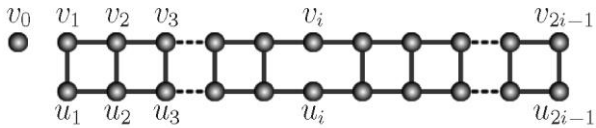

Figure 15
The graph $\alpha_{2i-1} - \{\overline{v_0v_1}, \overline{u_iv_i}\}$

Now, Player II can play according to Strategy 1.18. By Lemma 1.19, Player II is the player who isolates u_i, v_i and exactly half of the other vertices excluding v_0, u_i or v_i. Hence, in this case, the strategy in which Player II remove $\overline{u_iv_i}$ for their first move and play according to Strategy 1.18 for all their other moves is a winning strategy for Player II.

Case 2: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{1,2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and a graph isomorphic to β_{2i-2}, shown in Figures 16 and 17.

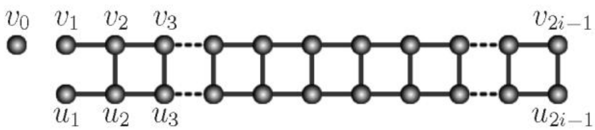

Figure 16
The graph $\alpha_{2i-1} - \{\overline{u_1v_1}, \overline{v_0v_1}\}$

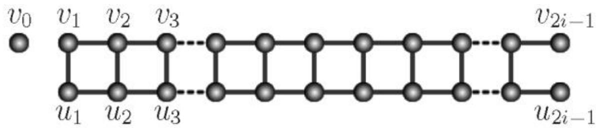

Figure 17
The graph $\alpha_{2i-1} - \{\overline{u_{2i-1}v_{2i-1}}, \overline{v_0v_1}\}$

By the induction hypothesis, $\alpha_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{v_0v_1}\} \approx \beta_{2i-2} \in \mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0}u_{i_0+1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of γ_{i_0-1} and a graph isomorphic to α_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 18.

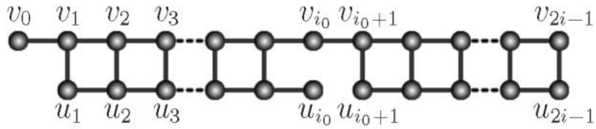

Figure 18
The graph $\alpha_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$

By the induction hypothesis, both γ_{i_0-1} and α_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\alpha_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{3,5,7,\ldots,2i-3\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0-1}u_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of δ_{i_0-1} and a graph isomorphic to β_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 19.

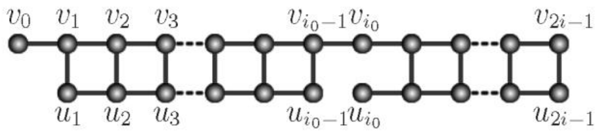

Figure 19
The graph $\alpha_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\}$, $i_0 \in \{3,5,7,\ldots,2i-3\}$

By the induction hypothesis, both δ_{i_0-1} and β_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\alpha_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\} \in \mathcal{W}_2$.

Case 5: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to α_{i_0-1} and a graph isomorphic to L_{2i-i_0}, sharing exactly one common vertex, shown in Figures 20 and 21.

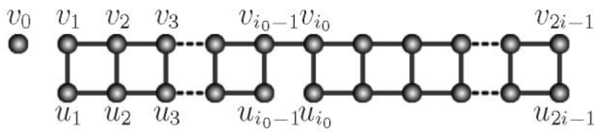

Figure 20

The graph $\alpha_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{2, 4, 6, \ldots, 2i-2\}$

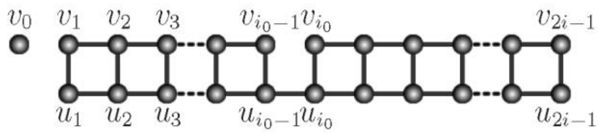

Figure 21

The graph $\alpha_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{2, 4, 6, \ldots, 2i-2\}$

By the induction hypothesis, both α_{i_0-1} and L_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\alpha_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \cong \alpha_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Case 6: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{3,5,7,\ldots,2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to L_{i_0-1} and a graph isomorphic to α_{2i-i_0}, sharing exactly one common vertex, shown in Figures 22 and 23.

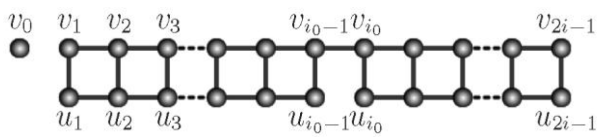

Figure 22

The graph $\alpha_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{3, 5, 7, \ldots, 2i-1\}$

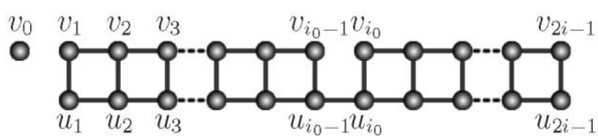

Figure 23

The graph $\alpha_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{3, 5, 7, \ldots, 2i-1\}$

By the induction hypothesis, both L_{i_0-1} and α_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\alpha_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \cong \alpha_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Consequently, $\alpha_{2i-1} \in \mathcal{W}_2$.

(C2) The graph β_{2i} is shown in Figure 24.

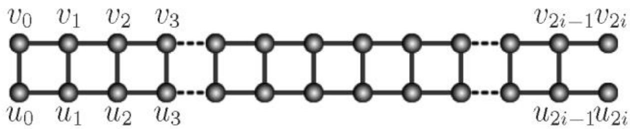

Figure 24
The graph β_{2i}

Consider $\mathbf{U}(\beta_{2i})$. There are eight cases, up to the symmetry of β_{2i}, for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{u_{2i-1}u_{2i}}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_{2i-1}v_{2i}}$. Then, the first move of each player isolates 1 vertex, and the resulting graph is the disjoint union of two trivial graphs and L_{2i}, shown in Figure 25.

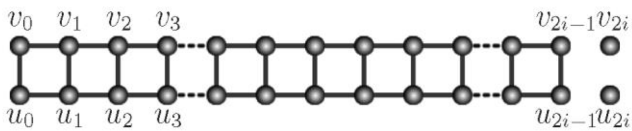

Figure 25
The graph $\beta_{2i} - \{\overline{u_{2i-1}u_{2i}}, \overline{v_{2i-1}v_{2i}}\}$

By **(A2)**, $\beta_{2i} - \{\overline{u_{2i-1}u_{2i}}, \overline{v_{2i-1}v_{2i}}\} \approx L_{2i} \in \mathcal{W}_2$.

Case 2: Player I's first move is the removal of $\overline{u_0v_0}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_1u_2}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of a graph isomorphic to γ_1 and a graph isomorphic to β_{2i-2}, sharing exactly one common vertex, shown in Figure 26.

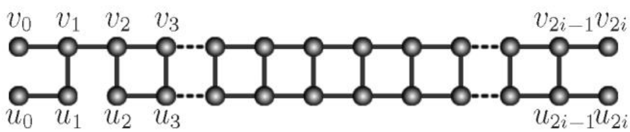

Figure 26
The graph $\beta_{2i} - \{\overline{u_0v_0}, \overline{u_1u_2}\}$

By the induction hypothesis, both γ_1 and β_{2i-2} belong to $\mathcal{W}_2$. By Theorem 2.5, $\beta_{2i} - \{\overline{u_0v_0}, \overline{u_1u_2}\} \in \mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{2i-1}v_{2i-1}}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{2i-1}u_{2i}}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to α_{2i-1} and a graph isomorphic to $G_{1,3}$, sharing exactly one common vertex, shown in Figure 27.

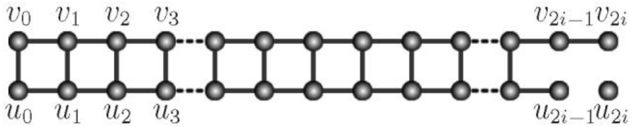

Figure 27
The graph $\boldsymbol{\beta_{2i} - \{\overline{u_{2i-1}v_{2i-1}}, \overline{u_{2i-1}u_{2i}}\}}$

By **(B2)** and Theorem 1.16 (1), both α_{2i-1} and $G_{1,3}$ belong to $\mathcal{W}_2$. By Theorem 2.5, $\beta_{2i} - \{\overline{u_{2i-1}v_{2i-1}}, \overline{u_{2i-1}u_{2i}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{1,3,5,\ldots,2i-3\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0-1}u_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of a graph isomorphic to α_{i_0} and a graph isomorphic to ζ_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 28.

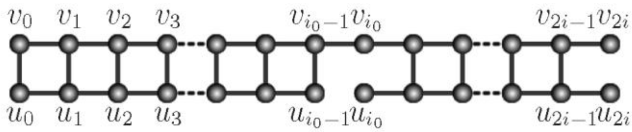

Figure 28
The graph $\boldsymbol{\beta_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\}}$, $\boldsymbol{i_0 \in \{1,3,5,\ldots,2i-3\}}$

By the induction hypothesis, both α_{i_0} and ζ_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\beta_{2i} - \left\{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\right\} \in \mathcal{W}_2$.

Case 5: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0}u_{i_0+1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of β_{i_0} and a graph isomorphic to γ_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 29.

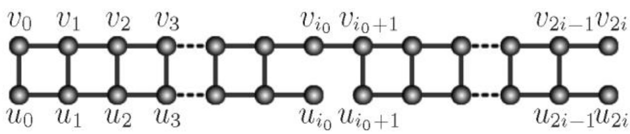

Figure 29
The graph $\boldsymbol{\beta_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\}}$, $\boldsymbol{i_0 \in \{2,4,6,\ldots,2i-2\}}$

By the induction hypothesis, both β_{i_0} and γ_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\beta_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\} \in \mathcal{W}_2$.

Case 6: Player I's first move is the removal of $\overline{u_0u_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_1v_1}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of a graph isomorphic to α_1 and a graph isomorphic to ζ_{2i-2}, sharing exactly one common vertex, shown in Figure 30.

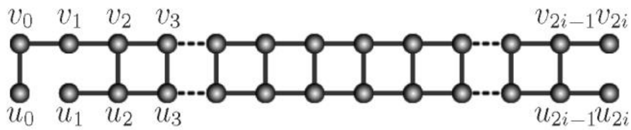

Figure 30
The graph $\boldsymbol{\beta_{2i} - \{\overline{u_0u_1}, \overline{u_1v_1}\}}$

By the induction hypothesis, both α_1 and ζ_{2i-2} belong to $\mathcal{W}_2$. By Theorem 2.5, $\beta_{2i} - \{\overline{u_0u_1}, \overline{u_1v_1}\} \in \mathcal{W}_2$.

Case 7: Player I's first move is the removal of $\overline{u_{i_0-1}u_{i_0}}$, $i_0 \in \{2,4,6,\dots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_{i_0-1}v_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the disjoint union of L_{i_0} and a graph isomorphic to β_{2i-i_0}, shown in Figure 31.

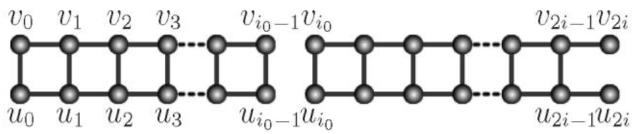

Figure 31
The graph $\boldsymbol{\beta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_{i_0-1}v_{i_0}}\},\ i_0 \in \{2,4,6,\dots,2i-2\}}$

By the induction hypothesis, both L_{i_0} and β_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.3, $\beta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_{i_0-1}v_{i_0}}\} \in \mathcal{W}_2$.

Case 8: Player I's first move is the removal of $\overline{u_{i_0-1}u_{i_0}}$, $i_0 \in \{3,5,7,\dots,2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0-1}v_{i_0-1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of β_{i_0-1} and a graph isomorphic to γ_{2i-i_0}, sharing exactly one common vertex, shown in Figure 32.

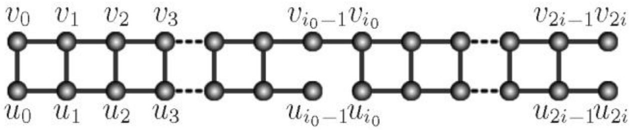

Figure 32

The graph $\boldsymbol{\beta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{u_{i_0-1}v_{i_0-1}}\}}$, $\boldsymbol{i_0 \in \{3, 5, 7, \ldots, 2i-1\}}$

By the induction hypothesis, both β_{i_0-1} and γ_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\beta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{u_{i_0-1}v_{i_0-1}}\} \in \mathcal{W}_2$.

Consequently, $\beta_{2i} \in \mathcal{W}_2$.

(D2) The graph γ_{2i-1} is shown in Figure 33.

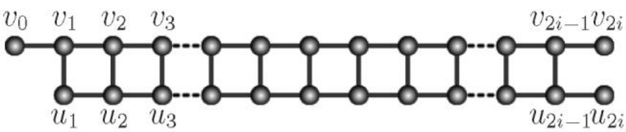

Figure 33

The graph $\boldsymbol{\gamma_{2i-1}}$

Consider $\mathbf{U}(\gamma_{2i-1})$. There are seven cases for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{v_0v_1}$, $\overline{u_{2i-1}u_{2i}}$ or $\overline{v_{2i-1}v_{2i}}$. In this case, there exists a Player II's first move which is the removal of one of edges incident to leaves, different from Player I's first move. Then, the first move of each player isolates 1 vertex, and the resulting graph is the disjoint union of two trivial graphs and a graph isomorphic to α_{2i-1}, shown in Figures 34, 35 and 36.

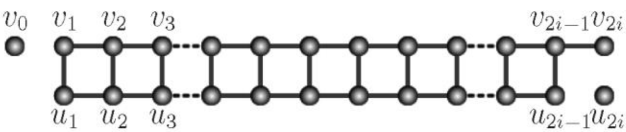

Figure 34

The graph $\boldsymbol{\gamma_{2i-1} - \{\overline{v_0v_1}, \overline{u_{2i-1}u_{2i}}\}}$

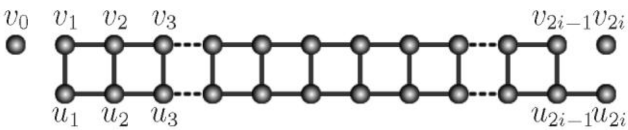

Figure 35

The graph $\boldsymbol{\gamma_{2i-1} - \{\overline{v_0v_1}, \overline{v_{2i-1}v_{2i}}\}}$

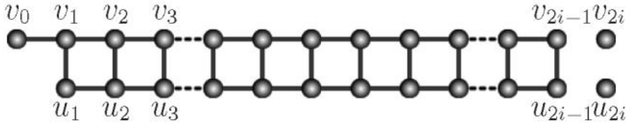

Figure 36
The graph $\gamma_{2i-1} - \{\overline{u_{2i-1}u_{2i}}, \overline{v_{2i-1}v_{2i}}\}$

By **(B2)**, $\gamma_{2i-1} - \{\overline{v_0v_1}, \overline{u_{2i-1}u_{2i}}\} \cong \gamma_{2i-1} - \{\overline{v_0v_1}, \overline{v_{2i-1}v_{2i}}\} \cong \gamma_{2i-1} - \{\overline{u_{2i-1}u_{2i}}, \overline{v_{2i-1}v_{2i}}\} \approx \alpha_{2i-1} \in \mathcal{W}_2$.

Case 2: Player I's first move is the removal of $\overline{u_1v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and a graph isomorphic to ζ_{2i-2}, shown in Figure 37.

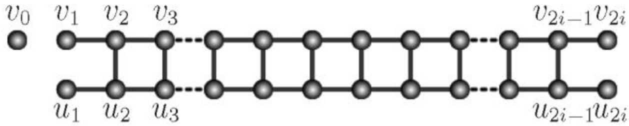

Figure 37
The graph $\gamma_{2i-1} - \{\overline{u_1v_1}, \overline{v_0v_1}\}$

By the induction hypothesis, $\gamma_{2i-1} - \{\overline{u_1v_1}, \overline{v_0v_1}\} \approx \zeta_{2i-2} \in \mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{2i-1}v_{2i-1}}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{2i-1}u_{2i}}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of ε_{2i-2} and a graph isomorphic to $G_{1,3}$, sharing exactly one common vertex, shown in Figure 38.

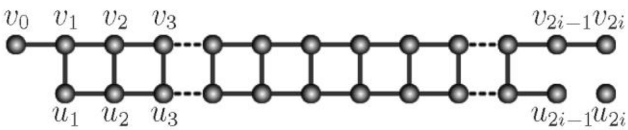

Figure 38
The graph $\gamma_{2i-1} - \{\overline{u_{2i-1}v_{2i-1}}, \overline{u_{2i-1}u_{2i}}\}$

By the induction hypothesis and Theorem 1.16 (1), both ε_{2i-2} and $G_{1,3}$ belong to $\mathcal{W}_2$. By Theorem 2.5, $\gamma_{2i-1} - \{\overline{u_{2i-1}v_{2i-1}}, \overline{u_{2i-1}u_{2i}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0}u_{i_0+1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of γ_{i_0-1} and a graph isomorphic to γ_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 39.

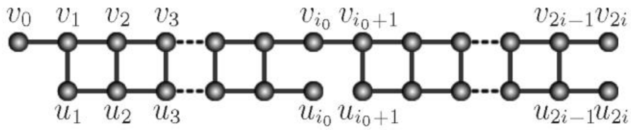

Figure 39
The graph $\gamma_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$

By the induction hypothesis, both γ_{i_0-1} and γ_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\gamma_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\} \in \mathcal{W}_2$.

Case 5: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{3,5,7,\ldots,2i-3\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0-1}u_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of δ_{i_0-1} and a graph isomorphic to ζ_{2i-i_0-1}, sharing exactly one common vertex, shown in Figure 40.

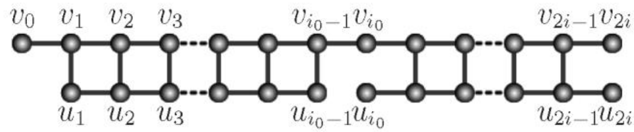

Figure 40
The graph $\gamma_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\}$, $i_0 \in \{3,5,7,\ldots,2i-3\}$

By the induction hypothesis, both δ_{i_0-1} and ζ_{2i-i_0-1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\gamma_{2i-1} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\} \in \mathcal{W}_2$.

Case 6: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to α_{i_0-1} and a graph isomorphic to β_{2i-i_0}, sharing exactly one common vertex, shown in Figures 41 and 42.

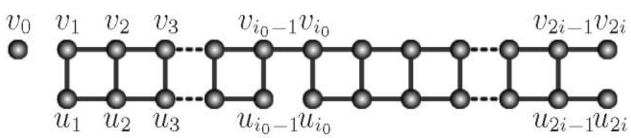

Figure 41
The graph $\gamma_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$

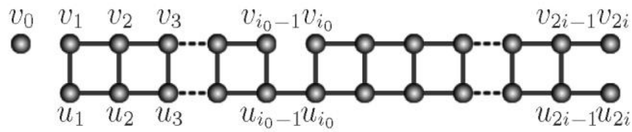

Figure 42
The graph $\gamma_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{2, 4, 6, \ldots, 2i-2\}$

By the induction hypothesis, both α_{i_0-1} and β_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\gamma_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \cong \gamma_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Case 7: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{3, 5, 7, \ldots, 2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to L_{i_0-1} and a graph isomorphic to γ_{2i-i_0}, sharing exactly one common vertex, shown in Figures 43 and 44.

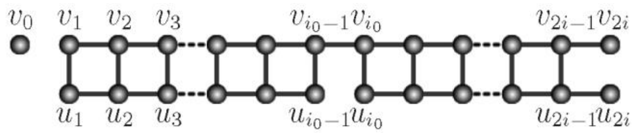

Figure 43
The graph $\gamma_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{3, 5, 7, \ldots, 2i-1\}$

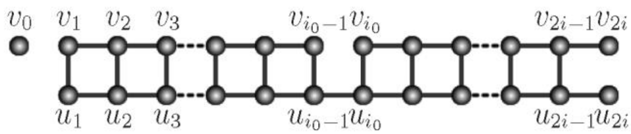

Figure 44
The graph $\gamma_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{3, 5, 7, \ldots, 2i-1\}$

By the induction hypothesis, both L_{i_0-1} and γ_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\gamma_{2i-1} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \cong \gamma_{2i-1} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Consequently, $\gamma_{2i-1} \in \mathcal{W}_2$.

(E2) The graph δ_{2i} is shown in Figure 45.

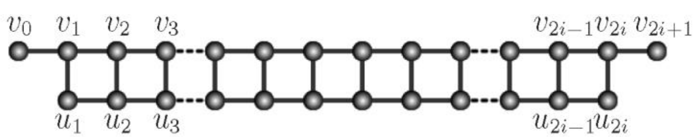

Figure 45
The graph δ_{2i}

Consider $\mathbf{U}(\delta_{2i})$. There are five cases, up to the symmetry of δ_{2i}, for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{v_0v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_{2i}v_{2i+1}}$. Then, the first move of each player isolates 1 vertex, and the resulting graph is the disjoint union of two trivial graphs and a graph isomorphic to L_{2i}, shown in Figure 46.

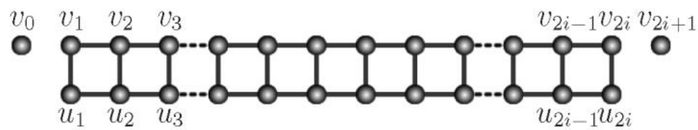

Figure 46
The graph $\delta_{2i} - \{\overline{v_0v_1}, \overline{v_{2i}v_{2i+1}}\}$

By **(A2)**, $\delta_{2i} - \{\overline{v_0v_1}, \overline{v_{2i}v_{2i+1}}\} \approx L_{2i} \in \mathcal{W}_2$.

Case 2: Player I's first move is the removal of $\overline{u_1v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and a graph isomorphic to γ_{2i-1}, shown in Figure 47.

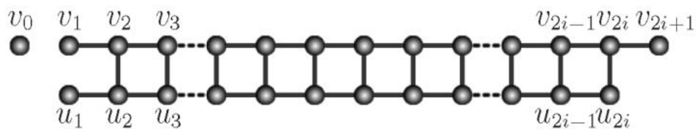

Figure 47
The graph $\delta_{2i} - \{\overline{u_1v_1}, \overline{v_0v_1}\}$

By **(D2)**, $\delta_{2i} - \{\overline{u_1v_1}, \overline{v_0v_1}\} \approx \gamma_{2i-1} \in \mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0}u_{i_0+1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of γ_{i_0-1} and a graph isomorphic to δ_{2i-i_0}, sharing exactly one common vertex, shown in Figure 48.

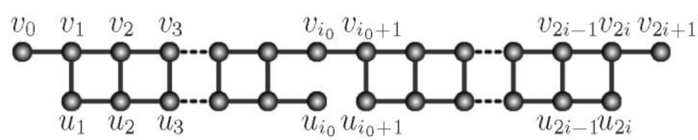

Figure 48
The graph $\delta_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$

By the induction hypothesis, both γ_{i_0-1} and δ_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\delta_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{2, 4, 6, \ldots, 2i\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to α_{i_0-1} and a graph isomorphic to α_{2i-i_0+1}, sharing exactly one common vertex, shown in Figures 49 and 50.

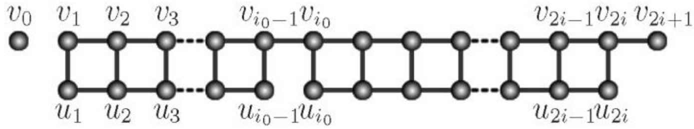

Figure 49
The graph $\boldsymbol{\delta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}}$, $\boldsymbol{i_0 \in \{2, 4, 6, \ldots, 2i\}}$

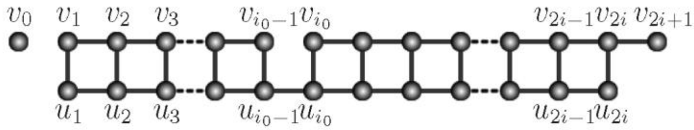

Figure 50
The graph $\boldsymbol{\delta_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}}$, $\boldsymbol{i_0 \in \{2, 4, 6, \ldots, 2i\}}$

By **(B2)** and the induction hypothesis, both α_{i_0-1} and α_{2i-i_0+1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\delta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$ and $\delta_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Case 5: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{3,5,7,\ldots,2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to L_{i_0-1} and a graph isomorphic to either δ_{2i-i_0+1} or ε_{2i-i_0+1}, sharing exactly one common vertex, shown in Figures 51 and 52.

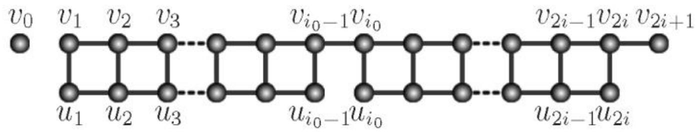

Figure 51
The graph $\boldsymbol{\delta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}}$, $\boldsymbol{i_0 \in \{3, 5, 7, \ldots, 2i-1\}}$

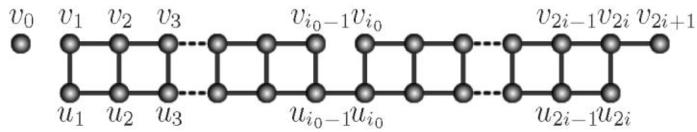

Figure 52
The graph $\delta_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{3, 5, 7, \ldots, 2i-1\}$

By the induction hypothesis, all L_{i_0-1}, δ_{2i-i_0+1} and ε_{2i-i_0+1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\delta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$ and $\delta_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Consequently, $\delta_{2i} \in \mathcal{W}_2$.

(F2) The graph ε_{2i} is shown in Figure 53.

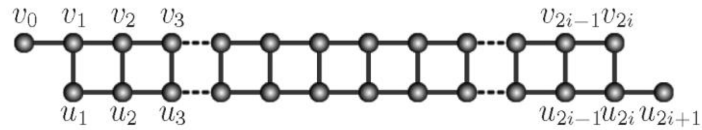

Figure 53
The graph ε_{2i}

Consider $\mathbf{U}(\varepsilon_{2i})$. There are five cases, up to the symmetry of ε_{2i}, for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{v_0v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{2i}u_{2i+1}}$. Then, the first move of each player isolates 1 vertex, and the resulting graph is the disjoint union of two trivial graphs and a graph isomorphic to L_{2i}, shown in Figure 54.

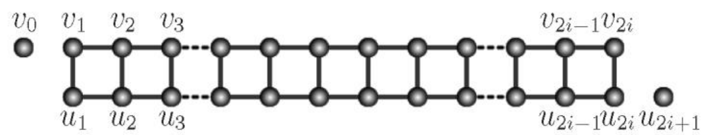

Figure 54
The graph $\varepsilon_{2i} - \{\overline{v_0v_1}, \overline{u_{2i}u_{2i+1}}\}$

By **(A2)**, $\varepsilon_{2i} - \{\overline{v_0v_1}, \overline{u_{2i}u_{2i+1}}\} \approx L_{2i} \in \mathcal{W}_2$.

Case 2: Player I's first move is the removal of $\overline{u_1v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and a graph isomorphic to γ_{2i-1}, shown in Figure 55.

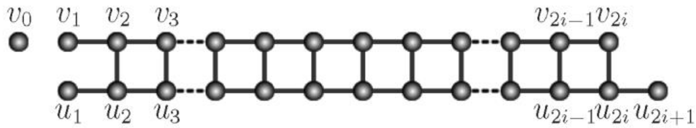

Figure 55
The graph $\varepsilon_{2i} - \{\overline{u_1v_1}, \overline{v_0v_1}\}$

By **(D2)**, $\varepsilon_{2i} - \{\overline{u_1v_1}, \overline{v_0v_1}\} \approx \gamma_{2i-1} \in \mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0}u_{i_0+1}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of γ_{i_0-1} and a graph isomorphic to ε_{2i-i_0}, sharing exactly one common vertex, shown in Figure 56.

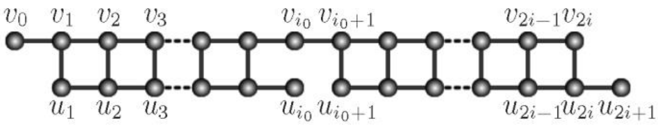

Figure 56
The graph $\varepsilon_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\}$, $i_0 \in \{2,4,6,\ldots,2i-2\}$

By the induction hypothesis, both γ_{i_0-1} and ε_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\varepsilon_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0}u_{i_0+1}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{2,4,6,\ldots,2i\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to α_{i_0-1} and a graph isomorphic to α_{2i-i_0+1}, sharing exactly one common vertex, shown in Figures 57 and 58.

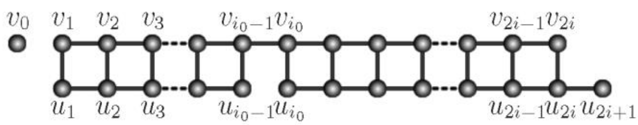

Figure 57
The graph $\varepsilon_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{2,4,6,\ldots,2i\}$

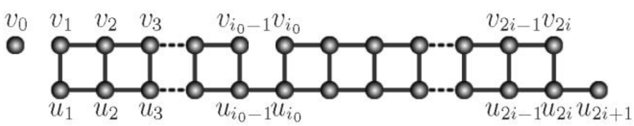

Figure 58
The graph $\varepsilon_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}$, $i_0 \in \{2,4,6,\ldots,2i\}$

By **(B2)** and the induction hypothesis, both α_{i_0-1} and α_{2i-i_0+1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\varepsilon_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$ and $\varepsilon_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Case 5: Player I's first move is the removal of either $\overline{u_{i_0-1}u_{i_0}}$ or $\overline{v_{i_0-1}v_{i_0}}$, $i_0 \in \{3,5,7,\ldots,2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_0v_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to L_{i_0-1} and a graph isomorphic to either δ_{2i-i_0+1} or ε_{2i-i_0+1}, sharing exactly one common vertex, shown in Figures 59 and 60.

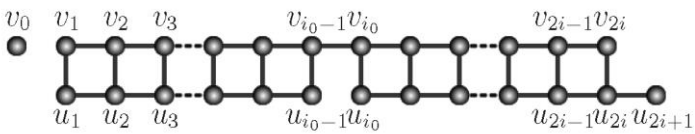

Figure 59
The graph $\boldsymbol{\varepsilon_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\}, i_0 \in \{3,5,7,\ldots,2i-1\}}$

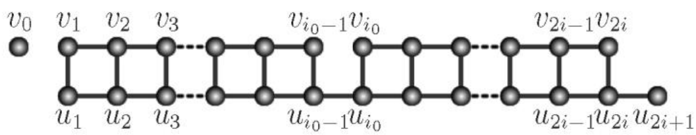

Figure 60
The graph $\boldsymbol{\varepsilon_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\}, i_0 \in \{3,5,7,\ldots,2i-1\}}$

By the induction hypothesis, all L_{i_0-1}, δ_{2i-i_0+1} and ε_{2i-i_0+1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\varepsilon_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$ and $\varepsilon_{2i} - \{\overline{v_{i_0-1}v_{i_0}}, \overline{v_0v_1}\} \in \mathcal{W}_2$.

Consequently, $\varepsilon_{2i} \in \mathcal{W}_2$.

(G2) The graph ζ_{2i} is shown in Figure 61.

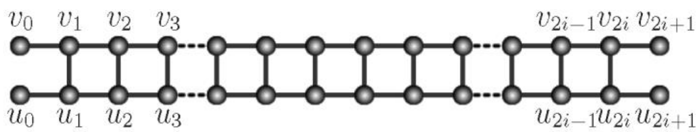

Figure 61
The graph $\boldsymbol{\zeta_{2i}}$

Consider $\mathbf{U}(\zeta_{2i})$. There are five cases, up to the symmetry of ζ_{2i}, for Player I's first move.

Case 1: Player I's first move is the removal of $\overline{u_0u_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{2i}u_{2i+1}}$. Then, the first move of each player isolates 1 vertex, and the resulting graph is the disjoint union of two trivial graphs and δ_{2i}, shown in Figure 62.

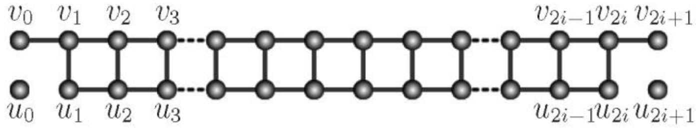

Figure 62
The graph $\zeta_{2i}-\{\overline{u_0u_1},\overline{u_{2i}u_{2i+1}}\}$

By **(E2)**, $\zeta_{2i}-\{\overline{u_0u_1},\overline{u_{2i}u_{2i+1}}\}\approx\delta_{2i}\in\mathcal{W}_2$.

Case 2: Player I's first move is the removal of $\overline{u_1v_1}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_0u_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of a graph isomorphic to γ_{2i-1} and a graph isomorphic to $G_{1,3}$, sharing exactly one common vertex, shown in Figure 63.

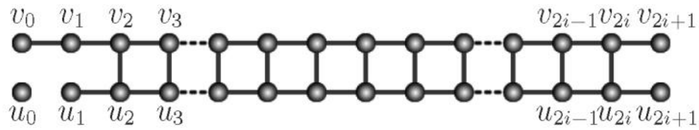

Figure 63
The graph $\zeta_{2i}-\{\overline{u_1v_1},\overline{u_0u_1}\}$

By **(D2)** and Theorem 1.16 (1), both γ_{2i-1} and $G_{1,3}$ belong to $\mathcal{W}_2$. By Theorem 2.5, $\zeta_{2i}-\{\overline{u_1v_1},\overline{u_0u_1}\}\in\mathcal{W}_2$.

Case 3: Player I's first move is the removal of $\overline{u_{i_0}v_{i_0}}$, $i_0\in\{2,4,6,\ldots,2i-2\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_{i_0-1}u_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the union of a graph isomorphic to γ_{i_0-1} and a graph isomorphic to ζ_{2i-i_0}, sharing exactly one common vertex, shown in Figure 64.

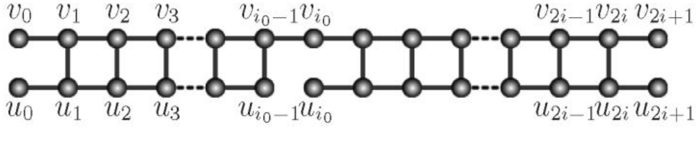

Figure 64
The graph $\zeta_{2i}-\{\overline{u_{i_0}v_{i_0}},\overline{u_{i_0-1}u_{i_0}}\}$, $i_0\in\{2,4,6,\ldots,2i-2\}$

By the induction hypothesis, both γ_{i_0-1} and ζ_{2i-i_0} belong to $\mathcal{W}_2$. By Theorem 2.5, $\zeta_{2i} - \{\overline{u_{i_0}v_{i_0}}, \overline{u_{i_0-1}u_{i_0}}\} \in \mathcal{W}_2$.

Case 4: Player I's first move is the removal of $\overline{u_{i_0-1}u_{i_0}}$, $i_0 \in \{2,4,6,\dots,2i\}$. In this case, there exists a Player II's first move which is the removal of $\overline{u_0u_1}$. Then, Player I's first move isolates no vertices, Player II's first move isolates 1 vertex, and the resulting graph is the disjoint union of a trivial graph and the union of α_{i_0-1} and a graph isomorphic to γ_{2i-i_0+1}, sharing exactly one common vertex, shown in Figure 65.

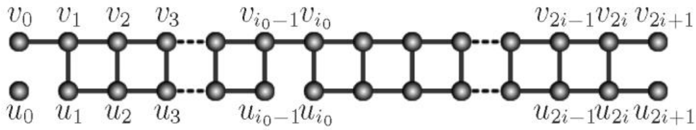

Figure 65
The graph $\zeta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{u_0u_1}\}$, $i_0 \in \{2,4,6,\dots,2i\}$

By **(B2)**, **(D2)** and the induction hypothesis, both α_{i_0-1} and γ_{2i-i_0+1} belong to $\mathcal{W}_2$. By Theorem 2.5, $\zeta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{u_0u_1}\} \in \mathcal{W}_2$.

Case 5: Player I's first move is the removal of $\overline{u_{i_0-1}u_{i_0}}$, $i_0 \in \{3,5,7,\dots,2i-1\}$. In this case, there exists a Player II's first move which is the removal of $\overline{v_{i_0-1}v_{i_0}}$. Then, the first move of each player isolates no vertices, and the resulting graph is the disjoint union of β_{i_0-1} and a graph isomorphic to β_{2i-i_0+1}, shown in Figure 66.

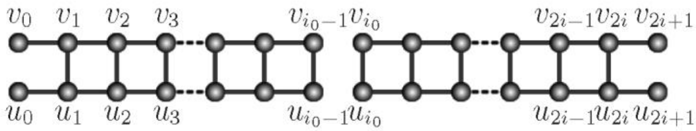

Figure 66
The graph $\zeta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_{i_0-1}v_{i_0}}\}$, $i_0 \in \{3,5,7,\dots,2i-1\}$

By the induction hypothesis, both β_{i_0-1} and β_{2i-i_0+1} belong to $\mathcal{W}_2$. By Theorem 2.3, $\zeta_{2i} - \{\overline{u_{i_0-1}u_{i_0}}, \overline{v_{i_0-1}v_{i_0}}\} \in \mathcal{W}_2$.

Consequently, $\zeta_{2i} \in \mathcal{W}_2$.

Corollary 2.8: *Let n be a positive odd integer. Then, L_n, α_{n+1}, β_n, γ_{n+1}, δ_n, ε_n and ζ_n belong to $\mathcal{W}_1$.*

Proof: Because n is odd, $n+1$ is even.

(A) By Theorem 1.16 (4), $L_n \in \mathcal{W}_1$.

(B) By Theorem 2.7, $\alpha_{n+1} - \overline{v_0 v_1} \approx L_{n+1} \in \mathcal{W}_2$. By Lemma 1.10, $\alpha_{n+1} \in \mathcal{W}_1$.

(C) By Theorem 2.7, $\beta_n - \overline{u_{n-1} u_n} \approx \alpha_n \in \mathcal{W}_2$. By Lemma 1.10, $\beta_n \in \mathcal{W}_1$.

(D) By Theorem 2.7, $\gamma_{n+1} - \overline{v_0 v_1} \approx \beta_{n+1} \in \mathcal{W}_2$. By Lemma 1.10, $\gamma_{n+1} \in \mathcal{W}_1$.

(E) By Theorem 2.7, $\delta_n - \overline{v_0 v_1} \approx \alpha_n \in \mathcal{W}_2$. By Lemma 1.10, $\delta_n \in \mathcal{W}_1$.

(F) By Theorem 2.7, $\varepsilon_n - \overline{v_0 v_1} \approx \alpha_n \in \mathcal{W}_2$. By Lemma 1.10, $\varepsilon_n \in \mathcal{W}_1$.

(G) By Theorem 2.7, $\zeta_n - \overline{u_0 u_1} \approx \gamma_n \in \mathcal{W}_2$. By Lemma 1.10, $\zeta_n \in \mathcal{W}_1$.

3. Conclusion and Discussion

This article extends the results of [5] by determining the outcomes of the Unshackle games on grid graphs in some remaining cases.

The first outcome concerns some cases of grid graphs, including ladder graphs $L_n = G_{n,2}$, which is derived from the results in [5], Theorem 2.2 and 2.7.

Outcome 1: Let m and n be positive integers such that $(m, n) \neq (1,1)$.

m	n	Condition	Class of $G_{m,n}$
odd	even	any	$\mathcal{W}_1$
even	odd	any	$\mathcal{W}_1$
odd	odd	any	$\mathcal{W}_2$
even	even	$m = n$ or $m = 2$ or $n = 2$	$\mathcal{W}_2$
even	even	Otherwise	$\mathcal{W}_2$ or $\mathcal{D}$

The outcome for the cases where $m \geq 4$ and $n \geq 4$ are both even with $m \neq n$ remain undetermined. However, this article poses the conjecture that if $m \geq 4$ and $n \geq 4$ are both even and $m \neq n$, then $G_{m,n} \in \mathcal{W}_2$.

The second outcome concerns all cases of ladder-like graphs, which is derived from Theorem 2.7 and Corollary 2.8.

Outcome 2: Let n be a positive integer.

k	α_k	β_k	γ_k	δ_k	ε_k	ζ_k
odd	$\mathcal{W}_2$	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_1$	$\mathcal{W}_1$	$\mathcal{W}_1$
even	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_2$	$\mathcal{W}_2$

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