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Triple connected certified domination number of triangular grid with boundaries

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Abstract

A dominating set S in a graph G is termed a Triple Connected Certified Dominating Set (TCCD-set) if, for each vertex v in S, either the number of neighbors of v in $V - S$ is zero or k where $k \geq 2$, and any three vertices of S are positioned on a path within the subgraph induced by S. The minimum size of a TCCD-set is known as the Triple connected certified domination Number (TCCD-number) and is symbolized as $\gamma_{TCC}(G)$. This study explores the TCCD number for triangular grid of rectangular and square border.

Subject Classification (2020): *05C38, 05C69.*

Keywords: *Triple connected, Triple connected certified domination number, Certified domination, Triangular grid, Boundaries.*

1. Introduction

This article delves into various graphs, including finite, non-trivial, and simple ones. The concept of triple connected graphs was initially introduced by Joseph et al. [1], followed by the introduction of the triple connected domination number [2]. A dominating set S is termed a triple connected dominating set if the

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subgraph induced by S is triple connected. The minimum size of such a set is called the triple connected domination number, denoted by $\gamma_{tc}(G)$. A dominating set S is considered as a certified dominating set if, for each vertex v in S, the number of neighbors of v in $V - S$ is either zero or k, where k is greater than or equal to 2. The minimum size among all certified dominating sets is the certified domination number, denoted by $\gamma_{cer}(G)$ was introduced by Dettlaff et al. [3]. Building on this previous research, a new domination parameter was studied [4]. A dominating set S is termed a triple connected certified dominating set (TCCD-set) if, for each vertex v in S, the number of neighbors of v in $(V - S)$ is either zero or k, where k is greater than or equal to 2, and the $\langle S \rangle$ is triple connected. The minimum size of a TCCD-set is called the triple connected certified domination number (TCCD-number), denoted by $\gamma_{TCC}(G)$. The TCCD number for a strong product of graphs was generalized in [5], which also extracted γ_{TCC} values for Cartesian, corona, and lexicographic products of paths and cycles. The γ_{TCC} number of power graphs of certain special graphs has been studied in [6]. This article defines the triple connected certified domination number for the triangular grids with boundaries. The triangular grid with a square boundary is known as a square triangular grid and denoted as $ST_{r,r}$, same as a triangular grid with rectangular boundaries with $r < s$ is known as a rectangular triangular grid and denoted as $RT_{r,s}$, where $V(ST_{r,r}) = \{v_{a,b}: 0 \leq a, b \leq r\}$ and $E(ST_{r,r}) = \{v_{a,b}v_{a,b+1}: 0 \leq a \leq r, 0 \leq b \leq r-1\} \cup \{v_{a,b}v_{a+1,b}: 0 \leq a \leq r-1, 0 \leq b \leq r\} \cup \{v_{a,b}v_{a+1,b+1}: 0 \leq a, b \leq r-1\}$. The $V(RT_{r,s}) = \{0 \leq a \leq r, 0 \leq b \leq s\}$, $E(RT_{r,s}) = \{v_{a,b}v_{a,b+1}: 0 \leq a \leq r, 0 \leq b \leq s-1\} \cup \{v_{a,b}v_{a+1,b}: 0 \leq a \leq r-1, 0 \leq b \leq s\} \cup \{v_{a,b}v_{a+1,b+1}: 0 \leq a \leq r-1, 0 \leq b \leq s-1\}$. There are some works in triangular grid with borders which can be referred in [7]-[11]. The relationship between a triangular grid and a network application lies in how triangular grids can be used to model, optimize, or visualize network structures especially when dealing with mesh networks, wireless communication, routing, and graph theory applications. Figure 2 is the example of square triangular grid and rectangular triangular grid. Some recent results of graphs and domination can be studied in [12]-[15].

2. TCCD- number on square triangular grid

Observation 2.1: For a square triangular grid $ST_{r,r}, r \geq 2$, then $\gamma_{TCC}(ST_{2,2}) = 3$, $\gamma_{TCC}(ST_{3,3}) = 6$.

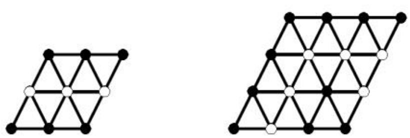

Figure 1
$ST_{2,2}$ **and** $ST_{3,3}$

Illustration: In Figure 1, the set of lightened vertices denote the TCCD-set.

Theorem 2.1: *For a square triangular grid graph $ST_{r,r}$, $r \geq 4$, the triple connected certified domination number*

$$\gamma_{TCC}(ST_{r,r}) = \begin{cases} (r-1)\frac{r}{3} + \frac{3r}{2} + 1 & if\ r = 6l, \\ (r-1)\left\lceil\frac{r}{3}\right\rceil + 6\left\lfloor\frac{r}{6}\right\rfloor & if\ r = 6l+1, \\ (r-1)\left\lceil\frac{r}{3}\right\rceil + 4\left\lfloor\frac{r}{6}\right\rfloor + 2\left\lceil\frac{r}{6}\right\rceil & if\ r = 6l+2, \\ (r-1)\left\lceil\frac{r}{3}\right\rceil + 3\left\lceil\frac{r}{6}\right\rceil + 3\left\lfloor\frac{r}{6}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor & if\ r = 6l+3, \\ (r-1)\left\lceil\frac{r}{3}\right\rceil + 3\left\lceil\frac{r}{6}\right\rceil + 3\left\lfloor\frac{r}{6}\right\rfloor + 2 & if\ r = 6l+4, \\ (r-1)\left\lceil\frac{r}{3}\right\rceil + 3\left\lfloor\frac{r}{6}\right\rfloor + 3\left\lfloor\frac{r}{6}\right\rfloor + 4 & if\ r = 6l+5. \end{cases}$$

Proof: Define the set S, constructed based on whether S based on modulo 6.

Case 1: If $r = 6l$, define $S_1 = \{v_{a,b}: 1 \leq a \leq r-1, b = 3l+1, 1 \leq b \leq r-3\} \cup \{v_{r,1}\}$, $S_2 = \{v_{a,b}: a = r-1, b = 6l\ or\ 6l+2\ or\ 6l+5, 2 \leq b \leq r-2\}$, $S_3 = \{v_a, b\}: a = 1, b = 6l\ or\ 6l+2\ or\ 6l+3, 2 \leq b \leq r-2\}$, $S_4 = \{v_{0,b}: b = r-1, r-2\} \cup \{v_{1,r}\} \cup \{v_{2,b}: b = r, r-1\}$, $S_5 = \{v_{a,b}: a = 1, b = r, r-2\} \cup \{v_{a,b}: b = r-1, a = 2l+1, 3 \leq a \leq r\}$, then $S = \bigcup_{i=1}^{5} S_i$. This ensures that for any vertex in $ST_{r,r}$ is either adjacent to S or in S without having one neighbor in $V-S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $ST_{r,r}$. The cardinality of S is given by $|S| = (r-1)\frac{r}{3} + \frac{3r}{2} + 1$.

Case 2: If $r = 6l+1$, define $S_6 = \{v_{a,b}: 0 \leq a \leq r-1, b = r-1\}$, then $S = \bigcup_{i=1}^{3} S_i \cup S_6$. This ensures that for any vertex in $ST_{r,r}$ is either adjacent to S or in S without having one neighbor in $V-S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $ST_{r,r}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{r}{3}\right\rceil + 6\left\lfloor\frac{r}{6}\right\rfloor$.

Case 3: If $r = 6l+2$, define $S_7 = \{v_{a,b}: 1 \leq a \leq r-1, b = 3l+1, 1 \leq b \leq r-1\} \cup \{v_{r,1}\}$, $S_8 = \{v_{a,b}: a = 1, b = 6l\ or\ 6l+2\ or\ 6l+3, 2 \leq b \leq r\}$, then $S = \bigcup_{i=7}^{8} S_i \cup S_2$. This ensures that for any vertex in $ST_{r,r}$ is either adjacent to S or in S without having one neighbor in $V-S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $ST_{r,r}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{r}{3}\right\rceil + 4\left\lfloor\frac{r}{6}\right\rfloor + 2\left\lceil\frac{r}{6}\right\rceil$.

Case 4: If $r = 6l+3$, define $S_9 = \{v_{a,b}: b = r-1, a = 2l, 2 \leq a \leq r\}$, then $S = \bigcup_{i=7}^{9} S_i \cup S_2$. This ensures that for any vertex in $ST_{r,r}$ is either adjacent to S

or in S without having one neighbor in $V-S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $ST_{r,r}$. The cardinality of S is given by $|S|=(r-1)\left\lceil\frac{r}{3}\right\rceil+3\left\lceil\frac{r}{6}\right\rceil+3\left\lfloor\frac{r}{6}\right\rfloor+\left\lfloor\frac{r}{2}\right\rfloor$.

Case 5: If $r=6l+4$, define $S_{10}=\{v_{a,b}: 1\le a\le r-1, b=3l+1, 1\le b\le r\}\cup\{v_{r,1}\}$, $S_{11}=\{v_{a,b}: a=0, b=r-1, r-2\}$, then $S=\bigcup_{i=2}^{3} S_i\cup S_j, j=10,11$. This ensures that for any vertex in $ST_{r,r}$ is either adjacent to S or in S without having one neighbor in $V-S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $ST_{r,r}$. The cardinality of S is given by $|S|=(r-1)\left\lceil\frac{r}{3}\right\rceil+3\left\lceil\frac{r}{6}\right\rceil+3\left\lfloor\frac{r}{6}\right\rfloor+2$.

Case 6: If $r=6l+5$, define $S_{12}=\{v_{a,b}: a=0, b=r-1, r-2\}\cup\{v_{a,b}: b=r, a=1,2\}$, $S_{13}=\{v_{a,b}: 1\le a\le r-1,\ b=3l+1,\ 1\le b\le r-1\}\cap\{v_{1,r-1}\}\cup\{v_{r,1}\}$, then $S=\bigcup_{i=2}^{3} S_i\cup S_j, j=12,13$. This ensures that for any vertex in $ST_{r,r}$ is either adjacent to S or in S without having one neighbor in $V-S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $ST_{r,r}$. The cardinality of S is given by $|S|=(r-1)\left\lceil\frac{r}{3}\right\rceil+3\left\lceil\frac{r}{6}\right\rceil+3\left\lfloor\frac{r}{6}\right\rfloor+4$. Thus a TCCD set S has been constructed for all the cases, and the cardinality of S is given by

$$|S|=\begin{cases}(r-1)\frac{r}{3}+\frac{3r}{2}+1 & if\ r=6l,\\ (r-1)\left\lceil\frac{r}{3}\right\rceil+6\left\lfloor\frac{r}{6}\right\rfloor & if\ r=6l+1,\\ (r-1)\left\lceil\frac{r}{3}\right\rceil+4\left\lfloor\frac{r}{6}\right\rfloor+2\left\lceil\frac{r}{6}\right\rceil & if\ r=6l+2,\\ (r-1)\left\lceil\frac{r}{3}\right\rceil+3\left\lceil\frac{r}{6}\right\rceil+3\left\lfloor\frac{r}{6}\right\rfloor+\left\lfloor\frac{r}{2}\right\rfloor & if\ r=6l+3,\\ (r-1)\left\lceil\frac{r}{3}\right\rceil+3\left\lceil\frac{r}{6}\right\rceil+3\left\lfloor\frac{r}{6}\right\rfloor+2 & if\ r=6l+4,\\ (r-1)\left\lceil\frac{r}{3}\right\rceil+3\left\lceil\frac{r}{6}\right\rceil+3\left\lfloor\frac{r}{6}\right\rfloor+4 & if\ r=6l+5.\end{cases}$$

Take $|S|=\beta$, Suppose that a TCCD set $D\subseteq ST_{r,r}$ exists such that $|D|\le d=\beta-1$.

Since no three vertices of D do not appear consecutively along a path in $\langle D\rangle$ or if there exists a vertex $u\in D$, then $|N(u)\cap(V-D)|=1$ or if there exists a vertex $v\in(V-D)$, $N(v)\cap D=\phi$. Therefore, the cardinality of any TCCD set D must satisfy $|D|\ge d+1=\beta$. Hence, the $\gamma_{TCC}(ST_{r,r})$ is at least $|S|$, and since it is already shown that $|S|$ is an upper bound for $\gamma_{TCC}(ST_{r,r})$, then $\gamma_{TCC}(ST_{r,r})=|S|$. This completes the proof.

2.1 *TCCD-number on rectangular triangular grid*

Observation 2.2: For a rectangular triangular grid $RT_{r,s}, r < s$, then $\gamma_{TCC}(RT_{2,s}) = s + 1$, $\gamma_{TCC}(RT_{3,s}) = s + \left\lceil \frac{s}{2} \right\rceil + 2$, $\gamma_{TCC}(RT_{4,s}) = 2s + 1$.

Theorem 2.2: *For a rectangular triangular grid graph $RT_{r,s}$, $r = 6l$, $r, s \geq 6$, the triple connected certified domination number*

$$\gamma_{TCC}(RT_{r,s}) = \begin{cases} (s-1)\frac{r}{3} + 4(\frac{r}{6} - 1) + \frac{r}{3} + \frac{s}{2} + 5 & if\ s = 6l\ or\ 6l + 4, \\ (r-1)\left\lceil \frac{s}{3} \right\rceil + 6\left\lfloor \frac{s}{6} \right\rfloor & if\ s = 6l + 1, \\ (r-1)\left\lceil \frac{s}{3} \right\rceil + 4\left\lfloor \frac{s}{6} \right\rfloor + 2\left\lceil \frac{s}{6} \right\rceil & if\ s = 6l + 2, \\ (r-1)\frac{s}{3} + 4\left\lfloor \frac{s}{6} \right\rfloor + 2\left\lceil \frac{s}{6} \right\rceil + \frac{r}{2} - 1 & if\ s = 6l + 3, \\ (r-1)\left\lceil \frac{s}{3} \right\rceil + 3\left\lfloor \frac{s}{6} \right\rfloor + 3\left\lceil \frac{s}{6} \right\rceil + 4 & if\ s = 6l + 5. \end{cases}$$

Proof: Define the set S, constructed based on whether S based on modulo 6.

Case 1: If $r = 6l\ or\ 6l + 4$, define $S_1 = \{v_{a,b}: 1 \leq b \leq s - 1, a = 3l + 1, 2 \leq a \leq r - 2\}$, $S_2 = \{v_{a,b}: a = r - 1,\ b = 2l + 1, 1 \leq b \leq s - 1\} \cup \{v_{r-1,0}\}$, $S_3 = \{v_{a,b}: b = 1, a = 6l\ or\ 6l + 3\ or\ 6l + 5, 5 \leq a \leq r - 2\}$, $S_4 = \{v_{a,b}: b = s - 1, a = 6l + 2\ or\ 6l + 3\ or\ 6l + 5, 2 \leq a \leq r - 2\}$, $S_5 = \{v_{a,b}: a = 0, b = s - 1, s - 2\} \cup \{v_{a,b}: a = 1, b = s\}$, then $S = \bigcup_{i=1}^{5} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (s-1)\frac{r}{3} + 4(\frac{r}{6} - 1) + \frac{r}{3} + \frac{s}{2} + 5$.

Case 2: If $r = 6l + 1$, define $S_6 = \{v_{a,b}: 1 \leq a \leq r - 1, b = 3l + 1, 1 \leq b \leq s - 1\} \cup \{v_{r,1}\}$, $S_7 = \{v_{a,b}: a = r - 1, b = 6l\ or\ 6l + 2\ or\ 6l + 5, 2 \leq b \leq s - 2\}$, $S_8 = \{v_{a,b}: a = 1, b = 6l\ or\ 6l + 2\ or\ 6l + 3, 2 \leq b \leq s - 2\}$, $S_9 = \{v_{a,b}: b = s - 1, 0 \leq a \leq r - 1\}$, then $S = \bigcup_{i=6}^{9} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil \frac{s}{3} \right\rceil + 6\left\lfloor \frac{s}{6} \right\rfloor$.

Case 3: If $r = 6l + 2$, define $S_{10} = \{v_{0,s-1}\}$, then $S = \bigcup_{i=6}^{8} S_i \cup S_{10}$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil \frac{s}{3} \right\rceil + 4\left\lfloor \frac{s}{6} \right\rfloor + 2\left\lceil \frac{s}{6} \right\rceil$.

Case 4: If $r = 6l + 3$, define $S_{11} = \{v_{a,b}: 1 \leq a \leq r-1, b = 3l+1, 1 \leq b \leq s-2\} \cup \{v_{r,1}\}$, $S_{12} = \{v_{a,b}: a = 10, b = 6l \text{ or } 6l+2 \text{ or } 6l+3, 2 \leq b \leq s\}$, $S_{13} = \{v_{a,b}: b = s-1, a = 2l+1, 3 \leq a \leq r-1\}$, then $S = \bigcup_{i=11}^{13} S_i \cup S_7$ This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \frac{r}{2} - 1$.

Case 5: If $r = 6l + 5$, define $S_{14} = \{v_{a,b} : a = 0, b = s-1, s-2\}$, $S_{15} = \{v_{a,b} : b = s, a = 1, 2\}$, $S_{16} = \{v_{a,b} : b = s-1, 2 \leq a \leq r-1\}$, then $S = S_i \cup S_j \cup S_k$, $14 \leq i \leq 16$, $j = 7, 8$, $k = 11, 12$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4$. Thus a TCCD set S has been constructed for all the cases, and the cardinality of S is given by

$$|S| = \begin{cases} (s-1)\frac{r}{3} + 4\left(\frac{r}{6}+1\right) + \frac{r}{3} + \frac{s}{2} + 5 & \text{if } s = 6l \text{ or } 6l+4, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 6\left\lfloor\frac{s}{6}\right\rfloor & \text{if } s = 6l+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & \text{if } s = 6l+2, \\ (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \frac{r}{2} - 1 & \text{if } s = 6l+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4 & \text{if } s = 6l+5. \end{cases}$$

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that a TCCD set $D \subseteq RT_{r,s}$ exists such that $|D| \leq d = \beta - 1$. Since no three vertices of D do not appear consecutively along a path in $\langle D \rangle$ or if there exists a vertex $u \in D$, then $|N(u) \cap (V - D)| = 1$ or if there exists a vertex $v \in (V - D)$, $N(v) \cap D = \phi$. Therefore, the cardinality of any TCCD set D must satisfy $|D| \geq d + 1 = \beta$. Hence, the $\gamma_{TCC}(RT_{r,s})$ is at least $|S|$, and since it is already shown that $|S|$ is an upper bound for $\gamma_{TCC}(RT_{r,s})$, then $\gamma_{TCC}(RT_{r,s}) = |S|$. This completes the proof.

Example 2.1:

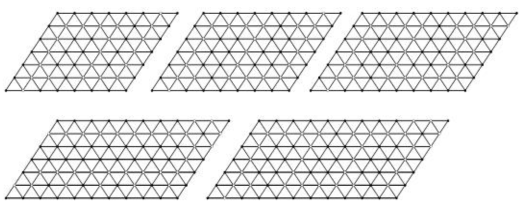

Figure 2
$RT_{6,7}$, $RT_{6,8}$, $RT_{6,9}$, $RT_{6,10}$ **and** $RT_{6,11}$

Illustration: In Figure 2, the set of lightened vertices denote the TCCD-set.

Theorem 2.3: *For a rectangular triangular grid graph* $RT_{r,s}$, $r = 6l + 1$, $r, s \geq 7$, *the triple connected certified domination number*

$$\gamma_{TCC}(RT_{r,s}) = \begin{cases} (r-1)\left\lfloor\frac{s}{3}\right\rfloor + 3\left\lfloor\frac{s}{6}\right\rfloor + 3(\left\lfloor\frac{s}{6}\right\rfloor - 1) + \left\lfloor\frac{r}{2}\right\rfloor + 4 & if\ s = 6l, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3(\left\lceil\frac{s}{6}\right\rceil - 1) + 3 & if\ s = 6l+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & if\ s = 6l+2, \\ (r-1)\frac{s}{3} + 3\left\lceil\frac{s}{6}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor & if\ s = 6l+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 2 & if\ s = 6l+4, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4 & if\ s = 6l+5. \end{cases}$$

Proof: Define the set S, constructed based on whether S based on modulo 6.

Case 1: If $r = 6l$, define $S_1 = \{a = r-1, b = 6l or 6l + 2 or 6l + 5, 2 \leq b \leq s-2\}$, $S_2 = \{v_{a,b}: b = 3l + 1, 1 \leq b \leq s-3, 1 \leq a \leq r-1\}$, $S_3 = \{v_{a,b}: a = 1, b = 6l\ or\ 6l+2\ or\ 6l+3, 2 \leq b \leq s-2\}$, $S_4 = \{v_{a,b}: a = 0, b = s-1, s-2\}$ $S_5 = \{v_{a,s-2}: 4 \leq a \leq s-1\} \cup \{v_{a,s}: a = 1,2\} \cup \{v_{1,s-2}\} \cup \{v_{3,s-1}\}$, $S_6 = \{v_{a,b}: a = 2l, b = s-1, 2 \leq a \leq r-1\}$, then $S = \bigcup_{i=1}^{6} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lfloor\frac{s}{3}\right\rfloor + 3\left\lfloor\frac{s}{6}\right\rfloor + 3(\left\lfloor\frac{s}{6}\right\rfloor - 1) + \left\lfloor\frac{r}{2}\right\rfloor + 4$.

Case 2: If $r = 6l + 1$, define $S_7 = \{v_{a,b}: b = 3l + 1, 1 \leq a \leq r-1, 1 \leq b \leq s-1\} \cup \{v_{r,1}\}$, $S_8 = \{v_{a,b}: b = s-1, 0 \leq a \leq r-1\}$, then $S = \bigcup_{i=7}^{8} S_i \cup S_1 \cup S_3$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3(\left\lfloor\frac{s}{6}\right\rfloor - 1) + 3$.

Case 3: If $r = 6l + 2$, define $S_9 = \{v_{a,b}: b = 3l + 1, 1 \leq a \leq r-1, 1 \leq b \leq s\} \cup \{v_{r,1}\}$, $S_{10} = \{v_{a,b}: a = 1, b = 6l\ or\ 6l+2\ or\ 6l+3, 2 \leq b \leq s\}$, then $S = \bigcup_{i=9}^{10} S_i \cup S_1$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil$.

Case 4: If $r = 6l + 3$, define $S_{11} = \{v_{a,b}: b = s - 1, a = 2l, 2 \leq a \leq r - 1\}$, then $S = \bigcup_{i=9}^{11} S_i \cup S_1$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\frac{s}{3} + 3\left\lceil\frac{s}{6}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor$.

Case 5: If $r = 6l + 4$, then $S = S_1 \cup S_3 \cup S_4 \cup S_9$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 2$.

Case 6: If $r = 6l + 5$, define $S_{12} = \{v_{a,b}: b = 3l + 1, 1 \leq a \leq r - 1, 1 \leq b \leq s - 2\} \cup \{v_{r,1}\}$, $S_{13} = \{v_{a,b}: b = s, a = 1, 2\} \cup \{v_{a,s-1}: 2 \leq a \leq r - 1\}$, then $S = S_i \cup S_j \cup S_1, 3 \leq i \leq 6, j = 12,13$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\frac{s}{3} + 3\left\lceil\frac{s}{6}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor$.

Thus a TCCD set S has been constructed for all the cases, and the cardinality of S is given by

$$|S| = \begin{cases} (r-1)\left\lfloor\frac{s}{3}\right\rfloor + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left(\left\lfloor\frac{s}{6}\right\rfloor - 1\right) + \left\lfloor\frac{r}{2}\right\rfloor + 4 & if\ s = 6l, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left(\left\lfloor\frac{s}{6}\right\rfloor - 1\right) + 3 & if\ s = 6l+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & if\ s = 6l+2, \\ (r-1)\frac{s}{3} + 3\left\lceil\frac{s}{6}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + \left\lfloor\frac{r}{2}\right\rfloor & if\ s = 6l+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 2 & if\ s = 6l+4, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4 & if\ s = 6l+5. \end{cases}$$

Take $|S| = \beta$, Suppose that a TCCD set $D \subseteq RT_{r,s}$ exists such that $|D| \leq d = \beta - 1$. Since no three vertices of D do not appear consecutively along a path in $\langle D\rangle$ or if there exists a vertex $u \in D$, then $|N(u) \cap (V - D)| = 1$ or if there exists a vertex $v \in (V - D)$, $N(v) \cap D = \phi$. Therefore, the cardinality of any TCCD set D must satisfy $|D| \geq d + 1 = \beta$. Hence, the $\gamma_{TCC}(RT_{r,s})$ is at least $|S|$, and since it is already shown that $|S|$ is an upper bound for $\gamma_{TCC}(RT_{r,s})$, then $\gamma_{TCC}(RT_{r,s}) = |S|$. This completes the proof.

Theorem 2.4: *For a rectangular triangular grid graph* $RT_{r,s}$, $r = 6l + 2$, $r, s \geq 8$, *the triple connected certified domination number*

$$\gamma_{TCC}(RT_{r,s}) = \begin{cases} (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \frac{r}{2} & if s = r+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4 & if s = r+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & if s = r+6, \\ (s-1)\left\lceil\frac{r}{3}\right\rceil + 5\left\lfloor\frac{r}{6}\right\rfloor + (\left\lfloor\frac{r}{6}\right\rfloor - 1) + 7 & otherwise. \end{cases}$$

Proof: **Case 1:** If $s = r + 1$, define $S_1 = \{v_{a,b}: b = 3l + 1, 1 \leq a \leq r - 1, 1 \leq b \leq s - 2\} \cup \{v_{r,1}\}$, $S_3 = \{a = r - 1, b = 6l$ or $6l + 2$ or $6l + 5, 2 \leq b \leq s - 2\}$, $S_4 = \{v_{a,b}: a = 1, b = 6l$ or $6l + 2$ or $6l + 3, 2 \leq b \leq s\}$, $S_6 = \{v_{a,b}: b = s - 1, a = 3l + 1, 3 \leq a \leq r - 1\}$, then $S = \bigcup_{i=1}^{4} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \frac{r}{2}$.

Case 2: If $s = r + 3$, define $S_5 = \{v_{a,b}: a = 1, b = 6l \; or \; 6l + 2 \; or \; 6l + 3, 2 \leq b \leq s - 2\}$, $S_6 = \{v_{a,b}: b = s - 1, 2 \leq a \leq r - 1\}$, $S_7 = \{v_{a,b}: a = 0, b = s - 1, s - 2\} \cup \{v_{a,b}: b = s, a = 1,2\}$, then $S = S_i \cup S_j, 5 \leq i \leq 7, j = 1,2$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4$.

Case 3: If $s = r + 6$, define $S_8 = \{v_{a,b}: b = 3l + 1, 1 \leq a \leq r - 1, 1 \leq b \leq s - 1\} \cup \{v_{r,1}\}$, then $S = \bigcup_{i=2}^{3} S_i \cup S_8$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil$.

Case 4: If $s \neq r + 1 \; or \; r + 3 \; or \; r + 6$, define $S_9 = \{v_{a,b}: \leq b \leq s - 1, a = 3l + 1, 4 \leq a \leq r - 2\} \cup \{v_{a,b}: a = 1, 0 \leq b \leq s - 2\} \cup \{v_{a,b}: a = r - 1, 2 \leq b \leq s\}$, $S_{10} = \{v_{a,b}: a = 0, b = s - 1, s - 2\} \cup \{v_{a,b}: b = s, a = 1,2\}$, $S_{11} = \{v_{a,b}: a = r - 1, r - 2, b = 0\} \cup \{v_{a,b}: a = r, b = 1,2\}$, $S_{12} = \{v_{a,b}: b = s - 1, a = 6l + 2 \; or \; 6l + 3 \; or \; 6l + 5, 2 \leq a \leq r - 4\}$, $S_{13} = \{v_{a,b}: b = 1, a = 6l \; or \; 6l + 3 \; or \; 6l + 5, 5 \leq a \leq r - 2\}$, then $S = \bigcup_{i=9}^{13} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of

S is given by $|S| = (s-1)\left\lceil\frac{r}{3}\right\rceil + 5\left\lfloor\frac{r}{6}\right\rfloor + \left(\left\lfloor\frac{r}{6}\right\rfloor - 1\right) + 7$. Thus a TCCD set S has been constructed for all the cases, and the cardinality of S is given by

$$|S| = \beta \begin{cases} (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \frac{r}{2} & if\ s = r+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 4 & if\ s = r+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & if\ s = r+6, \\ (s-1)\left\lceil\frac{r}{3}\right\rceil + 5\left\lfloor\frac{r}{6}\right\rfloor + (\left\lfloor\frac{r}{6}\right\rfloor - 1) + 7 & otherwise. \end{cases}$$

Take $|S| = \beta$, Suppose that a TCCD set $D \subseteq RT_{r,s}$ exists such that $|D| \le d = \beta - 1$. $d = \beta - 1$ Since no three vertices of D do not appear consecutively along a path in $\langle D \rangle$ or if there exists a vertex $u \in D$, then $|N(u) \cap (V - D)| = 1$ or if there exists a vertex $v \in (V - D)$, $N(v) \cap D = \phi$. Therefore, the cardinality of any TCCD set D must satisfy $|D| \ge d + 1 = \beta$. Hence, the $\gamma_{TCC}(RT_{r,s})$ is at least $|S|$, and since it is already shown that $|S|$ is an upper bound for $\gamma_{TCC}(RT_{r,s})$, then $\gamma_{TCC}(RT_{r,s}) = |S|$. This completes the proof.

Theorem 2.5: *For a rectangular triangular grid graph* $RT_{r,s}$, $r = 6l + 3\ or\ 6l + 4$, $r, s \ge 9$, *the triple connected certified domination number*

$$\gamma_{TCC}(RT_{r,s}) = \begin{cases} (r-1)\frac{s}{3} + \frac{s}{2} + 3\left(\frac{s}{6} - 1\right) + \left\lfloor\frac{r}{2}\right\rfloor + 4 & if\ s = 6l, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 6\left\lfloor\frac{s}{6}\right\rfloor & if\ s = 6l+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 6\left\lfloor\frac{s}{6}\right\rfloor + 2 & if\ s = 6l+2, \\ (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \left\lceil\frac{r}{2}\right\rceil & if\ s = 6l+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 2 & if\ s = 6l+4, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lfloor\frac{s}{6}\right\rfloor + 4 & if\ s = 6l+5. \end{cases}$$

Proof: Define the set S, constructed based on whether S based on modulo 6.

Case 1: If $s = 6l$, define $S_1 = \{v_{a,b}: 1 \le a \le r-1, b = 3l+1, 1 \le b \le s-3\} \cup \{v_{r,1}\}$, $S_2 = \{v_{a,b}: a = r-1, b = 6l\ or\ 6l+2\ or\ 6l+5, 2 \le b \le s-2\}$ $S_3 = \{v_{a,b}: a = 1, b = 6l\ or\ 6l+2\ or\ 6l+3, 2 \le b \le s-2\}$ $S_4 = \{v_{a,b}: 4 \le a \le r-1, b = s-2\} \cup \{v_{0,s-2}\}$ $S_5 = \{v_{a,b}: a = 2l, b = s-1\} \cup \{v_{4,s-1}\} \cup \{v_{a,b}: a = 1, 2, b = s\}$, then $S = \bigcup_{i=1}^{5} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\frac{s}{3} + \frac{s}{2} + 3(\frac{s}{6} - 1) + \left\lfloor\frac{r}{2}\right\rfloor + 4$.

Case 2: If $s = 6l + 1$, define $S_6 = \{v_{a,b}: 0 \leq a \leq r - 1, b = s - 1\}$, then $S = \bigcup_{i=1}^{3} S_i \cup S_6$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r - 1)\left\lceil \frac{s}{3} \right\rceil + 6\left\lfloor \frac{s}{6} \right\rfloor$.

Case 3: If $s = 6l + 2$, define $S_7 = \{v_{a,b}: 1 \leq a \leq r - 1, b = 3l + 1, 1 \leq b \leq s - 1\} \cup \{v_{r,1}\}$, $S_8 = \{v_{a,b}: a = 0, b = s - 1, s - 2\}$, then $S = \bigcup_{i=7}^{8} S_i \cup S_2$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r - 1)\left\lceil \frac{s}{3} \right\rceil + 6\left\lfloor \frac{s}{6} \right\rfloor + 2$.

Case 4: If $s = 6l + 3$, define $S_9 = \{v_{a,b}: a = 1, b = 6l \text{ or } 6l + 2 \text{ or } 6l + 3, 2 \leq b \leq s\}$ $S_{10} = \{v_{a,b}: a = 2l, 2 \leq a \leq r, b = s - 1\}$, then $S = \bigcup_{i=9}^{10} S_i \cup S_2 \cup S_7$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r - 1)\frac{s}{3} + 4\left\lfloor \frac{s}{6} \right\rfloor + 2\left\lceil \frac{s}{6} \right\rceil + \left\lceil \frac{r}{2} \right\rceil$.

Case 5: If $s = 6l + 4$, define $S_{11} = \{v_{a,b}: 1 \leq a \leq r - 1, b = 3l + 1, 1 \leq b \leq s - 1\} \cup \{v_{r,1}\}$, then $S = S_2 \cup S_3 \cup S_8 \cup S_{11}$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r - 1)\left\lceil \frac{s}{3} \right\rceil + 3\left\lfloor \frac{s}{6} \right\rfloor + 3\left\lceil \frac{s}{6} \right\rceil + 2$.

Case 6: If $s = 6l + 5$, define $S_{12} = \{v_{a,b}: a = 0, b = s - 1, s - 2\} \cup \{v_{a,b}: b = s, a = 1,2\}$ $S_{13} = \{v_{a,b}: 2 \leq a \leq r - 1, b = s - 1\}$, then $S = S_i \cup S_j, 1 \leq i \leq 3, j = 12,13$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r - 1)\left\lceil \frac{s}{3} \right\rceil + 3\left\lceil \frac{s}{6} \right\rceil + 3\left\lfloor \frac{s}{6} \right\rfloor + 4$. Thus a TCCD set S has been constructed for all the cases, and the cardinality of S is given by

$$|S| = \begin{cases} (r-1)\frac{s}{3} + \frac{s}{2} + 3\left(\frac{s}{6} - 1\right) + \left\lfloor\frac{r}{2}\right\rfloor + 4 & if\ s = 6l, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 6\left\lfloor\frac{s}{6}\right\rfloor & if\ s = 6l+1, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 6\left\lfloor\frac{s}{6}\right\rfloor + 2 & if\ s = 6l+2, \\ (r-1)\frac{s}{3} + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil + \left\lceil\frac{r}{2}\right\rceil & if\ s = 6l+3, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 3\left\lceil\frac{s}{6}\right\rceil + 2 & if\ s = 6l+4, \\ (r-1)\left\lceil\frac{s}{3}\right\rceil + 3\left\lceil\frac{s}{6}\right\rceil + 3\left\lfloor\frac{s}{6}\right\rfloor + 4 & if\ s = 6l+5. \end{cases}$$

Take $|S| = \beta$, Suppose that a TCCD set $D \subseteq RT_{r,s}$ exists such that $|D| \leq d = \beta - 1$. Since no three vertices of D do not appear consecutively along a path in $\langle D \rangle$ or if there exists a vertex $u \in D$, then $|N(u) \cap (V - D)| = 1$ or if there exists a vertex $v \in (V - D)$, $N(v) \cap D = \phi$. Therefore, the cardinality of any TCCD set D must satisfy $|D| \geq d + 1 = \beta$. Hence, the $\gamma_{TCC}(RT_{r,s})$ is at least $|S|$, and since it is already shown that $|S|$ is an upper bound for $\gamma_{TCC}(RT_{r,s})$, then $\gamma_{TCC}(RT_{r,s}) = |S|$. This completes the proof.

Theorem 2.6: For a rectangular triangular grid graph $RT_{r,s}$, $r = 6l + 5$, $r, s \geq 5$, the triple connected certified domination number

$$\gamma_{TCC}(RT_{r,s}) = \begin{cases} (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & if\ s = r+3, \\ (s-1)\left\lceil\frac{r}{3}\right\rceil + 3\left\lfloor\frac{r}{6}\right\rfloor + 3\left\lceil\frac{r}{6}\right\rceil + 4 & otherwise. \end{cases}$$

Proof: **Case 1:** If $s = r + 3$, define $S_1 = \{v_{a,b}: 1 \leq a \leq r - 1, b = 3l + 1, 1 \leq b \leq s - 1\} \cup \{v_{r,1}\}$, $S_2 = \{v_{a,b}: a = r - 1, b = 6l\ or\ 6l + 2\ or\ 6l + 5, 2 \leq b \leq s - 2\}$, $S_3 = \{v_{a,b}: a = 1, b = 6l\ or\ 6l + 2\ or\ 6l + 3, 2 \leq b \leq s\}$, then $S = \bigcup_{i=1}^{3} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil$.

Case 2: If $s \neq r + 3$, define $S_4 = \{a = 3l + 1, 4 \leq a \leq r, 1 \leq b \leq s - 1\}$, $S_5 = \{v_{a,b}: a = 1, 0 \leq b \leq s - 2\} \cup \{v_{r-1,0}\}$, $S_6 = \{a = 6l + 2or6l + 3or6l + 5, 2 \leq a \leq r - 1, b = s - 1\}$, $S_7 = \{v_{a,b}: a = 6lor6l + 3or6l + 5, 5 \leq a \leq r - 1, b = 1\}$, $S_8 = \{v_{a,b}: a = 0, b = s - 1, s - 2\} \cup \{v_{a,b}: a = 1,2, b = s\}$, then $S = \bigcup_{i=4}^{8} S_i$. This ensures that for any vertex in $RT_{r,s}$ is either adjacent to S or in S without having one neighbor in $V - S$ and any of the three vertices of S are located along a path, making S as a TCCD set. Hence the vertices in S forms a TCCD set in $RT_{r,s}$. The cardinality of S is given by $|S| = (s-1)\left\lceil\frac{r}{3}\right\rceil + 3\left\lfloor\frac{r}{6}\right\rfloor +$

$3\left\lceil\frac{r}{6}\right\rceil + 4$. Thus a TCCD set S has been constructed for all the cases, and the cardinality of S is given by

$$|S| = \begin{cases} (r-1)\left\lceil\frac{s}{3}\right\rceil + 4\left\lfloor\frac{s}{6}\right\rfloor + 2\left\lceil\frac{s}{6}\right\rceil & if\ s = r+3, \\ (s-1)\left\lceil\frac{r}{3}\right\rceil + 3\left\lfloor\frac{r}{6}\right\rfloor + 3\left\lceil\frac{r}{6}\right\rceil + 4 & otherwise. \end{cases}$$

Take $|S| = \beta$, Suppose that a TCCD set $D \subseteq RT_{r,s}$ exists such that $|D| \leq d = \beta - 1$. Since no three vertices of D do not appear consecutively along a path in $\langle D\rangle$ or if there exists a vertex $u \in D$, then $|N(u) \cap (V - D)| = 1$ or if there exists a vertex $v \in (V - D)$, $N(v) \cap D = \phi$. Therefore, the cardinality of any TCCD set D must satisfy $|D| \geq d + 1 = \beta$. Hence, the $\gamma_{TCC}(RT_{r,s})$ is at least $|S|$, and since it is already shown that $|S|$ is an upper bound for $\gamma_{TCC}(RT_{r,s})$, then $\gamma_{TCC}(RT_{r,s}) = |S|$. This completes the proof.

3. Conclusion

This article studies the γ_{TCC} number for triangular grids with square and rectangular boundaries. The parameter's definition and relevance are outlined, emphasizing its role in graph theory. Future articles will explore the behavior of γ_{TCC} under specific graph operations. Additionally, comparisons will be made with other key graph theoretical parameters to highlight its significance.

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