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Pairs of soft ideals in a ternary semigroup

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Abstract

The purpose of this paper is to introduce the concept of a pair soft left (resp. right, lateral) ideal of a ternary semigroup. Relationships between a pair soft left (resp. right, lateral) ideal, a soft left (resp. right, lateral) ideal, and a regular (quasi-regular) semigroup are described.

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1. Introduction

A soft set is a mathematical tool for handling uncertainty, introduced by D. Molodtsov [15]. Let U be an universal set, and let E be a set of parameters. A pair (F,Ω) is said to be a soft set over U if (1) Ω is a subset of E, and (2) $F:\Omega \to P(U)$ is a mapping ($P(U)$ is the power set of U). Let SSU denote the set of all soft sets over U.

Since then, mathematicians have studied algebraic structures on soft sets by algebrizing either the universal set or the parameter set. Ternary semigroups have been extensively studied (see [8], [9], [10], [11], [17], [19]). A ternary semigroup can be extended, by augmenting a compatibly patial relation to the structure, to be an ordered ternary semigroup (see [20], [21]). Soft algebraic structures have been widely studied (see [4], [5], [1], [2], [3], [6], [7], [14], [12], [13], [16], [18]). In this paper we introduce and study a pair soft ideal of a ternary semigroup.

The following will be used throughout the paper.

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Definition 1.1: $(F,\Omega) \in SSU$ is null if $F(\rho) = \emptyset$ for all $\rho \in \Omega$, and an empty if $\Omega = \emptyset$.

Definition 1.2: $(U,E) \in SSU$ such that $U: E \to P(U)$ defined by $U(e) = U$ for all $e \in E$ is an universal soft set.

Definition 1.3: For $(G,\Sigma),(F,\Omega) \in SSU$, $(G,\Sigma) \tilde{\subseteq} (F,\Omega)$ if $\Sigma \subseteq \Omega$ and $G(\rho) \subseteq F(\rho)$ for all $\rho \in \Sigma$; $(F,\Omega) = (G,\Sigma)$ if $(F,\Omega) \tilde{\subseteq} (G,\Sigma)$ and $(G,\Sigma) \tilde{\subseteq} (F,\Omega)$.

Definition 1.4: For $(F,\Omega),(G,\Sigma) \in SSU$, the restricted intersection is

$$(F,\Omega) \cap_R (G,\Sigma) = (H,\Xi),$$

where $\Xi = \Omega \cap \Sigma$ and $H(\varrho) = F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Xi$.

2. Preliminaries

A *ternary semigroup* T consists of a non-empty set T and an associative ternary operation $[\,]: T \times T \times T \to T$. For subsets A, B, C of T, define the set product $[ABC]$ by:

$$[ABC] = \{[x_1x_2x_3] | x_1 \in A, x_2 \in B, x_3 \in C\}.$$

A non-empty subset S of T is a *ternary subsemigroup* if $[SSS] \subseteq S$. T is *commutative* if $[x_1x_2x_3] = [x_{\alpha(1)}x_{\alpha(2)}x_{\alpha(3)}]$ for all permutation α on $\{1,2,3\}$.

Hereafter, we take T to be the universal set and E to be the set of parameters.

Let SST represent the collection of all soft sets defined over T.

Definition 2.1: The restricted ternary product of $(F,\Omega),(G,\Sigma),(H,\Xi) \in SST$ is

$$(F,\Omega) \odot (G,\Sigma) \odot (H,\varrho) = (K,\Delta),$$

where $\Delta = \Omega \cap \Sigma \cap \Xi$ and $K(\varrho) = [F(\varrho)G(\varrho)H(\varrho)]$ for all $\varrho \in \Delta$.

Definition 2.2: $(F,\Omega) \in SST$ is a soft ternary semigroup if $(F,\Omega) \odot (F,\Omega) \odot (F,\Omega) \subseteq (F,\Omega)$.

Definition 2.3: A non-null and non-empty $(F,\Omega) \in SST$ is said to be

- a *soft left ideal* if $(T,E) \odot (T,E) \odot (F,\Omega) \tilde{\subseteq} (F,\Omega)$;
- a *soft right ideal* if $(F,\Omega) \odot (T,E) \odot (T,E) \tilde{\subseteq} (F,\Omega)$;
- a *soft lateral ideal* if $(T,E) \odot (F,A) \odot (T,E) \tilde{\subseteq} (F,\Omega)$;
- a *soft ideal* if (F,Ω) is a soft left, a soft right and a soft lateral ideal.

Let SI (respectively SL, SR, SM) denote the set of all soft ideals (respectively left, right, and lateral) over T.

3. Main results

Following, we introduce pair soft ideals.

Definition 3.1: Let $(F,\Omega),(G,\Sigma)\in SST$ such that $(F,\Omega)\,\widetilde{\cap}_R\,(G,\Sigma)$ is non-empty. A pair $((F,\Omega),(G,\Sigma))$ is

- a *pair soft left ideal* if $(T,E)\odot(F,\Omega)\odot(G,\Sigma)\,\widetilde{\subseteq}\,(F,\Omega)\,\widetilde{\cap}_R\,(G,\Sigma)$;
- a *pair soft right ideal* if $(F,\Omega)\odot(G,\Sigma)\odot(T,E)\,\widetilde{\subseteq}\,(F,\Omega)\,\widetilde{\cap}_R\,(G,\Sigma)$;
- a *pair soft lateral ideal* if $(F,\Omega)\odot(T,E)\odot(G,\Sigma)\,\widetilde{\subseteq}\,(F,\Omega)\,\widetilde{\cap}_R\,(G,\Sigma)$;
- a *pair soft ideal* if $((F,\Omega),(G,\Sigma))$ is a pair soft left, right, and lateral ideal.

Let PSI (respectively PSL,PSR,PSM) denote the collection of all pair soft ideals (respectively left, right, and lateral) over T.

Proposition 3.2:

(1) If $(F,\Omega)\in SM$ and $(G,\Sigma)\in SL$, then $((F,\Omega),(G,\Sigma))\in PSL$.
(2) If $(F,\Omega)\in SR$ and $(G,\Sigma)\in SM$, then $((F,\Omega),(G,\Sigma))\in PSR$.
(3) If $(F,\Omega)\in SR$ and $(G,\Sigma)\in SL$, then $((F,\Omega),(G,\Sigma))\in PSM$.
(4) If $(F,\Omega)\in SL$, then $((F,\Omega),(F,\Omega))\in PSL\cap PSM$.
(5) If $(F,\Omega)\in SR$, then $((F,\Omega),(F,\Omega))\in PSR\cap PSM$.
(6) If $(F,\Omega)\in SM$, then $((F,\Omega),(F,\Omega))\in PSL\cap PSR$.
(7) If T is commutative and if $((F,\Omega),(G,\Sigma))\in PSM$, then $((G,\Sigma),(F,\Omega))\in PSM$.

Proof: To prove (1), we assume that $(F,\Omega)\in SM$ and $(G,\Sigma)\in SL$; then $(T,E)\odot(F,\Omega)\odot(T,E)\,\widetilde{\subseteq}\,(F,\Omega)$ and $(T,E)\odot(T,E)\odot(G,\Sigma)\,\widetilde{\subseteq}\,(G,\Sigma)$. We have

$$(T,E)\odot(F,\Omega)\odot(T,E)=(H,E\cap\Omega\cap E)=(H,\Omega)$$

where $H(\varrho)=[TF(\varrho)T]$ for all $\varrho\in\Omega$ and we have

$$(T,E)\odot(T,E)\odot(G,\Sigma)=(K,E\cap E\cap\Sigma)=(K,\Sigma)$$

where $K(\varrho)=[TTG(\varrho)]$ for all $\varrho\in\Sigma$. By assumption, $[TF(\varrho)T]\subseteq F(\varrho)$ for all $\varrho\in\Omega$, and $[TTG(\varrho)]\subseteq G(\varrho)$ for all $\varrho\in\Sigma$. We have to show that $((F,\Omega),(G,\Sigma))\in PSL$, i.e.,

$$(T,E)\odot(F,\Omega)\odot(G,\Sigma)\,\widetilde{\subseteq}\,(F,\Omega)\,\widetilde{\cap}_R\,(G,\Sigma).$$

Let $(M,\Xi)=(T,E)\odot(F,\Omega)\odot(G,\Sigma)$. Then $\Xi=E\cap\Omega\cap\Sigma=\Omega\cap\Sigma$ and $M(\varrho)=[TF(\varrho)G(\varrho)]$ for all $\varrho\in\Xi$. For $\varrho\in\Omega\cap\Sigma$, we have

$$M(\varrho)=[TF(\varrho)G(\varrho)]\subseteq[TF(\varrho)T]\subseteq F(\varrho)$$

and

$$M(\varrho)=[TF(\varrho)G(\varrho)]\subseteq[TTG(\varrho)]\subseteq G(\varrho).$$

Thus $H(\varrho)\subseteq F(\varrho)\cap G(\varrho)$. Hence $((F,\Omega),(G,\Sigma))\in PSL$.

That (2) and (3) hold can be proved in the same manner as (1).

To prove (4), we assume that $(F,\Omega)\in SL$. Then $(T,E)\odot(T,E)\odot(F,\Omega)\,\widetilde{\subseteq}\,(F,\Omega)$. We have

$$(T,E)\odot(T,E)\odot(F,\Omega)=(G,E\cap E\cap\Omega)=(G,\Omega)$$

where $G(\varrho) = [TTF(\varrho)]$ for all $\rho \in \Omega$ such that $G(\varrho) \subseteq F(\varrho)$ for all $\varrho \in \Omega$. We have to show that $((F,\Omega),(F,\Omega)) \in PSL$, i.e.,

$$(T,E) \odot (F,\Omega) \odot (F,\Omega) \mathrel{\tilde{\subseteq}} (F,\Omega) \mathbin{\tilde{\cap}_R} (F,\Omega).$$

Let $(H,\Sigma) = (T,E) \odot (F,\Omega) \odot (F,\Omega)$. Then $\Sigma = E \cap \Omega \cap \Omega = \Omega$ and $H(\varrho) = [TF(\varrho)F(\varrho)]$ for all $\varrho \in \Omega$. For $\varrho \in \Omega$, we have

$$H(\varrho) = [TF(\varrho)F(\varrho)] \subseteq [TTF(\varrho)] = G(\varrho) \subseteq F(\varrho)$$

Hence $((F,\Omega),(F,\Omega)) \in PSL$. We have to show that $((F,\Omega),(F,\Omega)) \in PSM$, i.e.,

$$(F,\Omega) \odot (T,E) \odot (F,\Omega) \mathrel{\tilde{\subseteq}} (F,\Omega) \mathbin{\tilde{\cap}_R} (F,\Omega).$$

Let $(H',\Sigma') = (F,\Omega) \odot (T,E) \odot (F,\Omega)$. Then $\Sigma' = \Omega \cap E \cap \Omega = \Omega$ and $H'(\varrho) = [F(\varrho)TF(\varrho)]$ for all $\varrho \in \Omega$. For $\varrho \in \Omega$, we have

$$H'(\varrho) = [F(\varrho)TF(\varrho)] \subseteq [TTF(\varrho)] = G(\varrho) \subseteq F(\varrho)$$

Therefore $((F,\Omega),(F,\Omega)) \in PSM$. Hence $((F,\Omega),(F,\Omega)) \in PSL \cap PSM$.

That (5) and (6) hold can be proved in the same manner as (4).

To prove (7), we assume that T is commutative and $((F,\Omega),(G,\Sigma)) \in PSM$. We have $(F,\Omega) \odot (T,E) \odot (G,\Sigma) \mathrel{\tilde{\subseteq}} (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma)$, i.e.,

$$(F,\Omega) \odot (T,E) \odot (G,\Sigma) = (H,\Omega \cap E \cap \Sigma) = (H,\Omega \cap \Sigma)$$

where $H(\varrho) = [F(\varrho)TG(\varrho)]$ for all $\varrho \in \Omega \cap \Sigma$ such that $H(\varrho) \subseteq F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Omega \cap \Sigma$. We have to show that $(G,\Sigma) \odot (T,E) \odot (F,\Omega) \mathrel{\tilde{\subseteq}} (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma)$. Let $(K,\Xi) = (G,\Sigma) \odot (T,E) \odot (F,\Omega)$; then $\Xi = \Omega \cap \Sigma$ and $K(\varrho) = [G(\varrho)TF(\varrho)]$ for all $\rho \in \Xi$. Since T is commutative,

$$K(\varrho) = [G(\varrho)TF(\varrho)] = [F(\varrho)TG(\varrho)] = H(\varrho) \subseteq F(\varrho) \cap G(\varrho)$$

for all $\varrho \in \Omega \cap \Sigma$. Hence $((G,\Sigma),(F,\Omega)) \in SM$.

Hereafter, let S denote a ternary subsemigroup of T. By writing (S,E), we mean a soft set whose universal set is S and whose parameter set is E.

Proposition 3.3: If $((S,E),(F,\Omega)) \in PSL$ (respectively PSR, PSM), then $(S,E) \mathbin{\tilde{\cap}_R} (F,\Omega) \in SL$ (respectively SR, SM).

Proof: The rest can be proved in the same manner. Assume that $((S,E),(F,\Omega)) \in PSL$; then $(T,E) \odot (S,E) \odot (F,\Omega) \mathrel{\tilde{\subseteq}} (S,E) \mathbin{\tilde{\cap}_R} (F,\Omega)$. We have

$$(T,E) \odot (S,E) \odot (F,\Omega) = (G, E \cap E \cap \Omega) = (G,\Omega)$$

where $G(\varrho) = [TSF(\varrho)]$ for all $\varrho \in \Omega$. And,

$$(S,E) \mathbin{\tilde{\cap}_R} (F,\Omega) = (H, E \cap \Omega) = (H,\Omega)$$

where $H(\rho) = S \cap F(\varrho)$ for all $\rho \in \Omega$. Then $[TSF(\varrho)] \subseteq S \cap F(\varrho)$ for all $\varrho \in \Omega$. We have to show that $(S,E) \mathbin{\tilde{\cap}_R} (F,\Omega) \in SL$, i.e., $(S,E) \odot (S,E) \odot ((S,E) \mathbin{\tilde{\cap}_R} (F,\Omega)) \mathrel{\tilde{\subseteq}} (S,E) \mathbin{\tilde{\cap}_R} (F,\Omega)$. Let $(K,\Delta) = (S,E) \odot (S,E) \odot ((S,E) \mathbin{\tilde{\cap}_R} (F,\Omega))$. So $\Delta = E \cap E \cap (E \cap \Omega) = \Omega$. Let $\varrho \in \Omega$; then

$K(\varrho) = [SS(S \cap F(\varrho))] \subseteq [SSS] \cap [SSF(\varrho)] \subseteq [SSF(\varrho)] \subseteq [TSF(\varrho)] \subseteq S \cap F(\varrho)$.

Hence $(S, E) \mathbin{\widetilde{\sqcap}_R} (F, \Omega) \in SL$.

Similarly.

Proposition 3.4: If $((F, \Omega), (S, E)) \in PSL$ (respectively PSR, PSM), then $(S, E) \mathbin{\widetilde{\sqcap}_R} (F, \Omega) \in SL$ (respectively SR, SM).

Proposition 3.5: If $((F, \Omega), (G, \Sigma)) \in PSL$ (respectively PSR, PSM), then $((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (T, E), (G, \Sigma) \mathbin{\widetilde{\sqcap}_R} (T, E)) \in PSL$ (respectively PSR, PSM).

Proof: The rest can be proved in the same manner. From $((F, \Omega), (G, \Sigma)) \in PSR$, it follows that

$$(F, \Omega) \odot (G, \Sigma) \odot (T, E) \mathrel{\widetilde{\subseteq}} (F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma).$$

We have

$$(F, \Omega) \odot (G, \Sigma) \odot (T, E) = (H, \Omega \cap \Sigma \cap E) = (H, \Omega \cap \Sigma)$$

where $H(\varrho) = [F(\varrho)G(\varrho)T]$ for all $\varrho \in \Omega \cap \Sigma$ such that $[F(\varrho)G(\varrho)T] \subseteq F(\varrho) \cap G(\varrho)$ for all $\rho \in \Omega \cap \Sigma$. We have to show that $((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (T, E)) \odot ((G, \Sigma) \mathbin{\widetilde{\sqcap}_R} (T, E)) \odot (S, E) \mathrel{\widetilde{\subseteq}} (F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)$. Let

$$(K, \Xi) = ((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (T, E) \odot ((G, \Sigma) \mathbin{\widetilde{\sqcap}_R} (T, E)) \odot (S, E).$$

We have $\Xi = \Omega \cap \Sigma$, and $K(\varrho) = [F(\varrho)G(\varrho)S]$ for all $\varrho \in \Omega \cap \Sigma$. Let $\varrho \in \Omega \cap \Sigma$; then

$$K(\varrho) = [F(\varrho)G(\varrho)S] \subseteq [F(\varrho)G(\varrho)T] \subseteq F(\varrho) \cap G(\varrho).$$

Hence $((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (T, E), (G, \Sigma) \mathbin{\widetilde{\sqcap}_R} (T, E)) \in PSR$.

Proposition 3.6: Let $((F, \Omega), (G, \Sigma)) \in PSL$. If $(H, \Lambda), (I, \Delta) \mathrel{\widetilde{\subseteq}} (F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)$, then

$$(((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta), ((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta)) \in PSL.$$

Proof: Assume that $(H, \Lambda), (I, \Delta) \mathrel{\widetilde{\subseteq}} (F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)$. Then $\Lambda, \Delta \subseteq \Omega \cap \Sigma$, $H(\varrho) \subseteq F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Lambda$, and $I(\varrho) \subseteq F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Delta$. Since $((F, \Omega), (G, \Sigma)) \in PSL$, $[TF(\varrho)G(\varrho)] \subseteq F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Omega \cap \Sigma$. We have to show that

$$(((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta), ((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta)) \in PSL,$$

i.e.,

$$\begin{aligned} &(T, E) \odot (((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta)) \odot (((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta)) \\ &\mathrel{\widetilde{\subseteq}} (((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta)) \mathbin{\widetilde{\sqcap}_R} (((F, \Omega) \mathbin{\widetilde{\sqcap}_R} (G, \Sigma)) \odot (H, \Lambda) \odot (I, \Delta)). \end{aligned}$$

We have

$$(T,E) \odot (((F,\Omega) \mathbin{\widetilde{\cap}_R} (G,\Sigma)) \odot (H,\Lambda) \odot (I,\Delta)) \odot (((F,\Omega) \mathbin{\widetilde{\cap}_R} (G,\Sigma)) \odot (H,\Lambda) \odot (I,\Delta)) = (K,\Lambda \cap \Delta)$$

where

$$K(\varrho) = [T[(F(\varrho) \cap G(\varrho))H(\varrho)I(\varrho)][(F(\varrho) \cap G(\varrho))H(\varrho)I(\varrho)]]$$

for all $\varrho \in \Lambda \cap \Delta$, and

$$(((F,\Omega) \mathbin{\widetilde{\cap}_R} (G,\Sigma)) \odot (H,\Lambda) \odot (I,\Delta)) \mathbin{\widetilde{\cap}_R} (((F,\Omega) \mathbin{\widetilde{\cap}_R} (G,\Sigma)) \odot (H,\Lambda)\mathrm{e}(I,\Delta)) = (L,\Lambda \cap \Delta)$$

where $[(F(\varrho) \cap G(\varrho))H(\varrho)I(\varrho)]$ for all $\varrho \in \Lambda \cap \Delta$. Let $\varrho \in \Lambda \cap \Delta$. We have

$$\begin{aligned} & [T[(F(\varrho) \cap G(\varrho))H(\varrho)I(\varrho)][(F(\varrho) \cap G(\varrho))H(\varrho)I(\varrho)]] \\ \subseteq\ & [T[TTI(\varrho)][G(\varrho)H(\varrho)I(\varrho)]] \\ =\ & [[TTT]I(\varrho)[G(\varrho)H(\varrho)I(\varrho)]] \\ \subseteq\ & [[TTT]F(\varrho)[G(\varrho)H(\varrho)I(\varrho)]] \\ \subseteq\ & [TF(\varrho)[G(\varrho)H(\varrho)I(\varrho)]] \\ \subseteq\ & [[TF(\varrho)G(\varrho)]H(\varrho)I(\varrho)] \\ \subseteq\ & [(F(\varrho) \cap G(\varrho))H(\varrho)I(\varrho)]. \end{aligned}$$

Hence

$$(((F,\Omega) \mathbin{\widetilde{\cap}_R} (G,\Sigma)) \odot (H,\Lambda) \odot (I,\Delta), ((F,\Omega) \mathbin{\widetilde{\cap}_R} (G,\Sigma)) \odot (H,\Lambda) \odot (I,\Delta)) \in PSL.$$

Proposition 3.7: Let $(F,\Omega) \in SM$ and $(G,\Sigma) \in SL$ such that

$$(F,\Omega) \odot (F,\Omega) \odot (H,\Lambda) \odot (F,\Omega) \odot (F,\Omega) = (F,\Omega) \odot (H,\Lambda) \odot (F,\Omega)$$

for all non-empty soft sets $(H,\Lambda) \mathbin{\widetilde{\subseteq}} (F,\Omega)$. Then

$$((F,\Omega) \odot (H,\Lambda) \odot (F,\Omega), (G,\Sigma) \odot (G,\Sigma) \odot (I,\Delta)) \in PSL$$

for any non-empty soft set $(I,\Delta) \mathbin{\widetilde{\subseteq}} (G,\Sigma)$.

Proof: Since $(F,\Omega) \in SM$, $[TF(\varrho)T] \subseteq F(\varrho)$ for all $\varrho \in \Omega$. Since $(G,\Sigma) \in SL$, $[TTG(\varrho)] \subseteq G(\varrho)$ for all $\varrho \in \Sigma$. We have to show that

$$((F,\Omega) \odot (H,\Lambda) \odot (F,\Omega), (G,\Sigma) \odot (G,\Sigma) \odot (I,\Delta)) \in PSL$$

for any non-empty soft set $(I,\Delta) \mathbin{\widetilde{\subseteq}} (G,\Sigma)$. Let $(I,\Delta) \mathbin{\widetilde{\subseteq}} (G,\Sigma)$ be a non-empty soft set. If $\varrho \in \Lambda \cap \Delta$, then

$$\begin{aligned} & [T[F(\varrho)H(\varrho)F(\varrho)][G(\varrho)G(\varrho)I(\varrho)]] \\ \subseteq\ & [T[F(\varrho)F(\varrho)H(\varrho)F(\varrho)F(\varrho)][G(\varrho)G(\varrho)I(\varrho)]] \\ \subseteq\ & [[TF(\varrho)T]H(\varrho)[TF(\varrho)T]] \\ \subseteq\ & [F(\varrho)H(\varrho)F(\varrho)] \end{aligned}$$

and

$$\begin{aligned} [T[F(\varrho)H(\varrho)F(\varrho)][G(\varrho)G(\varrho)I(\varrho)]] & \subseteq [TT[G(\varrho)G(\varrho)I(\varrho)]] \\ & = [[TTG(\varrho)]G(\varrho)I(\varrho)] \\ & \subseteq [G(\varrho)G(\varrho)I(\varrho)]. \end{aligned}$$

Hence $((F,\Omega) \odot (H,\Lambda) \odot (F,\Omega), (G,\Sigma) \odot (G,\Sigma) \odot (I,\Delta)) \in PSL$.

Definition 3.8: A non-null and non-empty $(F,\Omega) \in SST$ is a soft bi-ideal if

(1) $(F,\Omega) \odot (F,\Omega) \odot (F,\Omega) \mathrel{\tilde{\subseteq}} (F,\Omega)$; and

(2) $(F,\Omega) \odot (T,E) \odot (F,\Omega) \odot (T,E) \odot (F,\Omega) \mathrel{\tilde{\subseteq}} (F,\Omega)$.

Let SB denote the set of all soft bi-ideals of T.

Proposition 3.9: Let $((F,\Omega),(G,\Sigma)) \in PSM$. Then $(F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \in SB$.

Proof: From $((F,\Omega),(G,\Sigma)) \in PSM$, it follows that

$$(F,\Omega) \odot (T,E) \odot (G,\Sigma) \mathrel{\tilde{\subseteq}} (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma),$$

i.e., $[F(\varrho)TG(\varrho)] \subseteq F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Omega \cap \Sigma$. We have to show that

$$(F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \odot (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \odot (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \mathrel{\tilde{\subseteq}} (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma)$$

and

$$((F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma)) \odot (T,E) \odot ((F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma)) \odot (T,E) \odot ((F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma))$$
$$\mathrel{\tilde{\subseteq}} ((F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma)).$$

Let $\rho \in \Omega \cap \Sigma$; then

$$[(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))] \subseteq [F(\varrho)TG(\varrho)] \subseteq F(\varrho) \cap G(\varrho)$$

and

$$\begin{aligned} &[(F(\varrho) \cap G(\varrho))T(F(\varrho) \cap G(\varrho))T(F(\varrho) \cap G(\varrho))] \\ \subseteq\ &[F(\varrho)TG(\varrho)T(F(\varrho) \cap G(\varrho))] \\ \subseteq\ &[(F(\varrho) \cap G(\varrho))T(F(\varrho) \cap G(\varrho))] \\ \subseteq\ &[F(\varrho)TG(\varrho)] \\ \subseteq\ &F(\varrho) \cap G(\varrho). \end{aligned}$$

Hence $(F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \in SB$.

Corollary 3.10: *Let* $((F,\Omega),(G,\Sigma)) \in PSM$. *Then* $(F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \mathbin{\tilde{\cap}_R} (T,E) \in SB$.

Definition 3.11: T is said to be quasi-regular if for any $\eta \in T$ there exists $\nu \in T$ such that $\eta = [\eta[\eta\nu\eta]\eta]$.

Theorem 3.12: Let T be quasi-regular. If $((F,\Omega),(G,\Sigma)) \in PSR$ and $((G,\Sigma),(H,\Xi)) \in PSL$, then $(F,\Omega) \odot (G,\Sigma) \odot (H,\Xi) = (F,\Omega) \mathbin{\tilde{\cap}_R} (G,\Sigma) \mathbin{\tilde{\cap}_R} (H,\Xi)$.

Proof: By $((F,\Omega),(G,\Sigma)) \in PSR$,

$$[F(\varrho)G(\varrho)H(\varrho)] \subseteq [F(\varrho)G(\varrho)T] \subseteq F(\varrho) \cap G(\varrho)$$

for all $\varrho \in \Omega \cap \Sigma \cap \Xi$. Since $((G,\Sigma),(H,\Xi)) \in PSL$,

$$[F(\varrho)G(\varrho)H(\varrho)] \subseteq [TG(\varrho)H(\varrho)] \subseteq G(\varrho) \cap H(\varrho)$$

for all $\varrho \in \Omega \cap \Sigma \cap \Xi$. Thus

$$[F(\varrho)G(\varrho)H(\varrho)] \subseteq F(\varrho) \cap G(\varrho) \cap H(\varrho)$$

for all $\varrho \in \Omega \cap \Sigma \cap \Xi$. We claim that $F(\varrho) \cap G(\varrho) \cap H(\varrho) \subseteq [F(\varrho)G(\varrho)H(\varrho)]$ for all $\varrho \in \Omega \cap \Sigma \cap \Xi$. Let $\varrho \in \Omega \cap \Sigma \cap \Xi$. Let $\eta \in F(\varrho) \cap G(\varrho) \cap H(\varrho)$. Since T is quasi-regular, $\eta = [\eta[\eta\nu\eta]\eta]$ for some $\nu \in T$. We have

$$\eta = [\eta[\eta\nu\eta]\eta] = [[\eta[\eta\nu\eta]\eta][\eta\nu\eta]\eta] = [[\eta\eta\nu][\eta\eta\eta][\nu\eta\eta]] \in [F(\varrho)G(\varrho)H(\varrho)].$$

So we have the claim. Then $[F(\varrho)G(\varrho)H(\varrho)] = F(\varrho) \cap G(\varrho) \cap H(\varrho)$ for all $\varrho \in \Omega \cap \Sigma \cap \Xi$. Therefore

$$(F,\Omega) \odot (G,\Sigma) \odot (H,\Xi) = (F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma) \,\widetilde{\cap}_R\, (H,\Xi).$$

Theorem 3.13: *Let T be quasi-regular. If $((F,\Omega),(G,\Sigma)) \in PSM$, then*

$$((F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma)) \odot ((F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma)) \odot ((F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma)) = (F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma).$$

Proof: Assume that $((F,\Omega),(G,\Sigma)) \in PSM$. Then

$$[(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))] \subseteq [F(\varrho)TG(\varrho)] \subseteq F(\varrho) \cap G(\varrho)$$

for all $\varrho \in \Omega \cap \Sigma$. For the reverse inclusion, let $\eta \in F(\varrho) \cap G(\varrho)$. Then there exists $\nu \in T$ such that $\eta = [\eta[\eta\nu\eta]\eta]$. By assumption,

$$[\eta\nu\eta] \in [F(\varrho)TG(\varrho)] \subseteq F(\varrho) \cap G(\varrho).$$

Hence

$$\eta = [\eta[\eta\nu\eta]\eta] \in [(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))].$$

We then have $[(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))] = F(\varrho) \cap G(\varrho)$ for all $\varrho \in \Omega \cap \Sigma$. Therefore

$$((F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma)) \odot ((F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma)) \odot ((F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma)) = (F,\Omega) \,\widetilde{\cap}_R\, (G,\Sigma).$$

Simialrly,

Corollary 3.14: *Let T be quasi-regular. If $((F,\Omega),(G,\Sigma)) \in PSM$, then*

$$((F,\Omega) \odot (T,E) \odot (G,\Sigma)) \odot ((F,\Omega) \odot (T,E) \odot (G,\Sigma)) \odot ((F,\Omega) \odot (T,E) \odot (G,\Sigma)) = (F,\Omega) \odot (T,E) \odot (G,\Sigma).$$

Definition 3.15: An element $\eta \in T$ is said to be left (respectively right) regular if there exists $\nu \in T$ such that $\eta = [\nu\eta\eta]$ (respectively $\eta = [\eta\eta\nu]$).

Definition 3.16: A proper pair soft ideal $((F,\Omega),(G,\Sigma)) \in PSI$ is said to be completely semiprime if for any $\rho \in \Omega \cap \Sigma$, $[\eta\eta\eta] \in F(\varrho) \cap G(\varrho)$ implies $\eta \in F(\varrho) \cap G(\varrho)$ for all $\eta \in T$.

Theorem 3.17: T is left (respectively right) regular if and only if every pair soft left (respectively right) ideal of T is completely semiprime.

Proof: Assume that T is left regular. Let $((F,\Omega),(G,\Sigma)) \in PSL$; then

$$[TF(\varrho)G(\varrho)] \subseteq F(\varrho) \cap G(\varrho)$$

for all $\varrho \in \Omega \cap \Sigma$. Let $\varrho \in \Omega \cap \Sigma$. Let $\eta \in T$ be such that $[\eta\eta\eta] \in F(\varrho) \cap G(\varrho)$. Since T is left regular, there exists $\nu \in T$ such that $\eta = [\nu\eta\eta]$. From $\eta = [[\nu\nu[\nu\nu\nu]][\eta\eta\eta][\eta\eta\eta]]$, it follows that

$$\eta \in [[TT[TTT]](F(\varrho) \cap G(\varrho))(F(\varrho) \cap G(\varrho))] \subseteq [TF(\varrho)G(\varrho)] \subseteq F(\varrho) \cap G(\varrho).$$

Thus $((F,\Omega),(G,\Sigma))$ is completely semiprime. Conversely, assume that every pair soft left ideal of T is completely semiprime. To show that T is left regular, let $\eta \in T$. Since $((T,E),([T\eta\eta],E)) \in PSL$, $((T,E),([T\eta\eta],E))$ is completely semiprime. Then $\eta \in [T\eta\eta]$, and so $\eta = [\nu\eta\eta]$ for some $\nu \in T$. Therefore, T is left regular.

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