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On some specific order estimates for the Mertens function and its relation to the non-trivial zeros of ζ(s)

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Abstract

This article presents a brief overview of the Mertens function $M(n)$ and Redheffer matrices $\mathbb{A}_n$, deriving specific order estimates for $M(n)$ to investigate the distribution of non-trivial zeros of the Riemann zeta function $\zeta(s)$. Using these tools, we establish a necessary and sufficient condition for the validity of the Riemann Hypothesis.

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1. Introduction & Motivation

The *Riemann Hypothesis (RH)* remains one of the most celebrated unsolved problems in pure mathematics, particularly in number theory, with implications spanning fields such as quantum physics and computer science. While many equivalent formulations of RH are known, matrix-theoretic approaches are relatively rare yet remarkably insightful. This article focuses on a notable such approach introduced by *Redheffer* [43], who defined a special class of non-symmetric matrices $\mathbb{A}_n$, now known as *Redheffer matrices*, and proved that the bound,

$$\det(\mathbb{A}_n) = O(n^{\frac{1}{2}+\varepsilon}), \forall\, \varepsilon > 0$$

is both necessary and sufficient for RH to hold. A similar construction was later given by *Roesler* [49], involving matrices with comparable spectral properties.

The determinant behavior of these matrices is deeply linked to the *Mertens function* $M(n)$, which is the summatory function of the Möbius function $\mu(n)$

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(2.2.1) [50, pp. 91]. Sections 3 through 5 introduce these concepts, explore their interrelations, and examine the spectral properties of $\mathbb{A}_n$.

In Section 6, we derive order estimates for both $M(n)$ and its continuous analogue $M(x)$ (2.2.1), which are vital for analyzing the zero distribution of the Riemann zeta function $\zeta(s)$. In this context, recent work by *Alegri, Bonilla* and *Reddy* [10] provides complementary insights via *multiple zeta values* (MZVs) and *integer compositions* - tools that reinforce the analytic structures explored here.

Finally, Section 6 addresses the now-disproven *Mertens Hypothesis (MH)* through numerical and heuristic investigations, aimed at identifying the smallest counterexample. Although exact validation remains elusive, this remains an active field of inquiry. For further depth, readers are encouraged to consult the references.

2. A Survey on Mertens Function

2.1 *Notion of Arithmetic Functions*

We recall that, *Arithmetic Functions* are real or complex valued functions defined on the set of natural numbers $\mathbb{N}$. In this section, we shall define some specific examples of arithmetic functions pertinent to the concept of *Mertens Function*.

Definition 2.1.1: (Möbius Function) The Möbius Function $\mu: \mathbb{N} \to \{0, \pm 1\}$ is defined as follows:

$$\mu(n) := \begin{cases} (-1)^k & \text{if } n = \prod_{i=1}^{k} {p_i}^{a_i} \text{ such that, } gcd(p_i, p_j) = \forall\ i \neq j \\ 1 & \text{if, } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can in fact use *definition* (2.1.1) to deduce the following.

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & \text{if, } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

2.2 Formal Definition of $M(n)$

Definition 2.2.1: (Mertens Function) The Mertens Function [5] $M: \mathbb{N} \to \mathbb{Z}$ is defined as,

$$M(n) := \sum_{d=1}^{n} \mu(d) \tag{2.2}$$

Remark 2.2.1: In general, we can define the Extended Mertens Function as,

$$M(x) := \sum_{1 \le n \le x} \mu(n)\ ,\ \forall\ x \in \mathbb{R}$$

Mertens [5] conjectured that, for all $M(n)$ with $1 \le n \le 10^4$, we shall have, $|M(n)| < \sqrt{n}$, also known as the *Mertens Hypothesis* [2, Thm 14.28, pp. 374]. Extending *Mertens'* results further upto $n = 5 \times 10^6$, *Sterneck* [6] conjectured that,

$$|M(n)| < \frac{1}{2}\sqrt{n}\ ,\ \forall\ n > 200.$$

Mertens introduced the function $M(x)$ to study its connection to the zeros of the Riemann zeta function $\zeta(s)$, due to its profound implications for prime distribution—one of the central problems in analytic number theory.

3. The Riemann Hypothesis Revisited

We begin with an introduction to the Riemann Zeta Function $\zeta(s)$, initially studied by Euler and later extended by *Riemann*, which plays a central role in the Riemann Hypothesis (RH) and finds applications in modular forms, class field theory, and more. It can be defined formally as,

$$\zeta(s) := \sum_{n=1}^{\infty} \quad \text{where} \quad s \in \mathbb{C}, \Re(s) > 1. \tag{3.1}$$

By analytic continuation, $\zeta(s)$ extends to a meromorphic function on $\mathbb{C}$, with a simple pole at $s = 1$ and residue 1. For a detailed discussion, see *Ahlfors* [4]. Using the *Euler Product Formula*, we can pursue an alternative approach to define $\zeta(s)$ as,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=primes} \left\{\frac{1}{1-p^{-s}}\right\} \tag{3.2}$$

Where, the product on the R.H.S. is taken over all primes p, and converges for $\Re(s) > 1$.

Some of the analytic properties of $\zeta(s)$ can be summarized as follows.

- $\zeta(s)$ is *meromorphic* on $\mathbb{C}$.
- $\zeta(s)$ has a simple *pole* at $s = 1$ with *residue* 1.
- $\zeta(1)$ is a *Harmonic Series* that diverges to $+\infty$.
- $\zeta(2) = \frac{\pi^2}{6}$. (Also known as the *" Basel Problem"*)

In fact, *Definition* (2.1.1) does help us derive,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{3.3}$$

The functional equation for $\zeta(s)$ implies that its trivial zeros lie at negative even integers: $s = -2k$, $k \in \mathbb{N}$. In his seminal 1859 paper [3], Riemann conjectured that all non-trivial zeros lie within the critical strip $0 < \Re(s) < 1$, and are symmetric about the real axis and the critical line $\Re(s) = \frac{1}{2}$.

Theorem 3.0.1: (Riemann Hypothesis) *All the non-trivial zeroes of the Riemann Zeta Function* $\zeta(s)$ *lie on the critical line,* $\Re(s) = \frac{1}{2}$.

RH, one of the Millennium Prize Problems and Hilbert's 8^{th} problem, remains unsolved. One of its most significant implications lies in its connection to the prime counting function, $\pi(x)$, denoting the exact number of prime numbers $\leq x$, for every $x \geq 0$. In particular, *von Koch* [7] established that, RH is equivalent to,

$$\pi(x) = Li(x) + O(\sqrt{x} \ln x).$$

To be more specific, *Schoenfield* [8] proved that, under RH,

$|\pi(x) - Li(x)| \leq \frac{1}{8\pi}\sqrt{x}\ln x$, $\forall\ x \geq 2657$, where, $Li(x) := \int_2^x \frac{1}{\ln t}dt$.

Given the close resemblance between the explicit formula for $\pi(x)$ and $Li(x)$, the approach to RH using linear algebra, particularly via *Redheffer matrices* (discussed in the next section), offers a promising alternative analytic pathway.

4. Redheffer Matrices: An Introduction

4.1 *Definition*

Definition 4.1.1: (Redheffer Matrix) Often denoted by $\mathbb{A}_n$, Redheffer Matrix was first introduced by mathematician Redheffer (1977), and is defined to be a square $(0,1)$ matrix of the form, $\mathbb{A}_n := (a_{ij})_{n\times n}$ such that, for every $1 \leq i,j \leq n$,

$$a_{ij} := \begin{cases} 1 & \text{if, } i|j \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

A priori given the *invertibility* of the Redheffer Matrices being relatively complicated due to the existence of the initial column of 1's in the matrix, $\mathbb{A}_n$ is often expressed as,

$$\mathbb{A}_n := C_n + D_n$$

where, $C_n := (c_{ij})_{n\times n}$ and, $D_n := (d_{ij})_{n\times n}$ are the $(0,1)$ matrices defined as,

$$c_{ij} := \begin{cases} 1 & \text{if} f, i \neq 1, j = 1. \\ 0 & \text{otherwise.} \end{cases} \text{and, } \quad d_{ij} := \begin{cases} 1 & \text{if} f, i|j \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

for every $i = 1, 2, 3, \ldots, n$ and, $j = 1,2,3, \ldots, n$. For example, a 4×4 *Redheffer Matrix* can be represented in the following manner:

$$\mathbb{A}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, one can observe that,

$$C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and, } D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.2 *Relation with Mertens Function* $M(n)$

Theorem 4.2.1: (Determinant of $\mathbb{A}_n$) *We have, for the Redheffer Matrix* $\mathbb{A}_n$, $det(\mathbb{A}_n) = M(n)$.

Proof: Proof uses the following result.

Lemma 4.1: *The inverse of the matrix* D_n *has the expression,* $D_n^{-1} = \left(\delta_{ij}\right)_n$, *where,*

$$\delta_{ij} = \begin{cases} \mu\left(\frac{j}{i}\right) & \text{if, } i|j \\ 0 & \text{otherwise.} \end{cases}$$

Lemma (4.1) yields, $det(D_n{}^{-1}) = 1$. Hence, $det(\mathbb{A}_n) = det(D_n{}^{-1}\mathbb{A}_n) = det(D_n{}^{-1}C_n + I_n)$.

Hence, $D_n{}^{-1}C_n$ is *lower triangular*, so is $(D_n{}^{-1}C_n + I_n)$, which implies that, $det(D_n{}^{-1}C_n + I_n)$ is the product of its diagonal entries. Subsequently,

$$det(D_n{}^{-1}C_n + I_n) = \{\sum_{k=2}^{n} \mu(k) + 1\} = M(n).$$

4.3 *Characteristic Polynomial of* $\mathbb{A}_n$

We start with recalling that, A **directed graph**, also called a digraph, is a graph in which each of the edges has a direction. Furthermore, a **cycle** is apparently a set of directed edges $\{(i_1, j_1), (i_2, j_2), \ldots\ldots, (i_k, j_k)\}$ is called a cycle of length k if, $j_r = i_{r+1}$, $\forall\ r = 1(1)k - 1$, and $j_k = i_1$, and $i_1, i_2, \ldots\ldots, i_k$ are distinct.

One method to derive the *characteristic polynomial* of $\mathbb{A}_n$ is to use the concept of *directed graph* $\mathcal{D}(M_n)$ corresponding to the matrix, $M_n := xI_n - \mathbb{A}_n$. Observe that, $\mathcal{D}(M_n)$ has vertex set, $\Lambda = \{1,2, \ldots\ldots, n\}$ and a *directed edge* from vertex i to vertex j *iff*, $m_{ij} \neq 0$. In other words, $\mathcal{D}(M_n)$ will have a directed edge from i to j *iff*, $i|j$ or, $j = 1$. We denote it by the ordered pair (i, j). Moreover, suppose $\mathfrak{c} = \{(i_1, i_2), (i_2, i_3), \ldots\ldots, (i_k, i_1)\}$ is a *cycle* in $\mathcal{D}(M_n)$.

We set, $\mathcal{P}[\mathfrak{c}] = m_{i_1 i_2} m_{i_2 i_3} \ldots\ldots m_{i_k i_1}$. Also, for any $\mathcal{I} \subseteq \Lambda$, we define $M_n[\mathcal{I}]$ as the *Principal Submatrix* of M_n with rows and columns in $\mathcal{I}$. Assume $\mathcal{C}_1$ to be the set of *cycles* in $\mathcal{D}(M_n)$ containg the vertex 1, and for every $\mathfrak{c} \in \mathcal{C}_1$, denote its *length* by $l(\mathfrak{c})$. Also, suppose $\mathfrak{c}'$ be the *complement* of the vertices connected by $\mathfrak{c}$ in Λ.

Using [12, Theorem 2, pp. 503], we can assert that,

$$det(M_n) = \sum_{\mathfrak{c} \in \mathcal{C}_1} (-1)^{l(\mathfrak{c})+1}.\mathcal{P}[\mathfrak{c}].det(M_n[\mathfrak{c}']).$$

We can further claim that, the principal submatrix $M_n[\mathfrak{c}']$ is *upper-triangular* [a priori since, each $\mathfrak{c} \in \mathcal{C}_1$ contains the vertex 1] having determinant equal to $(x-1)^{n-l(\mathfrak{c})}$. Moreover, for $l(\mathfrak{c}) \geq 2$, $\mathcal{P}[\mathfrak{c}]$ doesn't contain any diagonal entry and precisely is equal to $(-1)^{l(\mathfrak{c})}$. Since, the only cycle of *length* 1 is, $\mathfrak{c} = \{(1,1)\}$, hence, $\mathcal{P}[\mathfrak{c}] = x - 1$.

Therefore, we obtain the *characteristic equation* of $\mathbb{A}_n$ as,

$$det(M_n) = (x-1)^n - \sum_{\substack{\mathfrak{c} \in \mathcal{C}_1 \\ l(\mathfrak{c}) \geq 2}} (x-1)^{n-l(\mathfrak{c})}$$

which can be rewritten as,

$$det(M_n) = (x-1)^n - \sum_{k=2}^{n} (x-1)^{n-k} v_{n,k-1}.$$

where, $v_{n,k}$ denotes the number of *cycles* starting at vertex 1, of length $k+1$, in the graph $\mathcal{D}(M_n)$. In other words, if a *k-list* in Λ is a list $\{1, i_1, i_2, \ldots\ldots, i_k\}$ with $i_r | i_{r+1}$, $\forall\ \ r = 1(1)k - 1$ and $1 < i_1 < i_2 < \ldots\ldots < i_k \leq n$, then, $v_{n,k}$ shall denote the number of k-lists in Λ.

Remark 4.3.1: Similar computation for a particular case $\mathbb{A}_6$ can be found in [13, pp. 676].

Remark 4.3.2: It has been deduced that, $v_{n,1} = n - 1$, and, $v_{n,2} = \sum_{k=2}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - 1 \right)$. Also, we can conclude that, each $v_{n,k}$ is positive and bounded above, as evident from the relation,

$$v_{n,k} < n.\frac{(\log n)^{k-1}}{(k-1)!} \tag{4.3}$$

4.4 *Eigenvalues of* $\mathbb{A}_n$

Theorem 4.4.1: *The Redheffer Matrix* $\mathbb{A}_n$ *has eigenvalue* 1 *with multiplicity* $n - \lfloor log_2 n \rfloor - 1$.

Proof: Here, instead of computing the eigenvalues using the characteristic polynomial of $\mathbb{A}_n$ given by, $p_n(x) = det(xI_n - \mathbb{A}_n)$, we work with a certain closely related polynomial, $q_n(x) = det\big(xI_n + (\mathbb{A}_n - I_n)\big)$. Thus using properties of determinants, we obtain,

$$p_n(x) = (-1)^n det\{(1-x)I_n + (\mathbb{A}_n - I_n)\} = (-1)^n q_n(1-x) \ .$$

Implying that, each eigenvalue α_j of $\mathbb{A}_n$ can be expressed as $(1 - \alpha'_j)$, where, α'_j is a root of $q_n(x)$, $\forall\ j = 1,2,3, \ldots., n$. Hence it only suffices to compute roots of $q_n(x)$ in order to deduce the eigenvalues of $\mathbb{A}_n$.

Using properties of *directed graphs*, we can associate a directed graph with any given square matrix, say, $B := (b_{ij})_{n\times n}$, all of whose entries are either 0 or 1, by assigning a vertex to each row (or column equivalently) and including an edge between the i^{th} and the j^{th} vertices iff, $b_{ij} = 1$, $\forall\ i,j = 1,2,3, \ldots., n$. Therefore, corresponding to the matrix $(\mathbb{A}_n - I_n)$, we can infer,

$$q_n(x) = x^n + \sum_{k=1}^{n} (-1)^{k-1} c(n,k) x^{n-k}$$

Where, $c(n,k)$ denotes the cycles of length k in the directed graph corresponding to the matrix $(\mathbb{A}_n - I_n)$. Now, for the proof of *Theorem 4.4.1*, observe that, in the directed graph of $(\mathbb{A}_n - I_n)$, the longest cycle is,

$$1 \to 2 \to 2^2 \to \ldots\ldots.. \to 2^{\lfloor \log_2 n \rfloor} \to 1 \ .$$

(Since, each entry in the cycle must be a non-trivial multiple of the previous one, and multiplying by 2 each time increases the cycle's elements as slowly as possible .)

The cycle contains $\lfloor \log_2 n \rfloor + 1$ elements, which yields, $c(n,k) = 0$ for $k > \lfloor \log_2 n \rfloor + 1$. The lowest power of x occuring in $q_n(x)$ is, $x^{n - \lfloor \log_2 n \rfloor - 1}$. Thus, q_n has a root 0 with multiplicity $n - \lfloor \log_2 n \rfloor - 1$, implying that, p_n has a root 1 with multiplicity $n - \lfloor \log_2 n \rfloor - 1$.

With this information, we can make further deductions about the spectral radius of $\mathbb{A}_n$.

4.5 *Spectral Radius of* $\mathbb{A}_n$

We recall that, **Spectral Radius** of a matrix is defined to be the maximum absolute value of its eigenvalues.

Remark 4.5.1: $\mathbb{A}_n$ has a positive spectral radius. (A priori given the **"Perron-Frobenius Theorem"**, we can deduce that, $\mathbb{A}_n$ clearly satisfies the irreducibility criterion mentioned in the theorem, hence, the statement follows.)

Barrett, *Forcade* and *Follington* [1] deduced the upper bounds for the co-efficients of the *characteristic polynomial* of $\mathbb{A}_n$. If ρ_n denotes the *spectral radius* of $\mathbb{A}_n$, then using these bounds, they proved the following asymptotic relation for ρ_n.

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{n}} = 1.$$

Moreover, *Barrett* [9] proposed several interesting conjectures based on numerical evidence regarding the other eigenvalues of $\mathbb{A}_n$. One of the significant among them was,

Theorem 4.5.2: *$\mathbb{A}_n$ has a negative (real) eigenvalue of magnitude $-\sqrt{n}$, and the remaining eigenvalues are bounded and are close to the origin.*

In [11, theorem 2, pp. 147], the existence of such a negative real eigenvalue has been proven. Furthermore, we can conclude that it is asymptotically equal to $-\sqrt{n}$ as $n \to \infty$, and also that, the remaining eigenvalues of $\mathbb{A}_n$ have magnitude $O\left(\frac{\sqrt{n}}{\log n}\right)$.

Again, given that most of the eigenvalues of $\mathbb{A}_n$ are 1, we can comment on their corresponding eigenspaces.

4.6 *Eigenspaces of $\mathbb{A}_n$ corresponding to its Eigenvalues*

Theorem 4.6.1: *The eigenspace of $\mathbb{A}_n$ corresponding to the eigenvalue 1 has dimension $\left\lfloor\frac{n}{2}\right\rfloor - 1$. In particular, $\mathbb{A}_n$ is non-diagonalizable for $n \geq 5$.*

Proof: Follows by obtaining the *row-reduced echelon form* of the matrix $(\mathbb{A}_n - I_n)$ and then determining its nullity.

A priori from the fact that, the *eigenspaces* of $\mathbb{A}_n$ and $\mathbb{A}_n^T$ are *isomorphic*, we can comment further about the eigenspaces corresponding to the eigenvalues of $\mathbb{A}_n$ that are not equal to 1 .

Theorem 4.6.2: *Every eigenspace of $\mathbb{A}_n$ with eigenvalue not equal to 1 has dimension 1.*

Proof: As for the proof, we use the following result.

Lemma 4.2: *$(w_1, w_2, \ldots\ldots, w_n)$ is an eigenvector of $\mathbb{A}_n^T$ with λ as an eigenvalue iff,*

- $\sum_{d|n} w_d = \lambda w_n$
- $\sum_{j=1}^{n} w_j = \lambda w_1$

Observe that, if $\lambda \neq 1$ be an *eigenvalue* of $\mathbb{A}_n^T$, then, given a value for w_1, we can *uniquely* determine the remaining w_j's using *Lemma (4.2)* and the following recursive relation,

$$w_j = \frac{1}{(\lambda-1)} \sum_{\substack{d|j \\ d\neq j}} w_d$$

Hence the result follows.

As a result of the above concept, we can introduce another arithmetic function in the next section.

4.7 *The Arithmetic Function "ν_λ"*

Define the function $\nu_\lambda : \mathbb{N} \to \mathbb{R}$ for arbitrary $\lambda \neq 1$ using the following recursive formula,

$$\nu_\lambda(n) := \begin{cases} 1 & \text{if}, n = 1 \\ \frac{1}{(\lambda-1)} \sum_{d|n, d\neq n} \nu_\lambda(d) & \text{otherwise.} \end{cases} \tag{4.4}$$

Clearly, (4.4) allows us to conclude that, λ is an *eigenvalue* of $\mathbb{A}_n$ iff, $\sum_{k=1}^{n} \nu_\lambda(k) = \lambda$. Moreover, given the definition of ν_λ, we can obtain the following estimate regarding it's corresponding *Dirichlet Series*.

$$V_\lambda(s) = \sum_{n=1}^{\infty} \frac{\nu_\lambda(n)}{n^s} \tag{4.5}$$

As a consequence, we obtain,

$$V_\lambda(s) = \sum_{n=1}^{\infty} \frac{\nu_\lambda(n)}{n^s} = \frac{\lambda-1}{\lambda-\zeta(s)} \tag{4.6}$$

If $a_k(n)$ denotes the number of ways of expressing a natural number n as a product of k factors in some particular order, then, we have the following recursive definition given by,

$$a_k(n) = \begin{cases} 1 & \text{if}, k = 0, n = 1 \\ 0 & \text{if}, k = 0, n \neq 1 \\ \sum_{d|n} a_{k-1}\left(\frac{n}{d}\right) & \text{if}, k > 0. \end{cases} \tag{4.7}$$

We can have another expression for ν_λ applying (4.7).

$$\nu_\lambda(n) = \left(1 - \frac{1}{\lambda}\right) \sum_{k=0}^{\infty} \frac{a_k(n)}{\lambda^k} \text{ for any } n \in \mathbb{N} \text{ where, } \lambda > 1. \tag{4.8}$$

Remark 4.7.1: The Redheffer Matrix does provide an alternative approach on different notions of divisibility. Being closely related to the Riemann Zeta Functions, it effectively bundles together the divisibility properties of many numbers into a single term, which can then be studied using techniques of Linear Algebra .

5. Order Estimates for $M(x)$ and its relation to the non-trivial zeros of $\zeta(s)$

As noted in the introduction, only a few among the many equivalent formulations of the Riemann Hypothesis have been explored through linear algebra and advanced matrix theory. *Mertens* [5] introduced the notion of the *extended Mertens Function*, $M(x)$ and suggested a completely different method to analyze the zeros of the location of the *Riemann Zeta Function* $\zeta(s)$. A priori for $\Re(s) > 1$, one can infer that,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{M(n)-M(n-1)}{n^s} = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{(n++1)^s}\right) = s\int_1^{\infty} \frac{M(x)}{x^{s+1}}dx \tag{5.1}$$

If we assume, $M(x) = O(x^{\varepsilon})$ for some $\varepsilon > 0$, then the integral on the R.H.S. of (5.1) represents an analytic function in the half-plane, $\Re(s) > \varepsilon$. Thus, $\frac{1}{\zeta(s)}$ shall also be analytic in that region. Consequently, we can in fact conclude, $\zeta(s) \neq 0$, which, in turn will imply [14, Thm 30, pp. 83], $\pi(x) = Li(x) + O(x^{\varepsilon}\log x)$. Clearly this relation justifies the significance of studying the order of $M(x)$. In particular, it seems obvious that, RH follows from the statement of MH, which can be generalized as follows:

$$M(x) = O\left(x^{\frac{1}{2}}\right) \tag{5.2}$$

A slightly more rigorous computation of the statement in (5.2) yields [2, theorem 14.25C, pp. 370] a necessary and sufficient condition for the *Riemann Hypothesis* to be true is,

$$M(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right), \text{for every } \varepsilon > 0. \tag{5.3}$$

Remark 5.0.1: Given the definition (4.1.1) of $\mathbb{A}_n$ and the underlying relation between the redheffer matrices and $M(n)$ (cf. Thm (4.2.1)), we can relate to the result proved by Redheffer [43] that the statement of RH is true iff,

$$det(\mathbb{A}_n) = O\left(n^{\frac{1}{2}+\varepsilon}\right), \text{for every } \varepsilon > 0. \tag{5.4}$$

Roesler also defined a special kind of matrix, $\mathbb{B}_n := (b_{ij})_{2\le i,j\le n}$ such that, for every $2 \le i,j \le n$,

$$b_{ij} := \begin{cases} i-1 & \text{if, } i|j \\ -1 & \text{otherwise.} \end{cases} \tag{5.5}$$

Known as the *Roesler Matrices*, using these matrices, he established that in a similar manner as for *Redheffer Matrices*, RH holds true iff,

$$det(\mathbb{B}_n) = O\left(n!.\,n^{-\frac{1}{2}+\varepsilon}\right), \text{for every } \varepsilon > 0. \tag{5.6}$$

Important to note that, both $\mathbb{A}_n$ and $\mathbb{B}_n$ are extensively related to $M(n)$ via their determinants. It can be checked that, these matrices are not *symmetric*, and one has to compute many of their eigenvalues to estimate their respective determinants.

6. Conjectures based on the Order of $M(x)$

This section surveys key numerical estimates and conjectures on bounds for $M(x)$, developed over the past half-century—many of which contributed to the disproof of the *Mertens Hypothesis*.

Neubauer [15] deduced four *isolated* values of $M(x)$ for which *von Sterneck*'s conjecture (2.2) doesn't hold true. One of the smallest such values of n is, $M(7760000000) = 47465$. Later on, *Cohen* and *Dress* further claimed and proved [16] that, the first violation of *von Sterneck*'s conjecture in the *positive direction* is, $M(7725038629) = 43947$, whereas, credit for deriving the first violation of the theorem in the *negative direction* belongs to *Dress* [17], as he showed, $M(330486258610) = -287440$. Towards the end of the 20^{th} century, *Odlyzko* and *te Riele* [18] astonishingly came up with the following bounds which which was sufficient to conclude tha, the statemnt of MH is indeed *false* along both the directions.

$$\liminf_{x\to\infty}\frac{|M(x)|}{\sqrt{x}} < -1.009 \text{ and, } \limsup_{x\to\infty}\frac{|M(x)|}{\sqrt{x}} > 1.06$$

Although, *Pintz* established in his paper [19], that the violation occurs for some, $x \lesssim^{3.21\times10^{64}} \backsimeq 10^{1.4\times10^{64}}$. As for further research in this area, interested readers can refer to the works of *Jurkat* [20], [21], *Spira* [22], *Jurkat* and *Peyerimhoff* [23], *te Riele* [24], *Möller* [25] and *Anderson* [26]. Many claim that, a priori assuming RH to be true, (5.3) is satisfied, although the statement of the *generalized Mertens Hypothesis* (5.2) is indeed false. One of the conjectures which supports this claim is as follows.

$$\limsup_{x\to\infty}\frac{|M(x)|}{\sqrt{x\log\log x}} = C \tag{6.1}$$

where one can obtain estimates for C in the form of, $C = \frac{6\sqrt{2}}{\pi^2}$ (*Lévy* in a comment to *Saffari* [44]), and $C = \frac{\sqrt{12}}{\pi}$ (*Good* and *Churchhouse* [27]). *El Marraki* [28] did provide one of the strongest unconditional order estimate for $M(x)$. He claimed,

$$M(x) = O\left(\frac{x}{\log^{\frac{236}{75}} x}\right) \tag{6.2}$$

A similar result can be found in *Walfisz*'s article [29],

$$M(x) = O\left(x\exp\left(-A\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right) \tag{6.3}$$

for some costant $A > 0$. One valid estimate for A can be obtained as, $A = 0.2098$. (cf.[30])

There have been significant progress in the direction of obtaing estimates for $M(x)$ of the form, $|M(x)| < \frac{x}{K}$, for some constant $K > 0$ and $\forall\ x > x_0$ for some $x_0 \in \mathbb{R}$. Interested readers can read from the papers of *Hackel* [31], *MacLeod* [32], *Dress* [33], *Diamond* and *McCurley* [34], *Costa Pereira* [35] and *Dress* and *El Marraki* [36].

Some more article pertinent to this topic can be cited as *Landau* [37], [38], *te Riele* [39], [40] and, *Odlyzko* and *te Riele* [41]. Some of the authors even opted for rigorous computation of the function $M(n)$ and the ratio $\frac{M(n)}{\sqrt{n}}$ with high hopes of finding possible counterexample to MH. For refernce, one can study articles by *Mertens* [5], *von Sterneck* [45], [46], [], *Neubauer* [15], *Yorinaga* [47], *Cohen* and *Dress* [16], *Dress* [17], *Lioen* and *van de Lune* [48].

7. Research Prospects in Mertens Hypothesis

In Section 6, both theoretical arguments and numerical evidence strongly indicate that the *Mertens Hypothesis* (MH) is false. This naturally raises the question:

Is it still worthwhile to study the Mertens Hypothesis?

Surprisingly, the answer is **yes**. Despite its disproof, MH continues to inspire research due to the depth of ideas involved. A key objective is to determine the smallest value of n for which the first violation occurs. Recall that, *Pintz* showed that such an n satisfies $n < 10^{1.4\times 10^{64}}$. Earlier conjectures by *Good, Churchhouse*, and *Le′vy* suggest values near $n \approx 10$ and $n \approx 48$, though these are far from realistic. Computational evidence instead shows that $n > 10^{14}$, highlighting the gap between theory and verified bounds. *Odlyzko* and *te Riele* [41] claimed that, the first violation for MH won't occur for $n < 10^{20}$, and probably not even for $n < 10^{30}$. We recall the following theorem from *Titchmarsh* [2, Theorem 14.27, p. 372].

Theorem 7.0.1; *A priori assuming RH, suppose we denote the zeros of* $\zeta(s)$ *on the critical line,* $\Re(s) = \frac{1}{2}$ *by,* $\rho = \frac{1}{2} + it$*. Further consider all the zeros to be simple. Then,* $\exists$ *a sequence,* T_k*, satisfying,* $k \leq T_k \leq k+1$*, such that,*

$$M(x) = 2\lim_{k\to\infty} \sum_{0<t<T_k} \Re\left(\frac{x^\rho}{\rho\zeta'(\rho)}\right) + O(1) \tag{7.1}$$

Kotnik and *van de Lune* [42] illustrated an experimentdepending on the evaluation of *partial sums* of the series on the R.H.S. of (7.1). Thus, eventually, we can obtain,

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x\log\log\log x}} = C$$

where, a valid estimate for C can be derived as, $C \approx 0.5$. As a consequence, we deduce the following estimate for n as, $n \simeq 10^{2.3\times 10^{23}}$ for the occurance of the first violation of MH.

Remark 7.0.2: This approximation for n is significantly better than the bound derived by Pintz, although being too large makes it difficult for mathematicians to verify all the details by direct computation. It is the very reason why people are still motivated as ever to investigate such a violation of MH by computing $M(n)$ consistently.

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Conflicts of Interest Statement

I as the author of this article declare no conflicts of interest.

Data Availability Statement

I as the sole author of this aarticle confirm that the data supporting the findings of this study are available within the article [and/or] its supplementary materials.

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