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On mixed metric dimension of flower and prism type graphs

Sunny Kumar Sharma [§]
School of Mathematics
Shri Mata Vaishno Devi University
Katra 182320
Jammu and Kashmir
India

Yousef Al-Qudah [†]
Department of Mathematics
Faculty of Arts and Science
Amman Arab University
Amman 11953
Jordan

Vijay Kumar Bhat *
School of Mathematics
Shri Mata Vaishno Devi University
Katra 182320
Jammu and Kashmir
India

Abstract

In this article, we consider two interesting classes of an infinite planar graphs (viz., the flower graph $S_n = \mathcal{F}_{n\times 3}$ and the graph D_n^*, constructed using prism graph) and for these, we study a graph parameter, called the mixed metric dimension (MMD). We prove that both of these are the families of planar graphs with an unbounded and non-constant MMD. We also discuss the concept of an independent MMD and for these classes of infinite graphs, we show that the mixed metric basis set $\mathfrak{L}_M$ is independent. We also compare the results of

[§] *ORCID-ID:* 0000-0001-9376-252X
[†] *ORCID-ID:* 0000-0001-5952-4990
* *ORCID-ID:* 0000-0001-8423-9067

[§] *E-mail:* sunnysrrm94@gmail.com
[†] *E-mail:* y.alqudah@aau.edu.jo;
alquyousef82@gmail.com
* *E-mail:* vijaykumarbhat2000@yahoo.com (Corresponding Author)

MMD, with some of the already obtained results regarding the variants of metric dimension for these two planar graphs.

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1 Introduction

The metric dimension (MD), a key graph invariant, has emerged as one of the most extensively studied topics in graph theory and finds relevance across numerous areas. The MD has applications in almost every computational branch of sciences, like mastermind games, chemical science, computer science, etc. Firstly, metric generators (resolving sets) were called as locating set by Slater in [1], where he introduced the concept of MD and consider an issue of identifying uniquely the position of an intruder in the given complex network. On the other side, the idea of the MD of a network was autonomously presented by Melter and Harary in [2], where the resolving sets are treated by giving another name i.e., the resolving sets.

After the introduction of this significant concept of metric dimension in the initial two articles [1, 2], various interesting results both theoretical and applicative for this graph parameter have been published. Slater in his first paper [1] focused on the location of a thief present in the network, while others researchers incorporated this idea to pattern recognition and image processing [3]. Some other explore its applications in chemical sciences [4] and while some have investigated the route of a navigating robots in a considered network [5]. The notion of MD contributed towards the issues related to network discovery [6] and has marked its applications in enhancement of combinatorial optimization [7]. For in depth knowledge of MD and its related work, one can refer [8, 9, 10] and references therein.

Suppose the graph $\Gamma = (\mathbb{V}(\Gamma), \mathbb{E}(\Gamma))$ to be undirected, connected and simple, where $\mathbb{E}(\Gamma)$ interprets the edges and $\mathbb{V}(\Gamma)$ be the vertices of the graph Γ. Let $d_\Gamma(x, y)$ and $d_\Gamma(x, e)$ represents the distance between the two vertices x, y and an edge $e = ab$ and vertex x respectively, where $d_\Gamma(x, e) = min\{d_\Gamma(x, a), d_\Gamma(x, b)\}$. A single vertex $x \in \mathbb{V}(\Gamma)$ resolves (determines or distinguishes) two elements $a, b \in \mathbb{V}(\Gamma) \cup \mathbb{E}(\Gamma)$ if $d_\Gamma(x, a) \neq d_\Gamma(x, b)$. A subset $\mathfrak{L}$ ($\mathfrak{L}_E$ for edge resolving set or $\mathfrak{L}_M$ for mixed resolving set) of $V(\Gamma)$, is called as the resolving set (RS) in Γ, if elements of the set $\mathfrak{L}$ determines each pair of distinct vertices of a graph Γ. Mathematically, for an ordered subset $\mathfrak{L} = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_p\}$ of vertices of Γ, and a member $v \in \mathbb{V}(\Gamma)$, the representation \code of v regarding $\mathfrak{L}$ is the ordered p-tuple $\zeta(v|\mathfrak{L}) = (d_\Gamma(v, \alpha_1), d_\Gamma(v, \alpha_2), d_\Gamma(v, \alpha_3), \dots, d_\Gamma(v, \alpha_p))$.

If for any two distinct vertices a and b of Γ, $\zeta(a|\mathfrak{L}) \neq \zeta(b|\mathfrak{L})$, then $\mathfrak{L}$ is called as the RS of Γ. The set $\mathfrak{L}$ with resolving property and minimum cardinality is name as the metric basis (MB) of Γ, and the MD of Γ is nothing the cardinality of its MB set, and which is represented by $\beta(\Gamma)$ or $\dim(\Gamma)$.

After the introduction and in depth exploration of the notion of resolving set and corresponding MD for connected graphs, various other variants of this concept have been introduced. Several prominent researchers have introduced distinct variants of resolving sets like for instance, partition resolving , edge resolving, edge-partition resolving, local resolving, fault-tolerant resolving, strong resolving, mixed resolving, dominant resolving, fractional resolving sets, etc. Then, these newly introduced concepts have been examined both theoretically and from the applicative view points. These graphs parameters are collectively known as resolvability parameters. For in depth knowledge of these notions, one can refer [9, 11, 12, 13] and references there in. Also, there are infinitely many graph families for which some of these resolvability parameters have not been studied, few of them are listed in these papers [14, 15, 16].

Definition 1: Independent Resolving Set (IRS): In Γ, a subset $L \subseteq V(\Gamma)$ is IRS, if L is both a RS and independent.

From the definition of MD, one can clearly find that the MD deals with the vertices of graph only. Similar to MD, an interesting concept known as EMD was initially defined by Kelenc, Tratnik, and Yero, [16], which consider only the edges present in a graph under consideration. In particular, the notion of MD uniquely identifies the vertices of a graph whereas the notion of EMD identifies all the edges of a graph. In order to define the notion of EMD, we have to consider the geodesic between a vertex t and an element $e = ry$ form $E(\Gamma)$, which is define as: $d_\Gamma(t, e) = min\{d_\Gamma(t, r), d_\Gamma(t, y)\}$. Then, a subset $\mathfrak{L}_E$ consisting of vertices from $V(\Gamma)$ is said to be an ERS if for each pair of distinct edges e_1 and e_2 in $E(\Gamma)$, there exist a vertex s in $\mathfrak{L}_E$, such that $d_\Gamma(s, e_1) \neq d_\Gamma(s, e_2)$. The set $\mathfrak{L}_E$ with edge resolving property and minimum cardinality is name as an edge metric basis (EMB) of Γ, and the EMD of Γ is nothing the cardinality of its EMB set, and which is represented by $\beta_E(\Gamma)$ or edim(Γ).

Definition 2: Independent Edge Resolving Set (IERS): In Γ, a subset $L_E \subseteq V(\Gamma)$ is an IERS, if L_E is both a ERS and independent.

After the concept of metric basis and edge basis, an interesting graph invariant, known as mixed metric basis (MMB) was originally formulated by Kelenc, Tratnik, and Yero, [16]. This new notion brings together the concepts of MD and EMD, and termed it as the MMD of Γ. For this, let $\mathfrak{L}_M \subseteq V(\Gamma)$ and is said to be a MRS for Γ, if for any two distinct members of $\mathbb{E}(\Gamma) \cup \mathbb{V}(\Gamma)$ are recognized by some vertex in $\mathfrak{L}_M$. The mixed metric code/representation (MMR) for any member k of $\mathbb{V}(\Gamma) \cup \mathbb{E}(\Gamma)$ corresponding to the set $\mathfrak{L}_M = \{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_p\}$ in Γ, is an ordered p-tuple $\zeta_M(k|\mathfrak{L}_M) = (d_\Gamma(k, \gamma_1), d_\Gamma(k, \gamma_2), d_\Gamma(k, \gamma_3), \dots, d_\Gamma\ (y, \gamma_p))$. If $\zeta_M(k_1|\mathfrak{L}_M) \neq \zeta_M(k_2|\mathfrak{L}_M)$, for any k_1 and k_2 in $\mathbb{E}(\Gamma) \cup \mathbb{V}(\Gamma)$, then $\mathfrak{L}_M$ is said to be a MRS for Γ. The set $\mathfrak{L}_M$ with mixed resolving characteristics and minimum cardinality is name as a mixed metric basis (MMB) of Γ, and the MMD of Γ is nothing the cardinality of its MMB set. Symbolically, it is represented by $mdim(\Gamma)$. Now, for this graph invariant we define the following:

Definition 3: Independent Mixed Resolving Set (IMRS): In Γ, a subset $L_M \subseteq V(\Gamma)$ is an IMRS, if L_M is both a MRS and independent.

In this work, we investigate MMB and MMD for two interesting infinite planar graphs viz., D_n^* and $\mathcal{F}_{n\times m}$. For these planar graphs, authors have computed the MD as well as EMD, providing the base to this paper. Regarding MD and EMD, for D_n^* and $\mathcal{F}_{n\times m}$, we have the following listedobtained results for them.

Theorem 1: [17] *Let D_n^* be a planar graph with $5n$ edges and $3n$ vertices. Then, $\beta(D_n^*) = 3$, for any $n \geq 6$.*

Theorem 2: [18] *Let $\mathcal{F}_{n\times m}$ be a planar graph with $3n$ edges and $2n$ vertices. Then, for every $n \geq 6$, we have*

$$\beta(\mathcal{F}_{n\times m}) = \begin{cases} 2, & \text{if } n \text{ is even natural;} \\ 3, & \text{othewise.} \end{cases}$$

and regarding the EMD, we have

Theorem 3: [17] *Let D_n^* be a planar graph with $5n$ edges and $3n$ vertices. Then, for every $n \geq 3$, we have*

$$\beta_E(D_n^*) = \begin{cases} 4, & \text{if } n = 3,4,; \\ \left\lceil \frac{n}{2} \right\rceil + 1, & \text{otherwise.} \end{cases}$$

Theorem 4: [18] *Let $\mathcal{F}_{n\times m}$ be a planar graph with $3n$ edges and $2n$ vertices. Then, for every $n \geq 3$, we have*

$$\beta_E(\mathcal{F}_{n\times m}) = \begin{cases} 3, & \text{if } n = 4,; \\ 4, & \text{if } n = 3,5,; \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

By using definition 2 and EMD of $\mathcal{F}_{n\times m}$ and D_n^*, we obtain the following two results respectively.

Theorem 5: *Let $\mathcal{F}_{n\times m}$ be a planar graph of size $3n$ and order $2n$. Then, its IEMD is $\left\lceil \frac{n}{2} \right\rceil$, for every $n \geq 6$.*

Proof: For detailed proof, consider Theorem 2.3[1] for reference.

and

Theorem 6: *Let D_n^* be a planar graph of size $5n$ and order $3n$. Then, its IEMD is $\left\lceil \frac{n}{2} \right\rceil + 1$, for every $n \geq 6$.*

Proof: For detailed proof, consider Theorem 3[17] for reference.

The organisation of this manuscript is as under: Section 2 determine the MMD of an infinite planar graph D_n^*, and also study MRS $\mathfrak{L}_M$ for it with independence characteristic. Section 3 determine the MMD of the second infinite planar graph $\mathcal{F}_{n\times m}$, and also study MRS $\mathfrak{L}_M$ for it with independence characteristic. In our last part of the manuscript, we summarizes our findings and results related to the above said families of the infinite planar graphs.

2. MMD of Infinite Graph D_n^*

The planar graph D_n^* [17] comprises of n four-sided faces, n triangular faces, an n-sided face, and a $2n$-sided face, refer Fig. 1. This planar network is generated from D_n (the prism graph) in a manner that, it is constructed from D_n by inserting a new vertex between the vertices $q_{ß}$ and $q_{ß+1}$ (say, $r_{ß}$) in the outer cycle of the prism graph and again adding the edges between the vertices $q_{ß}$ and $q_{ß+1}$. The set of edges has cardinality $5n$ and the set of vertices has cardinality $3n$ for D_n^* and are presented by $\mathbb{E}(D_n^*)$ and $\mathbb{V}(D_n^*)$, where $\mathbb{E}(D_n^*) = \{p_{ß}q_{ß}, q_{ß}r_{ß}, p_{ß}p_{ß+1}, q_{ß}q_{ß+1}, r_{ß}q_{ß+1}: 1 \le ß \le n\}$ and $\mathbb{V}(D_n^*) = \{p_{ß}, q_{ß}, r_{ß}: 1 \le ß \le n\}$ (we take $£ = \varpi$, in Figures 1 and 2).

For easy working, we name the vertex set $\{p_{ß}: 1 \le ß \le n\}$ in D_n^* present on the innermost cycle as the p-cycle vertices, we name the set of vertices $\{q_{ß}: 1 \le ß \le n\}$ in D_n^* present on the middle cycle as the q-cycle vertices, and finally we name the set of vertices $\{r_{ß}: 1 \le ß \le n\}$ present on the outer side of D_n^* as the outer vertices. For simplicity, we will take $p_{n+1} = p_1$, $q_{1+n} = q_1$, and $r_{n+1} = r_1$. In the present section, we will investigate MRS for D_n^* and prove that such a set has minimum of $n+1$ number of vertices. Furthermore, we will prove that the MRS is also independent for D_n^*. Next, in order to obtain our desired results regarding MMD of D_n^*, we strongly need the following three important Lemmas:

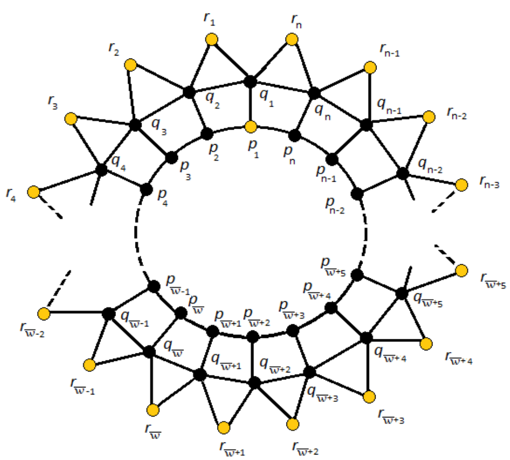

Figure 1
Infinite Planar Graph D_n^*, for $n \ge 6$

Lemma 1: *Let $\mathfrak{L}_M$ be any MRS for the plane graph D_n^*. Then, the set of outer vertices $\{r_ß | 1 \leq ß \leq n\} \subset \mathfrak{L}_M$.*

Proof: For the contradiction, we suppose that at least one vertex from the set $\{r_ß | 1 \leq ß \leq n\}$, is not in the MRS $\mathfrak{L}_M$. WLOG, for any ß, we suppose that $r_ß \not\subset \mathfrak{L}_M$. Then, we have $\zeta_M(q_ß | \mathfrak{L}_M) = \zeta_M(r_ß q_ß | \mathfrak{L}_M)$ and $\zeta_M(q_{ß+1} | \mathfrak{L}_M) = \zeta_M(r_ß q_{ß+1} | \mathfrak{L}_M)$, a contradiction.

Lemma 2: *Let $\mathfrak{L}_M$ be any MRS for the plane graph D_n^*. Then, at least one vertex from the p-cycle is in $\mathfrak{L}_M$.*

Proof: Suppose on the contrary that no vertex from the p-cycle is in $\mathfrak{L}_M$ i.e., for any ß, $p_ß \not\subset \mathfrak{L}_M$. Then, we have $\zeta_M(q_ß | \mathfrak{L}_M) = \zeta_M(p_ß q_ß | \mathfrak{L}_M)$, a contradiction.

In Lemma 3, we set the lower possible bound for the plane graph D_n^*, regarding its MMD.

Lemma 3: *For planar graph D_n^*, we have $\beta_M(D_n^*) \geq n + 1$, for each $n \geq 4$.*

Proof: Suppose the MMD of D_n^* is equals to n i.e., $\beta_M(D_n^*) = n$ (on contrary). Then, by considering Lemma 1 and Lemma 2, we will always obtain a contradiction, as the cardinality of the collection $\{r_ß | 1 \leq ß \leq n\}$ is n. So, we always find that $\beta_M(D_n^*) \geq n + 1$.

Finally, for the upper bound of the graph D_n^*, regarding its MMD, we have the following interesting and important result:

Theorem 7: *Let D_n^* be a planar graph of size $5n$ and order $3n$. Then, we have $\beta_M(D_n^*) = n + 1$, for every $n \geq 4$.*

Proof: In order to prove that $\beta_M(D_n^*) \leq n + 1$, suppose $\mathfrak{L}_M = \{p_1, r_1, r_2, \ldots, r_{n-1}, r_n\} \subset \mathbb{V}(D_n^*)$. We will prove that $\mathfrak{L}_M$ is the MMB set for the Prism type graph D_n^*. Note that for $n = 4,5$, one can see that the set $\mathfrak{L}_M$ is the MMB set for D_n^* by simple manual evaluation. Next, for each $n \geq 6$, we have to prove that its MMD is unbounded, to attain this we consider two cases based on the integer n i.e., when n is odd and when it is even.

Case (1) n is odd.

For this situation, the integer n takes the shape $n = 2£ + 1$, where $£ \in \mathbb{N}$ and $£ \geq 3$. Let $\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\} \subset \mathbb{V}(D_n^*)$. Now, to see that $\mathfrak{L}_M^*$ is the resolving set in the mixed sense for D_n^*, we will represent the MMRs for every member present in $V(D_n^*) \cup E(D_n^*)$ corresponding to the set $\mathfrak{L}_M^*$.

Now, the MMRs for the vertices $\{v = p_ß, q_ß, r_ß | ß = 1,2,3, \ldots, n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(v\|\mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(p_ß\|\mathfrak{L}_M^*)$:$(ß = 1)$	$(ß - 1,2,3, £ - ß + 3, £ + 1)$
$\zeta_M(p_ß\|\mathfrak{L}_M^*)$:$(ß = 2)$	$(ß - 1, ß, 2, £ - ß + 3, £ - ß + 4)$
$\zeta_M(p_ß\|\mathfrak{L}_M^*)$:$(3 \le ß \le £ + 1)$	$(ß - 1, ß, ß - 1, £ - ß + 3, £ - ß + 4)$
$\zeta_M(p_ß\|\mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 2, 2£ - ß + 4, ß - 1, ß - £, £ - ß + 4)$
$\zeta_M(p_ß\|\mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 2, 2£ - ß + 4, 2£ - ß + 5, ß - £, ß - £ - 1)$

$\zeta_M(v\|\mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(q_ß\|\mathfrak{L}_M^*)$:$(ß = 1)$	$(ß, 1,2, £ - ß + 2, £)$
$\zeta_M(q_ß\|\mathfrak{L}_M^*)$:$(ß = 2)$	$(ß, ß - 1,1, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M^*)$:$(3 \le ß \le £ + 1)$	$(ß, ß - 1, ß - 2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 3, 2£ - ß + 3, ß - 2, ß - £ - 1, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 3, 2£ - ß + 3, 2£ - ß + 4, ß - £ - 1, ß - £ - 2)$

$\zeta_M(v\|\mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(r_ß\|\mathfrak{L}_M^*)$:$(ß = 1)$	$(ß + 1,0,2, £ - ß + 2, £ + 1)$
$\zeta_M(r_ß\|\mathfrak{L}_M^*)$:$(ß = 2)$	$(ß + 1, ß, 0, £ - ß + 2, £ - ß + 3)$
$\zeta_M(r_ß\|\mathfrak{L}_M^*)$:$(3 \le ß \le £)$	$(ß + 1, ß, ß - 1, £ - ß + 2, £ - ß + 3)$
$\zeta_M(r_ß\|\mathfrak{L}_M^*)$:$(ß = £ + 1)$	$(2£ - ß + 3, ß, ß - 1,0, £ - ß + 3)$
$\zeta_M(r_ß\|\mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 3, 2£ - ß + 3, ß - 1, ß - £, 0)$
$\zeta_M(r_ß\|\mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 3, 2£ - ß + 3, 2£ - ß + 4, ß - £, ß - £ - 1)$

and the MMRs for the edges $\{\varepsilon = p_ß p_{ß+1}, p_ß q_ß, q_ß q_{ß+1}, q_ß r_ß, r_ß q_{ß+1} \| ß = 1,2,3, \dots, n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(\varepsilon\|\mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M^*)$:$(ß = 1)$	$(ß - 1,2,3, £ - ß + 2, £ + 1)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M^*)$:$(ß = 2)$	$(ß - 1, ß, 2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M^*)$:$(3 \le ß \le £)$	$(ß - 1, ß, ß - 1, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M^*)$:$(ß = £ + 1)$	$(2£ - ß + 1, ß, ß - 1,2, £ - ß + 3)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 1, 2£ - ß + 3, ß - 1, ß - £, 2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 1, 2£ - ß + 3, 2£ - ß + 4, ß - £, ß - £ - 1)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M^*)$:$(ß = 1)$	$(ß - 1,1,2, £ - ß + 2, £)$

$\zeta_M(p_ß q_ß \| \mathfrak{L}_M^*)$:$(ß = 2)$	$(ß - 1, ß - 1, 1, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß q_ß \| \mathfrak{L}_M^*)$:$(3 \le ß \le £ + 1)$	$(ß - 1, ß - 1, ß - 2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß q_ß \| \mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 2, 2£ - ß + 3, ß - 2, ß - £ - 1, £ - ß + 3)$
$\zeta_M(p_ß q_ß \| \mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 2, 2£ - ß + 3, 2£ - ß + 4, ß - £ - 1, ß - £ - 2)$

$\zeta_M(\varepsilon \| \mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(q_ß q_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = 1)$	$(ß, 1, 1, £ - ß + 1, £)$
$\zeta_M(q_ß q_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = 2)$	$(ß, ß - 1, 1, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß q_{ß+1} \| \mathfrak{L}_M^*)$:$(3 \le ß \le £)$	$(ß, ß - 1, ß - 2, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß q_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = £ + 1)$	$(2£ - ß + 2, ß - 1, ß - 2, 1, £ - ß + 2)$
$\zeta_M(q_ß q_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 2, 2£ - ß + 2, ß - 2, ß - £ - 1, 1)$
$\zeta_M(q_ß q_{ß+1} \| \mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 2, 2£ - ß + 2, 2£ - ß + 3, ß - £ - 1, ß - £ - 2)$

$\zeta_M(\varepsilon \| \mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(q_ß r_ß \| \mathfrak{L}_M^*)$:$(ß = 1)$	$(ß, 0, 2, £ - ß + 2, £)$
$\zeta_M(q_ß r_ß \| \mathfrak{L}_M^*)$:$(ß = 2)$	$(ß, ß - 1, 0, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß r_ß \| \mathfrak{L}_M^*)$:$(3 \le ß \le £)$	$(ß, ß - 1, ß - 2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß r_ß \| \mathfrak{L}_M^*)$:$(ß = £ + 1)$	$(ß, ß - 1, ß - 2, 0, £ - ß + 3)$
$\zeta_M(q_ß r_ß \| \mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 3, 2£ - ß + 3, ß - 2, ß - £ - 1, 0)$
$\zeta_M(q_ß r_ß \| \mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 3, 2£ - ß + 3, 2£ - ß + 4, ß - £ - 1, ß - £ - 2)$

$\zeta_M(\varepsilon \| \mathfrak{L}_M^*)$	$\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}, r_{£+2}\}$
$\zeta_M(q_ß r_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = 1)$	$(ß + 1, 0, 1, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß r_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = 2)$	$(ß + 1, ß, 0, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß r_{ß+1} \| \mathfrak{L}_M^*)$:$(3 \le ß \le £)$	$(ß + 1, ß, ß - 1, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß r_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = £ + 1)$	$(2£ - ß + 2, 2£ - ß + 2, ß - 1, 0, £ - ß + 2)$
$\zeta_M(q_ß r_{ß+1} \| \mathfrak{L}_M^*)$:$(ß = £ + 2)$	$(2£ - ß + 2, 2£ - ß + 2, 2£ - ß + 3, ß - £, 0)$
$\zeta_M(q_ß r_{ß+1} \| \mathfrak{L}_M^*)$:$(£ + 3 \le ß \le 2£ + 1)$	$(2£ - ß + 2, 2£ - ß + 2, 2£ - ß + 3, ß - £, ß - £ - 1)$

From the above, for the MMR of edges and vertices for the graph D_n^* regarding the set $\mathfrak{L}_M^*$, we find that for $1 \le ß \le n$ and $ß \ne 1, 2, £ + 1, £ + 2$, $\zeta_M(q_ß | \mathfrak{L}_M^*) = \zeta_M(r_ß q_ß | \mathfrak{L}_M^*)$ and $\zeta_M(q_{ß+1} | \mathfrak{L}_M^*) = \zeta_M(r_ß q_{ß+1} | \mathfrak{L}_M^*)$. In the rest of the cases, there does not exist any two vertices and edges in D_n^* with the same mixed metric code. For every $1 \le ß \le n$ and $ß \ne 1, 2, £ + 1, £ + 2$, we see that $\zeta_M(q_ß | \mathfrak{L}_M^* \cup \{r_ß\}) \ne \zeta_M(r_ß q_ß | \mathfrak{L}_M^* \cup \{r_ß\})$ and $\zeta_M(q_{ß+1} | \mathfrak{L}_M^* \cup \{r_ß\}) \ne \zeta_M(r_ß q_{ß+1} |$

$\mathfrak{L}_M^* \cup \{r_ß\}$). From this, we have $\zeta_M(q_ß|\mathfrak{L}_M) \neq \zeta_M(r_ß q_ß|\mathfrak{L}_M)$ and $\zeta_M(q_{ß+1}|\mathfrak{L}_M) \neq \zeta_M(r_ß q_{ß+1}|\mathfrak{L}_M)$, for any $1 \leq ß \leq n$ and so $|\mathfrak{L}_M| \leq n+1$. Now, on combining Lemma 3 and the above discussion, we find that $\mathfrak{L}_M$ is the MMB set for the plane graph D_n^*, which implies that $\beta_M(D_n^*) = n+1$ in this case.

Case (2) n is even.

For this situation, the integer n takes the shape $n = 2£$, where $£ \in \mathbb{N}$ and $£ \geq 3$. Let $\mathfrak{L}_M^* = \{p_1, r_1, r_2, r_{£+1}\} \subset \mathbb{V}(D_n^*)$. Now, to see that $\mathfrak{L}_M^*$ is the resolving set in the mixed sense for D_n^*, we will represent the MMRs for every member present in $V(D_n^*) \cup E(D_n^*)$ corresponding to the set $\mathfrak{L}_M^*$.

Now, the MMRs for the vertices $\{v = p_ß, q_ß, r_ß | ß = 1,2,3,\ldots,n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(v\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{p_1, r_1, r_2, r_{£+1}\}$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = 1)$	$(ß-1,2,3,£+1)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß-1,ß,2,£-ß+3)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(3 \leq ß \leq £+1)$	$(ß-1,ß,ß-1,£-ß+3)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = £+2)$	$(2£-ß+1,2£-ß+3,ß-1,ß-£)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(£+3 \leq ß \leq 2£)$	$(2£-ß+1,2£-ß+3,2£-ß+4,ß-£)$

$\zeta_M(v\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{p_1, r_1, r_2, r_{£+1}\}$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = 1)$	$(ß,1,2,£)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß,ß-1,1,£-ß+2)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(3 \leq ß \leq £+1)$	$(ß,ß-1,ß-2,£-ß+2)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = £+2)$	$(2£-ß+2,2£-ß+2,ß-2,ß-£-1)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(£+3 \leq ß \leq 2£)$	$(2£-ß+2,2£-ß+2,2£-ß+3,ß-£-1)$

$\zeta_M(v\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{p_1, r_1, r_2, r_{£+1}\}$
$\zeta_M(r_ß\|\mathfrak{L}_M)$:$(ß = 1)$	$(ß+1,0,2,£-ß+2)$
$\zeta_M(r_ß\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß+1,ß,0,£-ß+2)$
$\zeta_M(r_ß\|\mathfrak{L}_M)$:$(3 \leq ß \leq £)$	$(ß+1,ß,ß-1,£-ß+2)$
$\zeta_M(r_ß\|\mathfrak{L}_M)$:$(ß = £+1)$	$(2£-ß+2,2£-ß+2,ß-1,0)$
$\zeta_M(r_ß\|\mathfrak{L}_M)$:$(£+2 \leq ß \leq 2£)$	$(2£-ß+2,2£-ß+2,2£-ß+3,ß-£)$

and the MMRs for the edges $\{\varepsilon = p_ß p_{ß+1}, p_ß q_ß, q_ß q_{ß+1}, q_ß r_ß, r_ß q_{ß+1} | ß = 1,2,3,\ldots,n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{p_1, r_1, r_2, r_{£+1}\}$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = 1)$	$(ß-1,2,3,£-ß+2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß-1,ß,2,£-ß+2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(3 \leq ß \leq £)$	$(ß-1,ß,ß-1,£-ß+2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = £+1)$	$(2£-ß,2£-ß+2,ß-1,2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(£+2 \leq ß \leq 2£)$	$(2£-ß,2£-ß+2,2£-ß+3,ß-£)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M=\{p_1,r_1,r_2,r_{£+1}\}$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß=1)$	$(ß-1,1,2,£)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß=2)$	$(ß-1,ß-1,1,£-ß+2)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(3\le ß\le £+1)$	$(ß-1,ß-1,ß-2,£-ß+2)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß=£+2)$	$(2£-ß+1,2£-ß+2,ß-2,ß-£-1)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(£+3\le ß\le 2£)$	$(2£-ß+1,2£-ß+2,2£-ß+3,ß-£-1)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M=\{p_1,r_1,r_2,r_{£+1}\}$
$\zeta_M(q_ß q_{ß+1}\|\mathfrak{L}_M)$:$(ß=1)$	$(ß,1,1,£-ß+1)$
$\zeta_M(q_ß q_{ß+1}\|\mathfrak{L}_M)$:$(ß=2)$	$(ß,ß-1,1,£-ß+1)$
$\zeta_M(q_ß q_{ß+1}\|\mathfrak{L}_M)$:$(3\le ß\le £)$	$(ß,ß-1,ß-2,£-ß+1)$
$\zeta_M(q_ß q_{ß+1}\|\mathfrak{L}_M)$:$(ß=£+1)$	$(2£-ß+1,2£-ß+1,ß-2,1)$
$\zeta_M(q_ß q_{ß+1}\|\mathfrak{L}_M)$:$(£+2\le ß\le 2£)$	$(2£-ß+1,2£-ß+1,2£-ß+2,ß-£-1)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M=\{p_1,r_1,r_2,r_{£+1}\}$
$\zeta_M(q_ß r_ß\|\mathfrak{L}_M)$:$(ß=1)$	$(ß,0,2,£)$
$\zeta_M(q_ß r_ß\|\mathfrak{L}_M)$:$(ß=2)$	$(ß,ß-1,0,£-ß+2)$
$\zeta_M(q_ß r_ß\|\mathfrak{L}_M)$:$(3\le ß\le £)$	$(ß,ß-1,ß-2,£-ß+2)$
$\zeta_M(q_ß r_ß\|\mathfrak{L}_M)$:$(ß=£+1)$	$(ß,ß-1,ß-2,0)$
$\zeta_M(q_ß r_ß\|\mathfrak{L}_M)$:$(ß=£+2)$	$(2£-ß+2,2£-ß+2,ß-2,ß-£-1)$
$\zeta_M(q_ß r_ß\|\mathfrak{L}_M)$:$(£+3\le ß\le 2£)$	$(2£-ß+2,2£-ß+2,2£-ß+3,ß-£-1)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M=\{p_1,r_1,r_2,r_{£+1}\}$
$\zeta_M(q_ß r_{ß+1}\|\mathfrak{L}_M)$:$(ß=1)$	$(ß+1,0,1,£-ß+1)$
$\zeta_M(q_ß r_{ß+1}\|\mathfrak{L}_M)$:$(ß=2)$	$(ß+1,ß,0,£-ß+1)$
$\zeta_M(q_ß r_{ß+1}\|\mathfrak{L}_M)$:$(3\le ß\le £)$	$(ß+1,ß,ß-1,£-ß+1)$
$\zeta_M(q_ß r_{ß+1}\|\mathfrak{L}_M)$:$(ß=£+1)$	$(2£-ß+1,2£-ß+1,ß-1,0)$
$\zeta_M(q_ß r_{ß+1}\|\mathfrak{L}_M)$:$(£+2\le ß\le 2£)$	$(2£-ß+1,2£-ß+1,2£-ß+2,ß-£)$

From the above, for the MMR of edges and vertices for the graph D_n^* regarding the set $\mathfrak{L}_M^*$, we find that for $1\le ß\le n$ and $ß\ne 1,2,£+1$, $\zeta_M(q_ß|\mathfrak{L}_M^*)=\zeta_M(r_ß q_ß|\mathfrak{L}_M^*)$ and $\zeta_M(q_{ß+1}|\mathfrak{L}_M^*)=\zeta_M(r_ß q_{ß+1}|\mathfrak{L}_M^*)$. In the rest of the cases, there does not exist any two vertices and edges in D_n^* with the same mixed metric code. For every $1\le ß\le n$ and $ß\ne 1,2,£+1$, we see that $\zeta_M(q_ß|\mathfrak{L}_M^*\cup\{r_ß\})\ne\zeta_M(r_ß q_ß|\mathfrak{L}_M^*\cup\{r_ß\})$ and $\zeta_M(q_{ß+1}|\mathfrak{L}_M^*\cup\{r_ß\})\ne\zeta_M(r_ß q_{ß+1}|\mathfrak{L}_M^*\cup\{r_ß\})$. From this, we have $\zeta_M(q_ß|\mathfrak{L}_M)\ne\zeta_M(r_ß q_ß|\mathfrak{L}_M)$ and $\zeta_M(q_{ß+1}|\mathfrak{L}_M)\ne\zeta_M(r_ß q_{ß+1}|\mathfrak{L}_M)$, for any $1\le ß\le n$ and so $|\mathfrak{L}_M|\le n+1$. Now, on combining Lemma 3 and the above discussion, we find that $\mathfrak{L}_M$ is the MMB set for graph D_n^*, which implies that $\beta_M(D_n^*)=n+1$, which completes the proof.

Theorem 8: *Let D_n^* be the prism type graph of size $5n$ and order $3n$. Then, its independent mixed metric number is $n+1$, for every $n \geq 4$.*

Proof: For detailed proof, consider Theorem 7 for reference

3. MMD of the Flower Graph $\boldsymbol{S_n = \mathcal{F}_{n\times 3}}$

The graph denoted by $\mathcal{F}_{n\times m}$ [1] and consisting of nm number of edges and $n(m-1)$ number of vertices is called a Flower graph if it has n vertices which induce n-cycle and n sets of $m-2$ vertices each of which forms the m-cycle around the n-cycle so that every m-cycle uniquely intersects the n-cycle on a single edge. The Flower graph $S_n = \mathcal{F}_{n\times 3}$ comprises of n 3-sided faces, an n-sided face, and a $2n$-sided face (see Figure 2). It is obtained from the Cycle graph C_n by including a new vertex between the vertices $p_ß$ and $p_{ß+1}$ (say, $q_ß$) in the p-cycle of the Cycle graph and again adding the edges between the vertices $p_ß$ and $p_{ß+1}$. The edge set of cardinality $3n$ and the vertex set of cardinality $2n$ for the graph S_n is denoted by $\mathbb{E}(S_n)$ and $\mathbb{V}(S_n)$ respectively, where $\mathbb{E}(S_n) = \{p_ß q_ß, q_ß p_{ß+1}, p_ß p_{ß+1}: 1 \leq ß \leq n\}$ and $\mathbb{V}(S_n) = \{p_ß, q_ß: 1 \leq ß \leq n\}$.

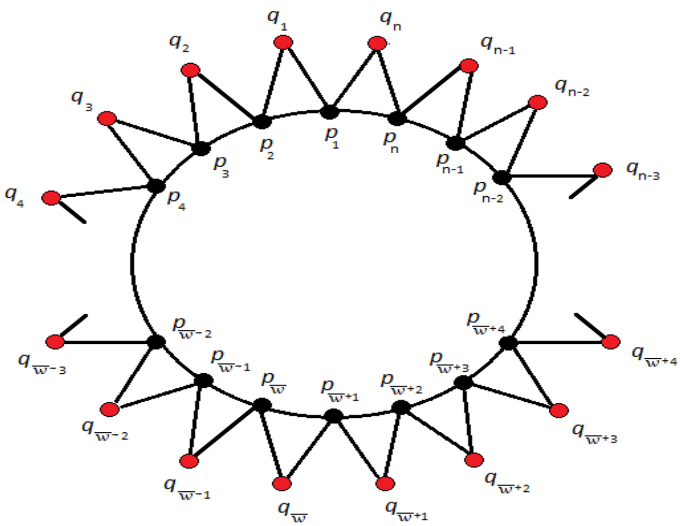

Figure 2
Flower Graph $\boldsymbol{S_n = \mathcal{F}_{n\times 3}}$, for $\boldsymbol{n \geq 4}$.

For easy working, we name the vertex set $\{p_ß: 1 \leq ß \leq n\}$ in S_n present on the innermost cycle as the p-cycle vertices and we name the set of vertices $\{q_ß: 1 \leq ß \leq n\}$ present on the outer side of S_n as the outer vertices. For simplicity, we will take $p_1 = p_{n+1}$ and $q_1 = q_{n+1}$. In the present section, we will investigate MRS for S_n and prove that such a set has minimum of n number of vertices. Furthermore, we will prove that the MRS is also independent for S_n.

Next, in order to obtain our desired results regarding MMD of S_n, we strongly need the following two important Lemmas:

Lemma 4: *Let $\mathfrak{L}_M$ be any MRS for the flower graph S_n. Then, the set of outer vertices $\{r_ß | 1 \leq ß \leq n\} \subset \mathfrak{L}_M$.*

Proof: For the contradiction, we suppose that at least one vertex from the set $\{q_ß | 1 \leq ß \leq n\}$, is not in the MRS $\mathfrak{L}_M$. Without loss of generality, for any ß, we suppose that $q_ß \not\in \mathfrak{L}_M$. Then, we have $\zeta_M(p_ß|\mathfrak{L}_M) = \zeta_M(q_ß p_ß|\mathfrak{L}_M)$ and $\zeta_M(p_{ß+1}|\mathfrak{L}_M) = \zeta_M(q_ß p_{ß+1}|\mathfrak{L}_M)$, a contradiction.

In Lemma 5, we set the lower bound for the flower graph S_n, regarding its MMD.

Lemma 5: *For the flower graph S_n, we have $\beta_M(S_n) \geq n$, where $n \geq 4$.*

Proof: Suppose on the contrary that the MMD of S_n equals $n-1$ i.e., $\beta_M(S_n) = n-1$. Then, by considering Lemma 4, we get contradiction, since the cardinality of the set $\{q_ß | 1 \leq ß \leq n\}$ is n. So, we always find that $\beta_M(S_n) \geq n$.

Finally, the following result provides the upper bound for the Flower graph S_n, regarding its MMD.

Theorem 9: *Let $S_n = \mathcal{F}_{n\times m}$ be a planar graph of size $3n$ and order $2n$. Then, we have $\beta_M(S_n = \mathcal{F}_{n\times 3}) = n$, for every $n \geq 4$.*

Proof: In order to show that $\beta_M(S_n) \leq n$, suppose $\mathfrak{L}_M = \{q_1, q_2, \ldots, q_{n-1}, q_n\} \subset \mathbb{V}(S_n)$. We will prove that $\mathfrak{L}_M$ is the MMB set for the Flower graph S_n. Note that for $n = 4,5$, one can see that the set $\mathfrak{L}_M$ is the MMB set for S_n by simple manual evaluation. Next, for each $n \geq 6$, we have to prove that its MMD is unbounded, to attain this we consider two cases based on the integer n i.e., when n is odd and when it is even.

Case (1) n is odd.

For this situation, the integer n takes the shape $n = 2£ + 1$, where $£ \in \mathbb{N}$ and $£ \geq 3$. Let $\mathfrak{L}_M^* = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\} \subset \mathbb{V}(S_n)$. Now, in order to see that $\mathfrak{L}_M^*$ is the resolving set in the mixed sense for S_n, we will represent the MMRs for every member present in $V(S_n) \cup E(S_n)$ corresponding to the set $\mathfrak{L}_M^*$.

Now, the MMRs for the vertices $\{v = p_ß, q_ß | ß = 1,2,3,\ldots,n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(v\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\}$
$\zeta_M(p_ß\|\mathfrak{L}_M)$: $(ß = 1)$	$(1,2,3,£-ß+2,£)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß-1,1,2,£-ß+2,£-ß+3)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = 3)$	$(ß-1,ß-2,1,£-ß+2,£-ß+3)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(4 \leq ß \leq £+1)$	$(ß-1,ß-2,ß-3,£-ß+2,£-ß+3)$

$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = £ + 2)$	$(ß - 1, ß - 2, ß - 3, ß - £ - 1, £ - ß + 3)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = £ + 3)$	$(2£ - ß + 3, ß - 2, ß - 3, ß - £ - 1, ß - £ - 2)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(ß = £ + 4)$	$(2£ - ß + 3, 2£ - ß + 4, ß - 3, ß - £ - 1, ß - £ - 2)$
$\zeta_M(p_ß\|\mathfrak{L}_M)$:$(£ + 5 \leq ß \leq 2£ + 1)$	$(2£ - ß + 3, 2£ - ß + 4, 2£ - ß + 5, ß - £ - 1, ß - £ - 2)$

$\zeta_M(\upsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\}$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = 1)$	$(0,2,3, £ - ß + 2, £ + 1)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß, 0,2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = 3)$	$(ß, ß - 1,0, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(4 \leq ß \leq £)$	$(ß, ß - 1, ß - 2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = £ + 1)$	$(ß, ß - 1, ß - 2,0, £ - ß + 3)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = £ + 2)$	$(2£ - ß + 3, ß - 1, ß - 2, ß - £, 0)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(ß = £ + 3)$	$(2£ - ß + 3, 2£ - ß + 4, ß - 2, ß - £, ß - £ - 1)$
$\zeta_M(q_ß\|\mathfrak{L}_M)$:$(£ + 4 \leq ß \leq 2£ + 1)$	$(2£ - ß + 3, 2£ - ß + 4, 2£ - ß + 5, ß - £, ß - £ - 1)$

and the MMRs for the edges $\{\varepsilon = p_ß p_{ß+1}, p_ß q_ß, q_ß p_{ß+1} | ß = 1,2,3, \ldots, n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\}$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = 1)$	$(1,1,2, £ - ß + 1, £)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß - 1,1,1, £ - ß + 1, £ - ß + 2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = 3)$	$(ß - 1, ß - 2,1, £ - ß + 1, £ - ß + 2)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\}$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(4 \leq ß \leq £)$	$(ß - 1, ß - 2, ß - 3, £ - ß + 1, £ - ß + 2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = £ + 1)$	$(ß - 1, ß - 2, ß - 3,1, £ - ß + 2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = £ + 2)$	$(2£ - ß + 2, ß - 2, ß - 3, ß - £ - 1,1)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(ß = £ + 3)$	$(2£ - ß + 2, 2£ - ß + 3, ß - 3, ß - £ - 1, ß - £ - 2)$
$\zeta_M(p_ß p_{ß+1}\|\mathfrak{L}_M)$:$(£ + 4 \leq ß \leq 2£ + 1)$	$(2£ - ß + 2, 2£ - ß + 3, 2£ - ß + 4, ß - £ - 1, ß - £ - 2)$

$\zeta_M(\varepsilon\|\mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\}$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß = 1)$	$(ß - 1,2,3, £ - ß + 2, £)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß = 2)$	$(ß - 1,0,2, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß = 3)$	$(ß - 1, ß - 2,0, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(4 \leq ß \leq £)$	$(ß - 1, ß - 2, ß - 3, £ - ß + 2, £ - ß + 3)$
$\zeta_M(p_ß q_ß\|\mathfrak{L}_M)$:$(ß = £ + 1)$	$(ß - 1, ß - 2, ß - 3,0, £ - ß + 3)$

$\zeta_M(p_ß q_ß \| \mathfrak{L}_M)$:$(ß = £ + 2)$	$(2£ - ß + 3, ß - 2, ß - 3, ß - £ - 1, 0)$
$\zeta_M(p_ß q_ß \| \mathfrak{L}_M)$:$(ß = £ + 3)$	$(2£ - ß + 3, 2£ - ß + 4, ß - 3, ß - £ - 1, ß - £ - 2)$
$\zeta_M(p_ß q_ß \| \mathfrak{L}_M)$:$(£ + 4 \leq ß \leq 2£ + 1)$	$(2£ - ß + 3, 2£ - ß + 4, 2£ - ß + 5, ß - £ - 1, ß - £ - 2)$

$\zeta_M(\varepsilon \| \mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}, q_{£+2}\}$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(ß = 1)$	$(0,1,2, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(ß = 2)$	$(ß, 0,1, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(ß = 3)$	$(ß, ß - 1, 0, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(4 \leq ß \leq £)$	$(ß, ß - 1, ß - 2, £ - ß + 1, £ - ß + 2)$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(ß = £ + 1)$	$(2£ - ß + 2, ß - 1, ß - 2, 0, £ - ß + 2)$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(ß = £ + 2)$	$(2£ - ß + 2, 2£ - ß + 3, ß - 2, ß - £, 0)$
$\zeta_M(q_ß p_{ß+1} \| \mathfrak{L}_M)$:$(£ + 3 \leq ß \leq 2£ + 1)$	$(2£ - ß + 2, 2£ - ß + 3, 2£ - ß + 4, ß - £, ß - £ - 1)$

From the above, for the MMR of edges and vertices for the graph S_n regarding the set $\mathfrak{L}_M^*$, we find that for $1 \leq ß \leq n$ and $ß \neq 1,2,3, £ + 1, £ + 2$, $\zeta_M(p_ß | \mathfrak{L}_M^*) = \zeta_M(q_ß p_ß | \mathfrak{L}_M^*)$ and $\zeta_M(p_{ß+1} | \mathfrak{L}_M^*) = \zeta_M(q_ß p_{ß+1} | \mathfrak{L}_M^*)$. In the rest of the cases, there does not exist any two vertices and edges in S_n with the same mixed metric code. For every $1 \leq ß \leq n$ and $ß \neq 1,2,3, £ + 1, £ + 2$, we see that $\zeta_M(p_ß | \mathfrak{L}_M^* \cup \{q_ß\}) \neq \zeta_M(q_ß p_ß | \mathfrak{L}_M^* \cup \{q_ß\})$ and $\zeta_M(p_{ß+1} | \mathfrak{L}_M^* \cup \{q_ß\}) \neq \zeta_M(q_ß p_{ß+1} | \mathfrak{L}_M^* \cup \{q_ß\})$. From this, we have $\zeta_M(p_ß | \mathfrak{L}_M) \neq \zeta_M(q_ß p_ß | \mathfrak{L}_M)$ and $\zeta_M(p_{ß+1} | \mathfrak{L}_M) \neq \zeta_M(q_ß p_{ß+1} | \mathfrak{L}_M)$, for any $1 \leq ß \leq n$ and so $|\mathfrak{L}_M| \leq n$. Now, on combining Lemma 5 and the above discussion, we obtain that $\mathfrak{L}_M$ is the MMB set for the Flower graph S_n, which implies that $\beta_M(S_n) = n$ in this case.

Case (2) n is even.

For this situation, the integer n takes the shape $n = 2£$, where $£ \in \mathbb{N}$ and $£ \geq 3$. Let $\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}\} \subset \mathbb{V}(S_n)$. Now, in order to see that $\mathfrak{L}_M^*$ is the resolving set in the mixed sense for S_n, we will represent the MMRs for every member present in $V(S_n) \cup E(S_n)$ corresponding to the set $\mathfrak{L}_M^*$.

Now, the MMRs for the vertices $\{v = p_ß, q_ß | ß = 1,2,3, \ldots, n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(v \| \mathfrak{L}_M)$	$\mathfrak{L}_M = \{q_1, q_2, q_3, q_{£+1}\}$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(ß = 1)$	$(1,2,3, £)$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(ß = 2)$	$(ß - 1, 1, 2, £ - ß + 2)$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(ß = 3)$	$(ß - 1, ß - 2, 1, £ - ß + 2)$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(4 \leq ß \leq £ + 1)$	$(ß - 1, ß - 2, ß - 3, £ - ß + 2)$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(ß = £ + 2)$	$(2£ - ß + 2, ß - 2, ß - 3, ß - £ - 1)$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(ß = £ + 3)$	$(2£ - ß + 2, 2£ - ß + 3, ß - 3, ß - £ - 1)$
$\zeta_M(p_ß \| \mathfrak{L}_M)$:$(£ + 4 \leq ß \leq 2£)$	$(2£ - ß + 2, 2£ - ß + 3, 2£ - ß + 4, ß - £ - 1)$

$\zeta_M(\upsilon\vert\mathfrak{L}_M)$	$\mathfrak{L}_M=\{q_1,q_2,q_3,q_{£+1}\}$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(ß=1)$	$(0,2,3,£-ß+2)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(ß=2)$	$(ß,0,2,£-ß+2)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(ß=3)$	$(ß,ß-1,0,£-ß+2)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(4\le ß\le £)$	$(ß,ß-1,ß-2,£-ß+2)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(ß=£+1)$	$(ß,ß-1,ß-2,0)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(ß=£+2)$	$(2£-ß+2,ß-1,ß-2,ß-£)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(ß=£+3)$	$(2£-ß+2,2£-ß+3,ß-2,ß-£)$
$\zeta_M(q_ß\vert\mathfrak{L}_M)$:$(£+4\le ß\le 2£)$	$(2£-ß+2,2£-ß+3,2£-ß+4,ß-£)$

and the MMRs for the edges $\{\varepsilon=p_ßp_{ß+1},p_ßq_ß,q_ßp_{ß+1}\vert ß=1,2,3,\ldots,n\}$ regarding $\mathfrak{L}_M^*$ are:

$\zeta_M(\varepsilon\vert\mathfrak{L}_M)$	$\mathfrak{L}_M=\{q_1,q_2,q_3,q_{£+1}\}$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=1)$	$(1,1,2,£-ß+1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=2)$	$(ß-1,1,1,£-ß+1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=3)$	$(ß-1,ß-2,1,£-ß+1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(4\le ß\le £)$	$(ß-1,ß-2,ß-3,£-ß+1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=£+1)$	$(ß-1,ß-2,ß-3,1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=£+2)$	$(2£-ß+1,ß-2,ß-3,ß-£-1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=£+3)$	$(2£-ß+1,2£-ß+2,ß-3,ß-£-1)$
$\zeta_M(p_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(£+4\le ß\le 2£)$	$(2£-ß+1,2£-ß+2,2£-ß+3,ß-£-1)$

$\zeta_M(\varepsilon\vert\mathfrak{L}_M)$	$\mathfrak{L}_M=\{q_1,q_2,q_3,q_{£+1}\}$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(ß=1)$	$(ß-1,2,3,£)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(ß=2)$	$(ß-1,0,2,£-ß+2)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(ß=3)$	$(ß-1,ß-2,0,£-ß+2)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(4\le ß\le £)$	$(ß-1,ß-2,ß-3,£-ß+2)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(ß=£+1)$	$(ß-1,ß-2,ß-3,0)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(ß=£+2)$	$(2£-ß+2,ß-2,ß-3,ß-£-1)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(ß=£+3)$	$(2£-ß+2,2£-ß+3,ß-3,ß-£-1)$
$\zeta_M(p_ßq_ß\vert\mathfrak{L}_M)$:$(£+4\le ß\le 2£)$	$(2£-ß+2,2£-ß+3,2£-ß+4,ß-£-1)$

$\zeta_M(\varepsilon\vert\mathfrak{L}_M)$	$\mathfrak{L}_M=\{q_1,q_2,q_3,q_{£+1}\}$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=1)$	$(0,1,2,£-ß+1)$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=2)$	$(ß,0,1,£-ß+1)$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=3)$	$(ß,ß-1,0,£-ß+1)$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(4\le ß\le £)$	$(ß,ß-1,ß-2,£-ß+1)$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=£+1)$	$(2£-ß+1,ß-1,ß-2,0)$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(ß=£+2)$	$(2£-ß+1,2£-ß+2,ß-2,ß-£)$
$\zeta_M(q_ßp_{ß+1}\vert\mathfrak{L}_M)$:$(£+3\le ß\le 2£)$	$(2£-ß+1,2£-ß+2,2£-ß+3,ß-£)$

From the above, for the MMR of edges and vertices for the graph S_n regarding the set $\mathfrak{L}_M^*$, we find that for $1 \leq ß \leq n$ and $ß \neq 1,2,3,£+1$, $\zeta_M(p_ß|\mathfrak{L}_M^*) = \zeta_M(q_ß p_ß|\mathfrak{L}_M^*)$ and $\zeta_M(p_{ß+1}|\mathfrak{L}_M^*) = \zeta_M(q_ß p_{ß+1}|\mathfrak{L}_M^*)$. In the rest of the cases, there does not exist any two vertices and edges in S_n with the same mixed metric code. For every $1 \leq ß \leq n$ and $ß \neq 1,2,3,£+1$, we see that $\zeta_M(p_ß|\mathfrak{L}_M^* \cup \{q_ß\}) \neq \zeta_M(q_ß p_ß|\mathfrak{L}_M^* \cup \{q_ß\})$ and $\zeta_M(p_{ß+1}|\mathfrak{L}_M^* \cup \{q_ß\}) \neq \zeta_M(q_ß p_{ß+1} |\mathfrak{L}_M^* \cup \{q_ß\})$. From this, we have $\zeta_M(p_ß|\mathfrak{L}_M) \neq \zeta_M(q_ß p_ß|\mathfrak{L}_M)$ and $\zeta_M(p_{ß+1}|\mathfrak{L}_M) \neq \zeta_M(q_ß p_{ß+1}|\mathfrak{L}_M)$, for any $1 \leq ß \leq n$ and so $|\mathfrak{L}_M| \leq n$. Now, on combining Lemma 5 and the above discussion, we find that $\mathfrak{L}_M$ is the MMB set for the Flower graph S_n, which implies that $\beta_M(S_n) = n$, which complete the proof.

Theorem 10: *Let $S_n = \mathcal{F}_{n\times m}$ be a planar graph on $3n$ and $2n$ edges and vertices respectively. Then, for $n \geq 4$, the cardinality of its minimum IMRS is n.*

Proof: For detailed proof, consider Theorem 9 for reference.

4. Conclusion

The selection of less number of vertices in a graph and using them to identify all the members (vertices & edges) of a graph is a very interesting problem. The concept of MMD, which is the combination of two concepts viz., MD and EMD, offers a versatile and powerful tool for effectively investigating and optimizing computational processes in complex networks. In this paper, we have explored the notion of MMD for two infinite classes of planar graphs, viz., D_n^* and $\mathcal{F}_{n\times m}$. For these two classes, we have determined that the cardinality of MMB depends upon the number of vertices present in them. This shows that the notion of MMD is unbounded for these two planar graphs. Moreover, for these two graphs, we found that the MMB $\mathfrak{L}_M$ and the EMB $\mathfrak{L}_E$ are independent. On comparing the results obtained for D_n^* and $\mathcal{F}_{n\times m}$ regarding MD, EMD, and MMD, we found that $\beta(\Gamma) < \beta_E(\Gamma) < \beta_M(\Gamma)$, for every $n \geq 7$.

Data Availability

No data were used to support this study.

Conflicts of Interest

No conflicts of interest.

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