Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 29 (2026), No. 7, pp. 2897–2904
DOI : 10.47974/JDMSC-2701

In-domination and out-domination numbers of unidirectional graphs

Sanal Kumar
Department of Sciences and Mathematics
Faculty of Mathematics
University of Technology and Applied Sciences
Sultanate of Oman

Abstract

The diverging unidirectional star graph is the unidirectional star graph with n+1 nodes(vertices) and n arcs, n nodes of in-degree 1 and out-degree 0. The converging unidirectional graph consisting of n+1 vertices and n arcs, where n vertices have out-degree one and in-degree zero, with all arcs oriented towards the single central vertex. A diverging unidirectional perfect binary tree is a directed tree where each internal vertex, except for the root, has an out-degree of two and in-degree of one, with all leaves at the same depth; conversely, a converging unidirectional perfect tree has the opposite structure. Here, we investigate the in-domination and out-domination numbers associated with these classes of directed graph. Explicit expressions for the same numbers are derived for the afore mentioned families of graphs as well as for unidirectional non-homogeneous super caterpillar graphs.

Subject Classification (2010): *05C20, 05C69, 05C99.*

Keywords: *Digraph, Domination number, In-domination, Out-domination, Unidirectional graph.*

I. Introduction

Graph theory is a well-established area of discrete mathematics, with extensive theoretical foundations and numerous practical applications. It plays a significant role in diverse fields such as social network analysis, communication networks, and the study of dominating sets in mobile ad hoc networks and wireless sensor networks, among others. Graphs are broadly classified into two categories: undirected graphs and directed graphs. In general, problems involving undirected graphs are more tractable than those concerning directed graphs, and consequently, much of the existing research has focused on undirected graphs, which can be viewed as a subclass of directed graphs. The studies in [1], [2], [3], and [4] introduced the fundamentals of domination. Subsequently, these ideas were extended to several areas, [5] investigated in polynomials, while [6] and [7] focused on perfect graphs. The notion was

E-mail: **drssanalkumar@gmail.com**

applied to permutations in [8] and to graph partitions in [9]. In particular, the works in [10] and [11] introduced the concepts of the clone hop domination number and the outer-connected domination number for graph products, and subsequent developments have followed similar lines. It was further generalized to directed graphs in [12] and [13]. Moreover, [14] introduced the concepts of in-domination and out-domination of graphs. Motivated by this, the present study aims to introduce new concepts in the context of directed graphs by defining the in-domination and out-domination numbers for certain classes of unidirectional graphs that commonly arise in network applications.

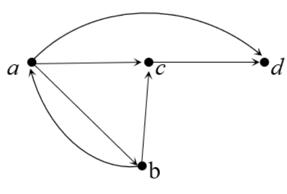
II. Priliminaries

A directed graph $G = (V, A)$ has a set of nodes V and arcs $A \subseteq V \times V$. A node u dominates v if (u, v) is an arc in G. The in-degree and out-degree of v are respectively the cardinality of $\{u/(u, v) \in A\}$ and the cardinality of $\{u/(v, u) \in A\}$. A is a dominating set if every vertex is either in A or adjacent to a vertex in A. A minimal dominating set has no proper subset that also dominates, while the minimal dominating set has the smallest size. This size is the domination number $\gamma(G)$.

III. In-domination and out-domination numbers

Let $G = (V, E)$ be a directed graph with node set V and arc set E. For any arc $(u, v) \in E$, u is a predecessor (or out-dominates) v, and v is a successor (or in-dominates) u. The open in-neighbourhood of a node v is $\mathcal{N}^-(v) = \{u \in V/(u, v) \in V\}$ and the closed in-neighbourhood is $\mathcal{N}^-[v] = \mathcal{N}^-(v) \cup \{v\}$. For a set $S \subseteq V$, these extended to $\mathcal{N}^-(S) = \bigcup_{v \in S} \mathcal{N}^-(v)$ and $\mathcal{N}^-[S] = \bigcup_{v \in S} \mathcal{N}^-[v]$. Similarly, the open-out neighbourhood of v is $\mathcal{N}^+(v) = \{u \in V/(v, u) \in V\}$ and the closed out-neighbourhood is $\mathcal{N}^+[v] = \mathcal{N}^+(v) \cup \{v\}$, with the corresponding $\mathcal{N}^+(S) = \bigcup_{v \in S} \mathcal{N}^+(v)$ and $\mathcal{N}^+[S] = \bigcup_{v \in S} \mathcal{N}^+[v]$. $S \subseteq V$ is an in-dominating set of G if $\mathcal{N}^-[S] = V$. The in-domination number $\gamma^-(G)$ is the size of the smallest in-dominating set. Similarly, $S \subseteq V$ is an out-dominating set if $\mathcal{N}^+[S] = V$, with the out-domination number $\gamma^+(G)$ defined as the cardinality of the smallest out-dominating set.

Consider the digraph G shown below,



The open-neighbourhood of a, b, c and d are $\mathcal{N}^+(a) = \{b, c, d\}$, $\mathcal{N}^+(b) = \{a, c\}$, $\mathcal{N}^+(c) = \{d\}$ and $\mathcal{N}^+(d) = \varphi$. Therefore, the out-dominating sets are $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$ and $\{a, b, c, d\}$. Hence $\gamma^+(G) = 1$, since $\{a\}$ is the only set of cardinality 1. Similarly, the open-neighbourhood of a, b, c and d are $\mathcal{N}^-(a) = \{b\}$, $\mathcal{N}^-(b) = \{a\}$, $\mathcal{N}^-(c) = \{a, b\}$ and $\mathcal{N}^-(d) = \{a, c\}$. Thus, the in-dominating sets are $\{c, d\}$, $\{a, d\}$,

$\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, and $\{a,b,c,d\}$. Therefore, $\gamma^-(G) = 2$ since $\{c,d\}$, $\{a,d\}$ are such sets of cardinalities 2.

IV. In-domination and out-domination numbers of unidirectional star graphs

A diverging unidirectional star graph (denoted S_n^+) is a unidirectional graph with $n+1$ nodes and n arcs, where n nodes have in-degree 1 and out-degree 0; hereafter, we refer to it as a diverging star graph. The unique vertex of out-degree n (the nucleus) out-dominates every other vertex of S_n^+, and no other vertex set has this property. Thus, the out-domination number of S_n^+ is 1. Conversely, no set of fewer than n vertices in-dominates S_n^+. However, the set consisting of all n non-nucleus vertices in-dominates the entire graph. Therefore, the in-domination number of S_n^+ is n. Hence, the in-domination and out-domination numbers of S_n^+ are n and 1, respectively.

Similarly, a converging unidirectional star graph (denoted S_n^-) has $n+1$ nodes and n arcs, with n nodes of out-degree 1 and in-degree 0; it will be referred to as a converging star graph. No subset of fewer than n vertices out-dominates S_n^-, while the set of all n non-nucleus vertices clearly out-dominates the graph. Thus, the out-domination number of S_n^- is n. Furthermore, the nucleus vertex in-dominates every other vertex of S_n^-, and no other singleton set achieves this. Therefore, $\gamma^-(S_n^-)$ is one. We thus, observe that γ^+ of S_n^+ coincides with γ^- of S_n^-, and vice versa.

Moreover, for a directed star graph with $n+m+1$ vertices $n+m$ arc, n vertices of out-degree one and in-degree zero, and m vertices of in-degree one and out-degree zero, then γ^+ is $n+1$ and γ^- is $m+1$.

V. In-domination and out-domination numbers of unidirectional tree graphs

A unidirectional perfect binary tree can be classified as either diverging or converging unidirectional perfect binary tree. In a diverging tree, each internal vertex except the root has an in-degree of one and an out-degree of two, whereas in a converging tree, each internal vertex except the root has an in-degree of two and an out-degree of one. In both types, all leaf vertices reside at the same level and the tree height h may be odd $h = 2n+1$ or even $h = 2n$.

Theorem 5.1: *The γ^+ and γ^- of a diverging unidirectional perfect binary tree of height h are respectively,*

$$\gamma^+(T_d) = \begin{cases} \frac{4^{n+1}-1}{3}, & if\ h\ is\ odd \\ \frac{2(4^n)+1}{3}, & if\ h\ is\ even \end{cases}$$

$$\gamma^-(T_d) = \begin{cases} \frac{2(4^{n+1}-1)}{3}, & if\ h\ is\ odd \\ \frac{4^{n+1}-1}{3}, & if\ h\ is\ even \end{cases}$$

The γ^+ and γ^- of a converging unidirectional perfect binary tree of height h are respectively,

$$\gamma^+(T_c) = \begin{cases} \frac{2(4^{n+1}-1)}{3}, & if\ h\ is\ odd \\ \frac{4^{n+1}-1}{3}, & if\ h\ is\ even \end{cases}$$

$$\gamma^-(T_c) = \begin{cases} \frac{4^{n+1}-1}{3}, & if\ h\ is\ odd \\ \frac{2(4^{n})+1}{3}, & if\ h\ is\ even \end{cases}$$

VI. In-domination and out-domination numbers of unidirectional caterpillar graph

The homogeneous caterpillar graph $C(m,n)$ is defined with $V\big(C(m,n)\big) = \{u_i/1 \le i \le m\} \cup \{v_{ij}/1 \le i \le m,\ 1 \le j \le n\}$ and $E\big(C(m,n)\big) = \{u_i v_{ij}/1 \le i \le m,\ 1 \le j \le n\} \cup \{v_i v_{i+1}/1 \le i \le m-1\}$. The homogeneous diverging caterpillar graph is $C_d(m,n)$, where $V\big(C_d(m,n)\big) = \{u_i/1 \le i \le m\} \cup \{v_{ij}/1 \le i \le m, 1 \le j \le n\}$ represents the vertex set and $E\big(C_d(m,n)\big) = \{(u_i, v_{ij})/1 \le i \le m, 1 \le j \le n\} \cup \{(v_i, v_{i+1})/1 \le i \le m-1\}$ represents the edge set. In contrast, the homogeneous converging caterpillar graph $C_c(m,n)$ is denoted with the edge set represented by $E\big(C_c(m,n)\big) = \{(v_{ij}, u_i)/1 \le i \le m, 1 \le j \le n\} \cup \{(v_{i+1}, v_i)/1 \le i \le m-1\}$.

Theorem 6.1: *The out-domination and in-domination numbers of a non-homogeneous diverging caterpillar graph are the sum of the leaves to the vertices of the central path and the number of vertices on the central path, whereas for a converging caterpillar graph, these values are reversed.*

Proof: The non-homogeneous caterpillar graph, denoted by $C(m; n_1, n_2, \dots n_r)$, generalizes the homogeneous caterpillar graph by allowing the vertices along the central directed path to have varying degrees. The non-homogeneous diverging caterpillar graph, denoted by $C_d(m; n_1, n_2, \dots n_r)$, is a diverging caterpillar graph represented in Figure 1 in which the vertices on the central directed path have differing in-degrees and out-degrees. Similarly, the non-homogeneous converging caterpillar graph, denoted by $C_c(m; n_1, n_2, \dots n_r)$, is defined analogously denoted by Figure 2, with edge directions opposite to those of the diverging caterpillar graph.

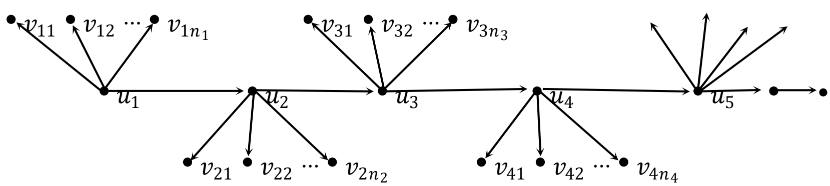


Figure 1
The non-homogeneous diverging caterpillar graph

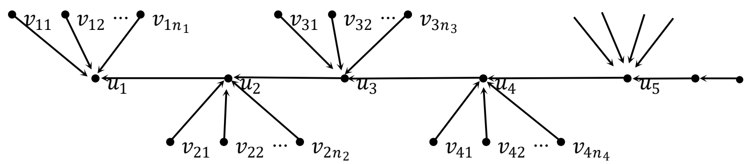


Figure 2
The non-homogeneous converging caterpillar graph

The set of vertices $\{v_{11}, v_{12}, \dots v_{1n_1}, v_{21}, v_{22}, \dots v_{2n_2}, v_{31}, v_{32}, \dots, v_{3n_3}, \dots, v_{m1}, v_{m2}, \dots v_{mn_r}\}$ associated with $C_d(m; n_1, n_2, \dots n_r)$ in-dominates the graph $C_d(m; n_1, n_2, \dots n_r)$ while the smaller set of vertices $\{u_1, u_2, \dots u_m\}$ out-dominates $C_d(m; n_1, n_2, \dots n_r)$. Hence, γ^- and γ^+ of the graph $C_d(m; n_1, n_2, \dots n_r)$ are $n_1 + n_2 + \cdots + n_r$ and m respectively. But for the converging caterpillar graph $C_c(m; n_1, n_2, \dots n_r)$, $\gamma^+(C_c(m; n_1, n_2, \dots n_r))$ and $\gamma^-(C_c(m; n_1, n_2, \dots, n_r))$ are m and $n_1 + n_2 + \cdots + n_r$ respectively.

Theorem 6.2: *In a diverging super caterpillar (DSC) graph, the out-domination number is the count of leaves to the vertices on the central path plus one, and in-domination number is the count of central path vertices plus one: for a converging super caterpillar graph (CSC), these are reversed.*

Proof: Let $\{C_i / 1 \le i \le k\}$ be a finite family of caterpillar graphs. Let u_{00} be a vertex not belonging to any C_i. For each i choose an end vertex of the central path of C_i. The graph obtained by adding the edges $u_{00}\, u_{i1}$, where $u_{i1}$is the chosen terminal vertex of the central path of C_i, is called a super caterpillar graph.

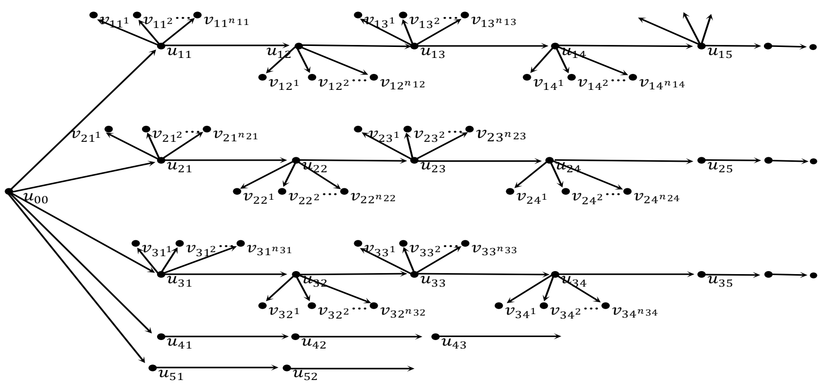


Figure 3
The diverging non-homogeneous super caterpillar graph

The diverging non-homogeneous super caterpillar graph is obtained by adding the directed edges $u_{00}\, u_{i1}$where $u_{i1}$is chosen end vertex of the central path of diverging non-homogeneous caterpillar graph $C_i^d(m_i; n_{i1}, n_{i2}, \dots n_{im_i})$, $1 \leq i \leq k$. Similarly, the converging non-homogeneous super caterpillar graph, denoted by $C_i^c(m_i; n_{i1}, n_{i2}, \dots n_{im_i})$, is defined analogously with edge directions opposite to those of the diverging non-homogeneous super caterpillar graph. An illustration of a diverging non-homogeneous super caterpillar graph is given in Fig. 3.

The vertices set $\{v_{11^1}, v_{11^2}, \dots, v_{11^{n_{11}}}, v_{12^1}, v_{12^2}, \dots, v_{12^{n_{12}}}, \dots, v_{1i^1}, v_{1j^2}, \dots, v_{1j^{n_{1i}}}, \dots, v_{1m_1^1}, v_{1m_1^2}, \dots, v_{1m_1^{n_{1m_1}}}, \dots, v_{i1^1}, v_{i1^2}, \dots, v_{ij^{n_{ij}}}, \dots, v_{im_j^{n_{im_j}}}, \dots, v_{k1^1}, \dots, v_{k1^2}, \dots, v_{km_k^{n_{km_k}}}\}$ together with any one vertex from the set $\{u_{i1}\}$ forms an in-dominating set of the DSC graph, since every vertex in the graph has at least one incoming arc from this set. On the other hand, the vertex set $\{u_{00}, u_{11}, u_{12}, \dots u_{1m_1}, u_{21}, u_{22}, \dots u_{2m_2}, \dots, u_{k1}, \dots, u_{km_k}\}$ constitutes an out-dominating set, as each vertex in the graph is reachable by an outgoing arc from at least one vertex in this set.

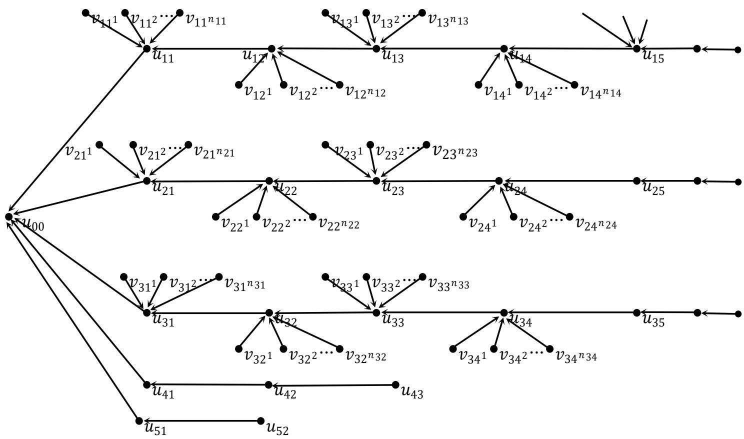


Figure 4
The converging non-homogeneous super caterpillar graph

Consequently, γ^- of the DSC graph is given by $\gamma^-(C^d) = 1 + \sum_{i=1}^{k} \sum_{j=1}^{m_i} n_{ij}$, while the out-domination number is $\gamma^+(C^d) = 1 + \sum_{i=1}^{k} m_i$. Furthermore, for the converging non-homogeneous super caterpillar graph illustrated in Fig. 4, the roles of in-domination and out-domination are reversed. That is, γ^- of CSC graph equals the γ^+ of the corresponding DSC graph, and vice versa.

VII. Summary And Future Work

The author derived in-domination and out-domination numbers of a unidirectional star graphs and binary trees. This work is further extended to investigate the same parameters for other classes of unidirectional caterpillar graphs under both converging and diverging orientations. Additionally, the author aims to determine a system-wide activation strategy that ensures full active using the minimum number of active nodes under selective activation patterns.

References

[1] J.F. Couturier, R. Letourneur, and M. Liedloff, "On the number of minimal dominating sets on some graph classes," Theoretical Computer Science, vol. 562, pp. 634–642 (2015).

[2] J. Ghoshal, R. Laskar, and D. Pillone, Domination in Graphs, 1st ed. New York, NY, USA: Routledge (1998).

[3] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of Domination in Graphs, 1st ed, CRC Press, Boca Raton (1998). doi.org/10.1201/9781482246582

[4] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Eds., Domination in Graphs: Advanced Topics, 1st ed, CRC Press, Boca Raton (1998). doi.org/10.1201/9781315141428

[5] S. Alikhani, "Dominating Sets and Domination Polynomials of Graphs", LAP Lambert Academic Publishing, Saarbrücken, Germany (Jan. 2012).

[6] D. G. Corneil and Y. Perl, "Clustering and domination in perfect graphs," *Discrete Applied Mathematics,* vol. 9, no. 1, pp. 27–39 (Sept. 1984).

[7] P. P. Hawkins, A. M. Anto, and T. S. I. Mary, "Perfect dominating sets and perfect domination polynomial of a pan graph," *Journal of Critical Reviews*, vol. 7, no. 4 (2020).

[8] A. Brandstadt and D. Kratsch, "On domination problems for permutation and other graphs," *Theoretical Computer Science*, vol. 54, no. 2–3, pp. 181–198 (1987).

[9] B. K. P. Arimbawa and E. T. Baskoro, "Partition dimension of some classes of trees," *Procedia Computer Science,* vol. 74, pp. 67–72 (2015).

[10] G. Mahadevan and V. Vijayalakshmi, “Clone hop domination number of a graph,” *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 22, no. 5, pp. 719–729 (2019), doi: 10.1080/09720529.2019.1681689.

[11] M. Hashemipour, M. R. Hooshmandasl, and A. Shakiba, “On the outer-connected domination number for graph products,” *Journal of Information and Optimization Sciences,* vol. 41, no. 5, pp. 1253–1267 (2020), doi: 10.1080/02522667.2019. 1645396.

[12] J. Ghoshal, R. Laskar, and D. Pillone, “Topics on domination in directed graphs,” in Domination in Graphs. New York, NY, USA: Routledge, pp. 401–437 (1998).

[13] C. Pang, R. Zhang, Q. Zhang, and J. Wang, “Dominating sets in directed graphs,” *Information Sciences*, vol. 180, no. 19, pp. 3647–3652 (2010).

[14] G. Chartrand, F. Harary, and B. Q. Yue, “On the out-domination and in-domination numbers of a digraph,” *Discrete Mathematics*, vol. 197–198, pp. 179–183 (1999).

Received June, 2023