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Generalized skew derivations on Posets

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Abstract

Introducing and examining the concept of generalized skew derivations on partially ordered sets (Posets) is the primary goal of this research. We use generalized skew derivations to demonstrate a few basic poset properties. Furthermore, examples are provided to illustrate and delimit our results.

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1. Introduction

$(S, \leq)$ consistently indicates a partially ordered set (poset) throughout this manuscript. The concept of derivations is overly broad; for instance, [1], [2], and [3]. The concepts of generalized skew derivations in rings, near-rings, and lattices served as the inspiration for the current investigation. Generalized skew derivations combine the concepts of skew and generalized derivations. Prime and semi-prime rings' (near-rings') commutativity accepting generalized skew derivations that fulfill fitting of algebraic requirements on certain subsets of the ring has been studied by several authors(viz.;[4],[5],[6],[7],[8], [9] and [10], where further references can be found). However, the investigation of generalized skew derivations in algebras gained momentum shortly after Herstein [11] and Posner [12] simultaneously achieved several outstanding outcomes, especially for prime rings. Research indicates that generalized skew derivations are essential for studying the commutativity of laws and defining the structures of algebras, rings, and near-rings. There are various studies that claim prime

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near-rings with limited generalized skew derivations behave like rings. Many mathematicians have explored generalized skew derivations in rings, near-rings and lattices in various ways. Applying derivations and their generalized forms to lattices (i.e., [13], [14], [15], [16], and references therein) has allowed some researchers to validate a number of significant conclusions in recent years.When Xin et al. [17] investigated the idea of fixed sets of derivations on lattices in 2012, they came up with some astounding findings. The fixed set $Fix_g(L)$ is an ideal of L, as demonstrated by the authors, when g is an isotone derivation of L. The systematic and mathematical properties of lattices have been studied by many authors. Also the notion of generalized skew derivations appeared in lattices sees, for example [13], [16] and [17]. Zhang and Li [18] introduced and investigated the ideas of derivations on posets in 2017. They offered a number of characterization theorems regarding poset derivations. In [19] the concept of skew derivations is introduced and studied on posets.

Nextly a few notations will introduced for the rest of the section. A poset is sometimes denoted by the abbreviation P. With respect to $w \in S$, $\downarrow w = \{v \in S: v \leq w\}$ and $\uparrow w = \{v \in S: w \leq v\}$. The upper cone of W is $u(W) = \{v \in S: w \leq v, forany\, w \in W\}$, while the lower cone is $l(W) = \{v \in S: v \leq w\, forall\, w \in W\}$. The compositions of $l(u(.))$ and $u(l(.))$ are monotone and antitones. The expressions $l(u(l(.))) = l(.)$ and $u(l(u(.))) = u(.)$ are from [20]. A finite subset of S is represented by $l(W) = l(w_1, w_2, \ldots, w_n)$ and $u(W) = u(w_1, w_2, \ldots, w_n)$, respectively, given $W = \{w_1, w_2, \ldots, w_n\}$. Additionally, for $W_1 \subseteq S$ and $W_2 \subseteq S$,1, the values $l(W_1, W_2)$ and $u(W_1, W_2)$ are displayed for $l(W_1 \cup W_2)$ and $u(W_1 \cup W_2)$, respectively. We write $\downarrow W = \{v \in S: v \leq w, forsome\, w \in W\}$. According to [21], if $W = \downarrow W$, then W is said to be a lower set. There is an upper bound in W for each finite subset of W if W is not empty. In this case, the subset is directed. Assuming that every possible combination of components in W has an upper bound in W, given non-emptiness, makes sense. If a subset J of S is directed lower set, it is called an ideal of S.

2. Preliminaries

The definitions that follow will be useful in the following sections. The following is the starting point for the discussion.

Definition 2.1: ([22]) Let $(S, \leq)$ represent a poset and $\mu: S \to S$ represent a map. μ is said to be increasing, if the following condition are met, if $s \leq t$, then $\mu(s) \leq \mu(t)$, for all $s, t \in S$.

Definition 2.2: ([22]) Consider $(S, \leq)$ as a poset and $\mu: S \to S$ as a map. An automorphism happens when μ is bijective and increasing while μ^{-1}, its inverse map, is also increasing.

Remark 2.3: $\mu(0) = 0$, happens if $(S, \leq)$ is a poset with the smallest element 0 and $\mu: S \to S$ is an automorphism.

Definition 2.4: ([18]) Suppose S is a poset. If a mapping $g: S \to S$ meets the following criteria, it is regarded as a derivation:

(i) $g(l(s,t)) = l(u(l(g(s),t), l(s,g(t))))$ for every $s,t \in S$;

(ii) $l(g(u(s,t))) = l(u(g(s),g(t)))$ for every $s,t \in S$.

Definition 2.5: ([23]) Under the assumption that S is a poset, $g, G: S \to S$ are functions such that g is a derivation of S. The following criteria must be met for G to be referred to as a generalized derivation on S.Under the assumption that S is a poset, $g, G: S \to S$ are functions such that g is a derivation of S. The following criteria must be met for G to be referred to as a generalized derivation on S.

(i) $G(l(r,s)) = l(u(l(G(r),s), l(r,g(s))))$, for each $r,s \in S$;

(ii) $l(G(u(r,s))) = l(u(G(r),G(s)))$ for each $r,s \in S$.

Definition 2.6 ([19]) Consider S be a poset and $g, \beta: S \to S$ are maps. The statements that follow show that g is a skew derivation on S associated with β.

(i) $g(l(r,s)) = l(u(l(g(r),\beta(s)), l(r,g(s))))$ for any $r,s \in S$;

(ii) $l(g(u(r,s))) = l(u(g(r),g(s)))$ for any $r,s \in S$.

The concept of generalized skew d-derivations on partially ordered sets (posets) is introduced in this work, along with some of its features. In addition to demonstrating the other classical results, we also demonstrate that if G_1 and G_2 are two generalized skew g-derivations of S related to skew derivation g and an automorphism β on S such that $G_1\beta = \beta G_1$ and $G_2\beta = \beta G_2$, then $G_1 \leq G_2$ if and only if $G_2G_1 = \beta G_1$. The situation of derivations [18] and skew derivations [19] are generalized by these results.

3. On generalized skew g-derivations of posets

This section examines generalized skew g-derivations on posets and is inspired by Lee [24] and Lee and Liu [25]. Inspired by the ideas behind generalized skew derivations in rings, algebras, and near-rings, we present and describe generalized skew derivations on posets.Let us start by talking about the following:

Definition 3.1: If S is a poset and $G, g, \beta: S \to S$ are functions, then g is a skew derivation on S that is connected to β of S. If G meets the following criteria, it is a generalized skew g-derivation of S connected to β on S:

(i) $G(l(r,s)) = l(u(l(G(r),\beta(s)), l(r,g(s))))$; for every $r,s \in S$;

(ii) $l(G(u(r,s))) = l(u(G(r),G(s)))$ for every $r,s \in S$.

Remark 3.2: Generalized skew d-derivations are a generalization of the concepts of derivations, skew derivations, and generalized derivations.

Remark 3.3: Of course, all skew derivations are generalized skew d-derivations on S. However, this is not always the case. The following example illustrates this.

Example 3.4: Consider the poset $(S, \leq) = ((1, \infty), \leq)$. Functions $G, g, \beta: S \to S$ must be defined by $g(s) = 0$, $\beta(s) = s + 1$ and $G(s) = 2s$ for all $s \in S$. The fact that d is a skew derivation of S associated with β on S can thus be easily verified. The map G represents a generalized skew g-derivation on S. G is not a skew derivation of β associated with S to be sure.

In the sequel, the following lemma is critical.

Lemma 3.5: *Examine a generalized skew derivation G that is associated with a skew derivation g on S and an automorphism $\beta: S \to S$ so that $id_S \leq G$ and $id_S \leq g \leq \beta$ of S. Then the claims listed below are true.:*

(i) $G(s_1) \leq G(s_2)$; *whenever* $s_1 \leq s_2$,

(ii) $g(s) \leq G(s)$ *for any* $s \in S$;

(iii) $G(s) \leq \beta(s)$ *for any* $s \in S$;

(iv) $G(I) \subseteq I$; *when* I *is an ideal of* S *and* $\beta(I) \subseteq I$,

(v) *Should* 0, *be the least element in* S, *then* $G(0) = 0$.

Proof:

(i) Let $s_1, s_2 \in S$ such that $s_1 \leq s_2$. Because of this, $l(u(G(s_1), G(s_2)) = l(G(u(s_2))) = l(G(u(s_1, s_2))$. Given that $G(s_1) \in l(u(G(s_1), G(s_2))$, we may conclude that $G(s_1) \in l(G(u(s_2))$ .Thus, for every $s_1, s_2 \in S$, $G(s_1) \leq G(s_2)$.

(ii) Considering that $t_1 \in S$,

$$\begin{aligned} l(g(t_1)) &= l(g(t_1), t_1)),, \text{since } id_s \leq g \text{ on } S, \\ &= l(u(l(g(t_1), t_1))), \\ &\subseteq l(u(l(G(t_1), \beta(t_1)), l(t_1, g(t_1)))), \\ &= l(u(l(G(t_1), \beta(t_1))))),, \text{by } id_s \leq G \text{ and } g \leq \beta \text{ on S}, \\ &= l(G(t_1), \beta(t_1)). \end{aligned}$$

The fact that $g(t_1) \in l(g(t_1))) \subseteq l(G(t_1), \beta(t_1))$ is also used. Hence, $g(t_1) \leq G(t_1)$ follows from $g(t_1) \in l(G(t_1), \beta(t_1)$.

(iii) Presume that $G: S \to S$ is a generalized skew g-derivation of S associated with β on S and a skew derivation g of S. Recall that β is an automorphism of S. Presuming that $t_1 \in S$, then

$$\begin{aligned} G(l(t_1, t_1)) &= l(u(l(G(t_1), \beta(t_1)), l(t_1, g(t_1)))), \\ &= l(u(l(G(t_1), \beta(t_1)))),, \text{by } id_s \leq G \text{ and } g \leq \beta \text{ on S}, \\ &= l\big(G(t_1), \beta(t_1)\big) \text{ for all } t_1 \in S. \end{aligned}$$

Therefore, $G(l(t_1)) = G(l(t_1, t_1)) \subseteq l(G(t_1), \beta(t_1))$ for all $t_1 \in S$. Also,using $G(t_1) \in G(l(t_1))$, leads to $G(t_1) \in l(G(t_1), \beta(t_1))$. Thus $G(t_1) \leq \beta(t_1)$ for all $t_1 \in S$.

(iv) Suppose that I is an ideal of S. $t_1 \in I$ exists such that $G(t_1) = t_2$ if $t_2 \in G(I)$. Applications of part (iii) gives $G(t_1) \leq \beta(t_1)$. As a result, the last relation yields $t_2 \leq \beta(t_1)$, but $\beta(I) \subseteq I$ and I is an ideal of S. This means that $t_2 \in I$., therefore $G(I) \subseteq I$.

(v) Applying Remark 2.3 yields $\beta(0) = 0$, assuming that S has the least element 0 and that β is an automorphism. Consequently, $0 \leq G(0) \leq \beta(0) = 0$., meaning that $G(0) = 0$. This demonstrates the lemma.

Lemma 3.6: *Given a skew derivation g of S and an automorphism β on S, let $G: S \to S$ represent a generalized skew g-derivation related to β. All of the following statements are true.*

- $G(r) = s$, *whenever* $G(l(r)) = l(s)$, *for every* $r, s \in S$*;*
- *If* $G(u(r)) = u(s)$, *then* $G(r) = s$ *for every* $r, s \in S$.

Proof:

(i) Suppose $G(l(r)) = l(s)$. Next, according to the definition, $s \in l(s)$, therefore we have $s \in G(l(r))$. Thus, $t \in l(r)$ exists in such a way that $G(t) = s$. The aforementioned relation yields $s \leq G(r)$ since $G(s) \leq G(r)$. An alternative is $G(r) \in G(l(r)) = l(s)$, so $G(r) \leq s$ and, for each $r, s \in S$, $G(r) = s$.

(ii) By employing a comparable strategy with the required modifications, we can demonstrate (ii).

This section's primary, crucial result is the following theorem.

Theorem 3.7: *Consider S to be a poset, and $G, g, \beta: S \to S$ are functions in which β is an automorphism on S and g is a skew derivation of S related to β on S and such that $id_S \leq G \leq \beta$ $id_S \leq g$ of S. As a result, G is a generalized skew g-derivation of S associated with β on S if and only if*

(i) $G(l(r, s)) = l(G(r), \beta(s)) = l(G(r), G(s))$, for all $r, s \in S$;

(ii) $l(G(u(r, s))) = l(u(G(r), G(s)))$ for all $r, s \in S$.

Proof: Essentially, it needs to show that condition (i) in this theorem is equivalent to condition (i) in Definition 2.6.

(1 Assuming that this theorem's condition is satisfied first, then for all $r, s \in S$

$$l(u(l(G(r), \beta(s)), l(r, g(s))) = l(u(l(G(r), \beta(s))))),$$

since $id_s \leq G$ and $g \leq \beta$ on S,

$$= l(G(r), \beta(s)),$$
$$= G(l(r, s)).$$

(2) Part 1: Alternatively, think of G as a generalized skew g-derivation of S associated with an automorphism β on S. So

$$\begin{aligned} G(l(t,s)) &= l(u(l(G(t),\beta(s)),l(t,g(s)))), \\ &= l(u(l(G(t),\beta(s)))),\ \text{since } id_s \leq G \text{ and } g \leq \beta \ on\ \text{S}, \\ &= l(G(t),\beta(s)). \end{aligned}$$

Part 2: We show that $G(l(t,s)) = l(G(r),G(s))$, for all $t,s \in S$. Let $t,s \in S$, and $z \in G(l(t,s))$. So there exists $r \in S$ such that $r \leq t, r \leq s$ and $G(r) = z$. By Lemma 3.5(1) we get $z = G(r) \leq G(t)$ and $z = G(r) \leq G(s)$. Thus $r \in l(G(t),G(s))$. Therefore $G(l(t,s)) \subseteq l(G(t),G(s))$ for all $t,s \in S$. Conversely, suppose that $t,s \in S$ and $z \in l(G(t),G(s))$. So by lemma 3.5(3) we have $z \leq G(t)$ and $z \leq G(s) \leq \beta(s)$. Hence $z \in l(G(t),\beta(s)$. Using part1 we obtain $z \in G(l(t,s))$. Consequently, for any $t,s \in S.$, $l(G(t),G(s)) \subseteq G(l(t,s)))$. Thus, the theorem's proof is completed.

The following results directly from Theorem 3.7.

Corollary 3.8: ([19], Theorem 2.10) *Let $G,\beta: S \to S$ be functions such that β is an automorphism and $id_S \leq G \leq \beta$, and let S be a poset. Then G is "a skew derivation" of S if and only if*

(i) $G(l(r,s)) = l(G(r),\beta(s)))$ for all $r,s \in S$;

(ii) $l(G(u(r,s))) = l(u(G(r),G(s)))$ for all $r,s \in S$.

Corollary 3.9: *A generalized skew g-derivation G associated with an automorphism β on S, and a skew derivation g of S such that $id_S \leq g \leq \beta$ and $id_S \leq G$ on S are given. Then $G(l(s)) = l(G(s))$ for all $s \in S$.*

Corollary 3.10: ([18], Theorem 2.2) *Assume that $g: S \to S$ represents a function and that S represents a poset. Then g is "a derivation" of S if and only if*

(i) $g(l(r,s)) = l(gr,s)) = l(r,gs)$ for each $r,s \in S$;

(ii) $l(g(u(r,s))) = l(u(gr,gs))$ for each $r,s \in S$.

Theorem 3.11: *Consider a generalized skew g-derivation G, a skew derivation g of S, and an automorphism β on S, such that $id_S \leq G$ and $id_S \leq g \leq \beta$ on S. Then $G(l(t_1,t_2)) = l(G(t_1),G(t_2))$ for all $t_1,t_2 \in S$.*

Proof: First suppose that if $w \in l(G(t_1),G(t_2))$, then $w \leq G(t_1)$, $w \leq G(t_2)$. By Lemma 3.5 (iii), $w \leq \beta(t_2)$. Then $w \in l(G(t_1),\beta(t_2))$. By Theorem 3.7(i), $l(G(t_1),G(t_2)) \subseteq G(l(t_1,t_2))$.

Conversely, suppose that $w \in G(l(t_1,t_2))$, Accordingly, $z \in l(t_1,t_2)$ occurs such that $G(z) = w$. Consequently, $G(z) \leq G(t_1), G(z) \leq G(t_2)$ may be obtained by applying Lemma 3.5(i). Hence, $w \in l(G(t_1),G(t_2))$. Therefore, for any $t_1,t_2 \in S$, $G(l(t_1,t_2) \subseteq l(G(t_1),G(t_2))$. This suggests that for any $t_1,t_2 \in S$, $G(l(t_1,t_2)) = l(G(t_1),G(t_2))$.

4. The β-fixed points of generalized skew g- derivations

The generalized skew g-derivations on a poset S will be discussed in this section along with their fixed points. Given a function $\beta: S \to S$ and a skew derivation g on S, such that $id_S \leq g \leq \beta$ of S. A generalized skew g-derivation associated with β on S. is denoted by G.We call s a β-fixed point of G in S if $G(s) = \beta(s)$, as stated in [19]." It is $Fix_{\beta,G}(S) = \{s \in S: G(s) = \beta(s)\}$ comprises every β-fixed point of G in S, and $G(S) = \{G(s): s \in S\}$. Obviously, $Fix_{id_S,G}(S) = Fix_G(S) = \{s \in S: G(s) = s\}$.

Theorem 4.1: *Consider the generalized skew g-derivation G, which is related to an automorphism β on S. Given a bijective map β on S such that $\beta G = G\beta$., let $id_S \leq G$, and $id_S \leq g \leq \beta$, β be Then, the following claims are accurate:*

(i) $G(t_1) \in Fix_{\beta,G}(S)$ *for any* $t_1 \in S$;

(ii) $Fix_{\beta,G}(S) = G(S)$.

Proof: (i) To demonstrate that $G(t_1) \in Fix_{\beta,G}(S)$ for each $t_1 \in S$. It suffices to show that, for any $t_1 \in S$., $G(G(t_1)) = \beta(G((t_1))$. Our findings are based on Lemma 3.5 and Theorem 3.7 (i).

$$\begin{aligned} G(l(G(t_1))) &= G(l(\beta(t_1), G(t_1))) \\ &= l(G(\beta(t_1)), \beta(G(t_1))) \\ &= l(\beta(G(t_1)), \beta(G(t_1))) \text{since } \beta G = G\beta \\ &= l(\beta(G(t_1))) \text{ for all } t_1 \in S. \end{aligned}$$

The use of Lemma 3.6 produces $G(G(t_1)) = \beta(G((t_1)))$ for any $t_1 \in S$. This means that $G(t_1) \in Fix_{\beta,G}(S)$ for any $t_1 \in S$.

To demonstrate (ii). Part (i) gives us $G(t_1) \in Fix_{\beta,G}(S)$ for all $t_1 \in S$. Accordingly, $G(S) \subseteq Fix_{\beta,G}(S)$. However, assuming that $t_1 \in Fix_{\beta,G}(S)$, we obtain $\beta(t_1) = G(t_1)$ for any $t_1 \in S$. Considering that β is a bijective map on S and $\beta G = G\beta$. Additionally, this suggests that $\beta^{-1}G = G\beta^{-1}$. Thus, we obtain $s_1 = \beta^{-1}(G(t_1)$ Equal to $G(\beta^{-1}(t_1))$ in S. for every t_1 Therefore, we find that $Fix_{\beta,G}(S) = G(S)$. The theorem is fully demonstrated by this.

The subsequent example illustrates that the Theorem 4.1 requires that β be bijective and commutative with G .

Example 4.2: Assume that $(S, \leq) = ((4, \infty), \leq)$ is a poset. Construct the functions $D, d, \beta: S \to S$ by $D(s) = 2s$, $d(s) = 0$ and $\beta(s) = s^2$ for all $s \in S$. Then, it becomes evident that g is a skew derivation associated with β on S. $G\beta \neq \beta G$, as well, but we discover that $G(G(s)) = 4s \neq 4s^2 = \beta(G(s))$ for any $s \in S$, that is, $G(s) \notin Fix_{\beta,G}(S)$. Thus, it is impossible to loosen the presumptive restrictions in Theorem 4.1's hypothesis.

Proposition 4.3: Given a bijective map β, let G_1 and G_2 be two generalized skew g-derivations associated with β on S such that $G_1\beta = \beta G_1$ and $G_2\beta = \beta G_2$. Consequently, $G_1 = G_2$ if and only if $Fix_{\beta,G_1}(S) = Fix_{\beta,G_2}(S)$.

Proof: There is just one clear way if $G_1 = G_2$. It is assumed, however, that $t \in S$. and $Fix_{\beta,G_1}(S) = Fix_{\beta,G_2}(S)$. Next, we wish to demonstrate that $G_1 = G_2$. With reference to Theorem 4.1 (i), we possess $G_1(t) \in Fix_{\beta,G_1}(S) = Fix_{\beta,G_2}(S)$. This forces that $G_2(G_1(t)) = \beta(G_1(t))$. By the assumption, we obtain $G_2(G_1(t)) = G_1(\beta(t))$. Similarly, $G_1(G_2(t)) = G_2(\beta(t))$ is obtained. We derive $G_1(\beta(t)) \leq G_2(\beta(t))$ and $G_2(\beta(t)) \leq G_1(\beta(t))$ from Lemma 3.5 (i), (iii). These two relations lead us to the conclusion that $G_1(\beta(t)) = G_2(\beta(t))$, and so $\beta(G_1(t)) = \beta(G_2(t))$. according to each $t \in S$." β is bijective, hence for all $t \in S$, the expression returns $G_1(t) = G_2(t)$, i.e., $G_1 = G_2$."

Proposition 4.4: The poset S is assumed to have the least element 0. Next, let G be a generalized skew g-derivation on S, linked to a skew derivation g on S and an automorphism map β, so that $G\beta = \beta G$. Then, the following claims are true:

(i) $0 \in Fix_{\beta,G}(S)$;

(ii) if $s \in Fix_{\beta,G}(S)$, and $r \leq s$ then $r \in Fix_{\beta,G}(S)$;

(iii) if S is directed, then, for any $r, s \in Fix_{\beta,G}(S)$, there exists $t \in Fix_{\beta,G}(S)$ such that $r \leq t, s \leq t$.

Proof:

(i) Since β is an automorphism, we obtain $G(0) = \beta(0) = 0$, via Lemma 3.5 (v). According to this, $0 \in Fix_{\beta,G}(S)$.

(ii) $G(s) = \beta(s)$ if $s \in Fix_{\beta,G}(S)$, $s \in S$, and $r \leq s$ are true. Then, we obtain $G(l(r)) = G(l(s,r)) = l(G(s), \beta(r)) = l(\beta(s), \beta(r)) = l(\beta(r))$ using Theorem 3.7(i).Using Lemma 3.6(i), we obtain $G(r) = \beta(r)$ for each $r \in S$. From this, we deduce that $r \in Fix_{\beta,G}(S)$.

(iii) S is assumed to be directed. There is a $t \in S$ such that $r \leq t$ and $s \leq t$. for every $r, s \in S$,. But $r, s \in Fix_{\beta,G}(S)$, then we write $G(r) = \beta(r)$ and $G(s) = \beta(s)$. But $G(r) = \beta(r) \leq G(t)$ and $G(s) = \beta(s) \leq G(t)$. We are given that β is an automorphism such that $G\beta = \beta G$ and therefore, we obtain $r \leq G(\beta^{-1}(t))$ and $s \leq G(\beta^{-1}(t))$. Applying Theorem 4.1(ii), we now obtain $z \in Fix_{\beta,G}(S)$ since $z = G(\beta^{-1}(t))$,. Thus, the outcome is demonstrated.

Corollary 4.5: *If S is a directed poset, then $Fix_{\beta,G}(S)$ is an ideal of S, with 0 as its least element.*

Theorem 4.6: *Given a skew derivation g on S and an automorphism $\beta: S \to S$, let G be a generalized skew g-derivation. Allow $G\beta = \beta G$ after that. Then, $Fix_{\beta,G}(S) \cap l(\beta(s)) = l(G(t))$ for all $t \in S$.*

Proof: $t_1 \in S$ and $t_2 \in Fix_{\beta,G}(S) \cap l(\beta(t_1))$ are assumed at first. Then, we get $\beta(t_2) = G(t_2)$ and $t_2 \leq \beta(t_1)$. This implies that $G(t_2) \leq G(\beta(t_1)) = \beta(G(t_1))$, i.e.; $\beta(t_2) \leq \beta(G(t_1))$. But β is an automorphism, henceforward we conclude $t_2 \leq G(t_1)$. Thus $t_2 \in l(G(t_1))$, and as a result, we get $Fix_{\beta,G}(S) \cap l(\beta(t_1)) \subseteq l(G(t_1))$. On the other hand we obtain $G(t_1) \leq \beta(t_1)$ and $G(t_1) \in Fix_{\beta,G}(S)$. This yields $G(t_1) \in Fix_{\beta,G}(S) \cap l(\beta(t_1))$. For any $t_1 \in S$., we thus obtain $l(G(t_1)) \subseteq Fix_{\beta,G}(S) \cap l(\beta(t_1))$ by Proposition4.4 (ii) . $Fix_{\beta,G}(S) \cap l(\beta(t_1)) = l(G(t_1))$ for any $t_1 \in S$.

An immediate result of the aforementioned theorem is as follows.

Corollary 4.7: [[18], Theorem 3.4] *Consider a derivation on S, g. Then, for every $x \in S$., $Fix_g(S) \cap l(x) = l(g(x))$*

5. The ideals and operations related with generalized skew g-derivations

The analysis of certain poset structural characteristics using generalized skew g-derivations is the focus of this section. Additionally, we give some basic results for ideals of S of these derivations and demonstrate that two generalized skew g-derivations on points always commute.From now on, S is considered a poset with the least element 0, and an automorphism is denoted by $\beta: S \to S$. We start with the subsequent outcome.

Theorem 5.1: *Let G be a generalized skew g-derivation on S associated with an automorphism $\beta: S \to S$ and a skew derivation g on S, such that G, β commute i.e. $G\beta = \beta G$. Next, consider two ideals of S, U and W. In that case, the following claims are true:*

(i) $G(W_1) \subseteq G(W_2)$, whenever $W_1 \subseteq W_2$,

(ii) $G(W_1 \cap W_2) = G(W_1) \cap G(W_2)$.

Proof:

(i) With G being a generalized skew g-derivation and $\beta: S \to S$ being an automorphism of S, we are provided with these facts. Assume $w \in G(U)$. If $u \in U \subseteq W$ exists, then $w = G(u)$. It follows from this that $w \in G(W)$. From this, we deduce that $G(U) \subseteq G(W)$.

(ii) A generalized skew g-derivation is G, and β is an automorphism of S. Obviously, $G(U \cap W) \subseteq G(U) \cap G(W)$ is one method. Conversely, let $v \in G(U) \cap G(W)$. $G(w_1) = v$ and $G(w_2) = v$ if $w_1 \in W_1, w_2 \in W_2$ exists. The following is the outcome of applying the theorem3.7 (i): $G(l(w_1, \beta^{-1}(G(w_2))) = l(G(w_1), G(w_2)) = l(v, v) = l(v)$. However, since $v \in l(v)$, we must deduce that $v \in G(l(w_1, \beta^{-1}(G(w_2))))$.

Consequently, $z \in l(w_1, \beta^{-1}(G(w_2))$ exists such that $G(z) = v$. We obtain $z \in W_1 \cap W_2$. by using $z \leq w_1$ and $z \leq \beta^{-1}(G(w_2)) = G(\beta^{-1}(w_2)) \leq w_2$. Therefore, $v \in G(W_1 \cap W_2)$. $G(W_1) \cap G(W_2) \subseteq G(W_1 \cap W_2)$, according to this $G(W_1 \cap W_2) = G(W_1) \cap G(W_2)$ is the result. The proof is now complete.

Theorem 5.2: G_1 and G_2 are two generalized skew g-derivations on S, related to an automorphism β of S and a skew derivation g on S such that $\beta\mathrm{G}_1 = \mathrm{G}_1\beta$ and $\beta\mathrm{G}_2 = \mathrm{G}_2\beta$. Afterwards, G_1 and G_2 have to commute.

Proof: We consider G_1 and G_2 to be two generalized skew g-derivations on S that are connected to a skew derivation g on S and an automorphism β of S. Thus, for each $s \in S$, we obtain

$$\begin{aligned} G_1(l(G_2(s))) &= G_1(l(\beta(s), G_2(s))) \\ &= l(G_1(\beta(s)), \beta(G_2(s))) \\ &= l(\beta(G_1(s)), \beta(G_2(s))) \text{ for each } s \in S, \end{aligned}$$

and

$$\begin{aligned} G_2(l(G_1(s))) &= G_2(l(\beta(s), G_1(s))) \\ &= l(G_2(\beta(s)), \beta(G_1(s))) \\ &= l(\beta(G_1(s)), \beta(G_2(s))) \text{ for each } s \in S. \end{aligned}$$

This implies that $G_1(l(G_2(s))) = G_2(l(G_1(s)))$. But $G_1(G_2(s)) \in G_1(l(G_2(s)))$, thus $G_1(G_2(s)) \in G_2(l(G_1(s)))$. Then there exists $r \in l(G_1(s))$ such that $G_1(G_2((s)) = G_2(r)$. Written by Lemma 3.7 (i), $G_2(s) \leq G_2(G_1(s))$, Accordingly, $G_1(G_2((s)) \leq G_2(G_1(s))$." Similarly, we can prove that $G_2(G_1(s)) \leq G_1(G_2(s))$. Henceforth, we conclude that $G_1(G_2(s)) = G_2(G_1(s))$ for any $s \in S$.

In the next example we show that the condition β commutes with G_1, G_2 in Theorem 5.2 is necessary and can not be omitted.

Example 5.3: Let $(S, \leq) = (-\infty, \infty), \leq)$ be a poset. Functions must be defined as follow $G_1, G_2, g, \beta: S \to S$ by $G_1(s) = 2s + 2, G_2(s) = 2s + 1, g(s) = s$ and $\beta(s) = 3s$, for any $s \in S$ which is an automorphisms on S. By doing this, we can confirm that g is a skew derivation associated with an automorphism β on S. Two generalized skew g-derivations associated with β of S are also G_1, G_2. However, we find that $G_1 G_2 \neq G_2 G_1$, indicating that G_1, G_2 do not commute. As a result, Theorem 5.2 requires that β to commutes with G_1, G_2.

Theorem 5.4: *The automorphism β and the skew derivation g on S are given. Let G_1 and G_2 be two generalized skew g-derivations of S such that $\beta G_1 = G_1\beta$ and $\beta G_2 = G_2\beta$. Then "$G_2 G_1 = \beta G_1$" if and only if "$G_1 \leq G_2$".*

Proof: We start by assuming that G_1 and G_2 are generalized skew g-derivations on S that correspond to an automorphism β on S in such a way that $G_1 \leq$.

Next, we have $G_1(s) \in Fix_{\beta,G_1}(S)$ for each $s \in S.$, which means that $\beta(G_1(s)) = G_1(G_1(s)) \leq G_2(G_1(s))$. Using Lemma 3.5 (iii) produces $G_2G_1(s) \leq \beta(G_1(s))$. for every $s \in S$. Consequently, for each $s \in S.$, $G_2(G_1(s)) = \beta(G_1(s))$ This demonstrates that $G_2(G_1(s)) = \beta(G_1(s))$. Conversely, we suppose that for every $s \in S.$, $\beta(G_1(s)) = G_2G_1(s) = G_1G_2(s) \leq \beta(G_2(s))$. Conversely, we suppose that for every $s \in S.$, $\beta(G_1(s)) = G_2G_1(s) = G_1G_2(s) \leq \beta(G_2(s))$. This indicates for every $s \in S$, $\beta(G_1(s)) \leq \beta(G_2(s))$. From the left side until the last relation, multiply by β^{-1} to get $G_1(s) \leq G_2(s)$ for all $s \in S$. This leads us to the conclusion that $G_1 \leq G_2$. This concludes the theorem's proof.

As a direct consequence of [[18], Theorem 4.5], the following corollary generalizes.

Corollary 5.5: *Consider two derivations of* S, G_1 *and* G_2. *In such case,* "$G_1 \leq G_2$" *if and only if* "$G_2G_1 = G_1$".

As seen by our final example, Theorem 5.4 requires that β *be an automorphism and commute with* G_1, G_2.

Example 5.6: Consider the poset $(S, \leq) = ([0, \infty], \leq)$. Functions $G_1, G_2, g, \beta : S \to S$ must be defined by $G_1(s) = G_2(s) = 2s$, $g(s) = 0$ and $\beta(s) = s^2$ for all $s \in S$. Once g is a skew derivation related with β on S, it is easy to see. Furthermore, G_1 and G_2 are generalized skew g-derivations associated with β on S. $G_1 \leq G_2$, but we find that $G_2(G_1)(s)) = 4s \neq 4s^2 = \beta G_1(s)$ for all $s \in S$. As a result, the assumed restriction cannot be ignored.

6. Conclusion

In this paper, let $(S, \leq)$, be a poset and $D, d, \beta : S \to S$ are maps such that $id_S \leq D$ and $id_S \leq d \leq \beta$ on S. This paper's primary goal is to describe generalized skew d-derivations on S that are connected to a skew derivation d on S and an automorphism β of S. Furthermore, we generalized all the results of the reference [19].

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