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Digraphs based on *BL*-algebras

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Abstract

This paper considers the notion of BL-algebras and converts any Galois finite field to a BL-algebra. It extends the class of BL-algebras based on finite fields and makes a connection between finite fields and BL-algebras. This study introduces the concept of a down digraph based on the notion of a down set (as a vertex) of BL-algebras, which states that any two vertices are adjacent if the related down sets are not joined. Characterization of the down digraphs is fundamental in this research, so we compute the number of edges in some down digraphs via combinators, to prove that down digraphs are connected under what conditions

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are connected and are Tournament. In the final, try to extract the BL-algebras from graphs, so show that any connected graph constructs a BL-algebra.

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1. Introduction

The mathematical modeling of network structures is a cornerstone of modern science, with applications spanning social networks, biological systems, computer science, and logistics [12]. A fundamental challenge in network science is to endow these complex structures with a formal, algebraic framework that allows for rigorous analysis and the discovery of inherent properties. While graph theory provides the structural language, logical algebras offer a powerful tool for encoding the relationships and interactions within a network, transforming a mere set of connections into a system with defined operational rules. Among these logical structures, BL-algebras (Basic Logic Algebras) have emerged as a significant class, originating from the study of fuzzy logic and serving as the algebraic counterpart of Hajek [7] Basic Logic. Defined by a rich structure involving lattice, monoidal, and implication operations, BL-algebras provide a natural setting for reasoning about graded concepts and uncertainty. Their structure has been extensively studied from a purely algebraic perspective [4, 11, 10], revealing connections to other logical systems and lattice-ordered algebraic objects. For more literature for the different topics on BL-algebras, see [3, 14, 15, 17].

A parallel line of inquiry seeks to bridge the gap between abstract algebra and graph theory. The construction of graphs from algebraic structures such as zero-divisor graphs of rings, Cayley graphs of groups, and annihilator graphs of various algebras has proven to be a fertile ground for research [2, 9, 6, 16]. These constructions often reveal hidden combinatorial properties of the algebra and provide intuitive, visual representations of their structure. Recently, there has been growing interest in leveraging logical algebras for this purpose. For instance, the work of connecting specific algebraic properties of BL-algebras to topological and order-theoretic concepts [5, 1, 13], suggests a deep, unexplored potential for graph-theoretic interpretations.

This paper is motivated by a synthesis of these two domains. We posit that the logical structure of BL-algebras can serve as a robust foundation for defining and analyzing networks. The core idea is to use the implication operation `→`, which intuitively captures a notion of "directed influence" or "conditional connection" to derive a canonical graph structure from any given BL-algebra. This leads to our central concept: the down digraph $\Gamma(X)$ of a BL-algebra X, where vertices are non-trivial elements, and directed edges are defined by the failure of a certain union of down sets to be closed under implication. This construction is fundamentally different from previous approaches [8], as it is intrinsically tied to the logical connective of implication rather than additive or multiplicative operations.

The principal aims and novel contributions of this work are fourfold. First, we introduce and formally define the down graph of a BL-algebra, establishing its core properties and demonstrating how it provides a graph-theoretic representation of the algebraic structure. Second, we explore the reverse direction by showing how certain graph classes, specifically path graphs and connected graphs, can be endowed with a BL-algebra structure, thereby establishing a two-way bridge between graph theory and algebraic logic. Third, we provide a detailed combinatorial characterization of down digraphs derived from finite fields, proving explicit formulas for their number of edges and establishing precise conditions under which they become tournaments (Theorem 4.16). Finally, we extend this analysis to Galois fields $GF(p^n)$, where we demonstrate that the corresponding down digraphs are *always* tournaments for $n \geq 2$ and compute their exact edge counts.

The potential applications of this framework are significant. By representing a network as a BL-algebra, one can use the algebra's logical operations to model dynamic processes like information flow, trust propagation, or resource allocation. The associated down graph then offers a static snapshot of the "influence relationships" within the system. Furthermore, this approach could lead to new methods for network classification using algebraic properties, enable the modeling of network dynamics as transformations in the underlying BL-algebra, and provide novel tools for robustness analysis by relating graph-theoretic concepts like connectivity to algebraic properties such as the existence of certain filters or ideals.

In summary, this work lays a novel foundation for a logic-based theory of networks. By forging a strong, two-way connection between BL-algebras and graph theory, we open up new avenues for analyzing complex systems through the lens of algebraic logic, with promising implications for both theoretical mathematics and applied network science.

2. Preliminaries

In this section, we recall some fundamental notions that are essential for our work.

A simple graph G is a finite, non-empty set V of objects are vertices (singular form: vertex), along with a set E of 2-element subsets from V, referred to as edges. It is denoted as $G = (V = V(G) = \{v_i\}_{i=1}^{n}, E = E(G) = \{e_i\}_{i=1}^{m})$. Two vertices $u, v \in V$ are adjacent if is an edge $e = \{u, v\}$ connecting them. A simple graph where every pair of distinct vertices are adjacent is a complete graph, represented as K_n for a graph with n vertices. A path in G is defined as a finite, distinct sequence of vertices and edges written as $x_0, e_1, x_1, e_2, x_2, \dots, x_{n-1}, e_n, x_n$. If the path ends where it started ($x_0 = x_n$), it is called a cycle. A graph is connected if there is a path between every pair of vertices. Conversely, a graph is disconnected if it is not connected. The number of connected components in a graph G is denoted by $t(G)$. A graph is also classified as bipartite if and only if every cycle within G has even length. Moving to directed graphs (or digraphs), a digraph is an ordered pair $D=(V(D),E(D))$, where $V(D)$ represents a set of vertices and $E(D)$ represents a set of edges. In this context, the edges are ordered pairs of vertices, $e=(u,v)$, where $u,v\in V(D)$. Here, u is referred to as the tail of the edge, v as the head, and both u and v as the endpoints of the edge. If there is an edge directed from u to v, it is represented as (u,v). An edge of the form $e=(u,u)$ (where tail and head coincide) is called a loop. Two edges e and e' are considered parallel if they share the same tail and head. A digraph $D=(V(D),E(D))$ is considered simple if it does not contain loops or parallel edges. For any digraph, there is an ordinary (undirected) graph G, called its underlying graph, which shares the same vertex and edge sets but ignores the direction of the edges. Conversely, the digraph can be referred to as an orientation of its underlying graph. If the underlying graph is complete, the digraph is termed a tournament. The outdegree of a vertex v, denoted as $deg^{+}(v)$, counts the number of edges originating from v. Similarly, the indegree of v, denoted as $deg^{-}(v)$, counts the number of edges ending at v. A directed walk in $D=(V(D),E(D))$ is defined as a sequence $v_0,e_1,v_1,\dots,e_n,v_n$, such that $v_i\in V(D)$ for every $0\le i\le n$, and each edge $e_i\in E(D)$ goes from vertex v_{i-1} to vertex v_i. This is referred to as a walk from v_0 to v_n. If $v_0=v_n$, the walk is called closed. If all vertices in the sequence $v_0,v_1,\dots,v_n$ are distinct, it is referred to as a directed path.

Finally, a digraph $D=(V(D),E(D))$ is considered strongly connected if for any two vertices $u,v\in V(D)$, there is a directed walk from u to v.

Definition 2.1: [7] An algebraic structure $(X,\wedge,\vee,*,\rightarrow,0,1)$ of type (2, 2, 2, 2, 0, 0) is called a BL-algebra if, for all $x,y,z\in X$, the following axioms hold:

(BL1) $(X,\wedge,\vee,0,1)$ is a bounded lattice with respect to the order $\le$.

(BL2) $(X,*,1)$ is a commutative monoid.

(BL3) $x*y\le z$ if and only if $x\le y\rightarrow z$. (Residuation)

(BL4) $x\wedge y=x*(x\rightarrow y)$. (Divisibility)

(BL5) $(x\rightarrow y)\vee(y\rightarrow x)=1$. (Prelinearity)

Theorem 2.2: [7] *Let* $(X,\wedge,\vee,*,\rightarrow,0,1)$ *be a* BL*-algebra and* $x,y,z\in X$. *Then*

(i) $x*y\le x\wedge y$,

(ii) $x\le y$ *if and only if* $x\rightarrow y=1$,

(iii) *if* $x\le y$, *then* $z\rightarrow x\le z\rightarrow y, y\rightarrow z\le x\rightarrow z$ *and* $x*z\le y*z$,

(iv) $x\rightarrow(y\rightarrow z)=(x*y)\rightarrow z=y\rightarrow(x\rightarrow z)$,

(v) $1\rightarrow x=x, x\rightarrow x=1$ *and* $x\rightarrow 1=1$,

(vi) $x\vee y=((x\rightarrow y)\rightarrow y)\wedge((y\rightarrow x)\rightarrow x)$.

3. Down sets of BL-algebras

In this section, we apply the notion of the down set of BL-algebra and inspect its properties. Indeed, the down set of any given BL-algebra is a set of elements just like those that are related to the top element 1 via the implication operation in BL-algebra. Also, we exhibit the idea of $(U.C)$-lattices and try to inspect the down set of any given BL-algebras based on the $(U.C)$-lattices. The notion of down set in BL-algebras is introduced as a special case of segment of BL-algebras as follows [1].

Let $(X,\wedge,\vee,*,\rightarrow,0,1)$ be a BL-algebra and $z \in X$. Then $\downarrow(z) = \{x \in X | x \rightarrow z = 1\}$ is defined as down set of z in X.

Let $(X,\wedge,\vee,*,\rightarrow,0,1)$ be a BL-algebra. From now on, for any BL-algebra X, we will denote $\downarrow(X) = \{\downarrow(x) | x \in X\}$ as the down set of X.

Proposition 3.1: Let $(X,\wedge,\vee,*,\rightarrow,0,1)$ be a BL-algebra and $z \in X$. Then

(i) $\{0, z\} \subseteq \downarrow(z)$.

(ii) $\downarrow(1) = X$.

(iii) $\downarrow(0) = \{0\}$.

(iv) for any $0 \neq z \in X, |\downarrow(z)| \geq 2$.

(v) $\bigcup_{z\in X} \downarrow(z) = X$.

(vi) $(\downarrow(X),\cup,\cap,\downarrow(0),\downarrow(1))$ is a bounded lattice.

Proof: We prove the items $(iv), (v), (vi)$, the other items are straightforward.

(iv) It is clear from item (i).

(v) Based on item (iv), $\bigcup_{z\in X} \downarrow(z) = \bigcup_{z\in X} \{z\} = X$.

(vi) Based item (v), $\bigcup_{z\in X} \downarrow(z) = X$ and for any $z, z' \in X, \downarrow(0) \subseteq (\downarrow(z) \cap \downarrow(z')) \neq \emptyset$. Further, for any $z \in X, \downarrow(0) \subseteq \downarrow(z) \subseteq \downarrow(1)$. It follows that $(\downarrow(X),\cup,\cap,\downarrow(0),\downarrow(1))$ is a bounded lattice.

Theorem 3.2: *Let $(X,\wedge,\vee,*,\rightarrow,0,1)$ be a BL-algebra and $x, y, z \in X$.*

(i) $x \leq y$ *if and only if* $\downarrow(x) \subseteq \downarrow(y)$.

(ii) $\downarrow(y) \subseteq \downarrow(x \rightarrow y)$.

(iii) $z \in \downarrow(x \rightarrow y)$ *if and only if* $x \in \downarrow(z \rightarrow y)$.

Proof: (i) Let $x \leq y$, by Theorem 2.2, get that $z \rightarrow x \leq z \rightarrow y$. If $z \in \downarrow(x)$, then $z \rightarrow x = 1$ and so $1 \leq z \rightarrow y$ indicates that $z \rightarrow y$. Hence $z \in \downarrow(y)$ and so $\downarrow(x) \subseteq \downarrow(y)$.

Conversely, let $\downarrow(x) \subseteq \downarrow(y)$. Then $x \in \downarrow(x)$ indicates that $x \in \downarrow(y)$ and so $x \rightarrow y = 1$. Thus $x \leq y$.

(ii) Let $z \in \downarrow(y)$. Thus $z \rightarrow y = 1$. Applying Theorem 2.2, $z \rightarrow (x \rightarrow y) = x \rightarrow (z \rightarrow y) = x \rightarrow 1 = 1$ and so $z \rightarrow (x \rightarrow y) = 1$. Hence $z \in \downarrow(x \rightarrow y)$ and so $\downarrow(y) \subseteq \downarrow(x \rightarrow y)$.

(iii) Let $x, y, z \in X$. Then by Theorem 2.2, $z \in \downarrow(x \rightarrow y)$ if and only if $z \rightarrow (x \rightarrow y) = 1$ if and only if $x \rightarrow (z \rightarrow y) = 1$, because of $z \rightarrow (x \rightarrow y) = x \rightarrow (z \rightarrow y)$. Hence $z \in \downarrow(x \rightarrow y)$ if and only if $x \in \downarrow(z \rightarrow y)$.

From Theorem 3.2, we immediately get the subsequent corollary.

Corollary 3.3: *Let* $(X,\wedge,\vee,*,\to,0,1)$ *be a* BL*-algebra and* $x,y\in X$.

(i) $x=y$ *if and only if* $\downarrow(x)=\downarrow(y)$.

(ii) $x\in\downarrow(y)$ *reason being* $1\in\downarrow(x\to y)$.

(iii) $\downarrow(x*(x\to y))\subseteq\downarrow(y)$.

Let $(X,\wedge,\vee,0,1)$ be a finite bounded lattice. We say that $(X,\wedge,\vee,0,1)$ is a $(U.C)$-lattice, if it is a complemented lattice and the complement of any it's element is unique. From now on, for any $x\in X$, will denote the complement of x by $\overline{x}$, which $x\wedge\overline{x}=0$ and $x\vee\overline{x}=1$.

Theorem 3.4: *Let* $(X,\wedge,\vee,*,\to,0,1)$ *be a* BL*-algebra, which* $(X,\wedge,\vee,0,1)$ *is a* $(U.C)$*-lattice. Then for any* $x\in X$,

(i) $x*\overline{x}=\overline{x}*x=0$,

(ii) $x\to\overline{x}=\overline{x}$,

(iii) $(x\to\overline{x})\to\overline{x}=(\overline{x}\to x)\to x$,

(iv) $\overline{x}\to x=x$.

Proof: (i) Let $x\in X$. By Theorem 2.2, $x*\overline{x}\leq x\wedge\overline{x}=0$. Thus $x*\overline{x}=\overline{x}*x=0$, because, $(L,*)$ is commutative.

(ii,iii) Using Theorem 2.2,

$$1=x\vee\overline{x}=((x\to\overline{x})\to\overline{x})\wedge((\overline{x}\to x)\to x)$$

and so $(x\to\overline{x})\to\overline{x}=(\overline{x}\to x)\to x$. It follows that $x\to\overline{x}\leq\overline{x}$. Further, Using Theorem 2.2,

$$\overline{x}=1\to\overline{x}\leq(x\to 1)\to(x\to\overline{x})=1\to(x\to\overline{x})=x\to\overline{x},$$

and so $\overline{x}\leq x\to\overline{x}$.

(iv) It is similar to (ii).

Theorem 3.5: *Let* $(X,\wedge,\vee,*,\to,0,1)$ *be a* BL*-algebra, which* $(X,\wedge,\vee,0,1)$ *is a* $(U.C)$*-lattice. Then for any* $x\in X$, $\downarrow(x)\cap\downarrow(\overline{x})=\{0\}$,

Proof: Let $z\in\downarrow(x)\cap\downarrow(\overline{x})$. Then $z\to x=1=z\to\overline{x}$. Using Theorem 2.2, we get that $z\leq x$ and $z\leq\overline{x}$ and so $z\leq x\wedge\overline{x}=0$. It follows that $z=0$.

4. Graph based on down sets in BL-algebras

In this section, we construct down graph by the down sets of BL-algebras and inspect their properties. Indeed, the down sets of BL-algebras are the vertices of graph and the edges extracted by the any two vertices that their down sets can not be joined with respect to the union and implication.

In what follows, we show that some connected graphs generate BL-algebras. A path graph is a graph $P_n=(V_n,E)$ whose vertices can be listed in the order $V_n=\{v_1,v_2,\dots,v_{n-1},v_n\}$ and for any $1\leq i\leq n-1,\{v_i,v_{i+1}\}\in E$ as Figure 1.

Figure 1
Path Graph P_n

Theorem 4.1: *Any path graph can be a BL-algebra.*

Proof: Consider the path graph $P_n = (V_n, E)$ whose vertices are listed in the order $V_n = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and for any $1 \leq i \leq n-1, \{v_i, v_{i+1}\} \in E$ as Figure 1. For any $1 \leq i \neq j \leq n$, define the binary operations $v_i \vee v_j = v_{\max\{i,j\}}, v_i * v_j = v_i \wedge v_j = v_{\min\{i,j\}}$ and $v_i \to v_j = \begin{cases} 1 & \text{if } i \leq j, \\ v_j & \text{otherwise} \end{cases}$.

$(BL1)$ It is clear that $(V_n, \wedge, \vee, v_1, v_n)$ is a bounded lattice relative to the order $\leq_{V_n}$, which for any $1 \leq i \neq j \leq n, v_i \leq_{V_n} v_j$ if and only if $i < j$.

$(BL2)$ $(V_n, *)$ is a commutative monoid, because for any $1 \leq i \neq j \leq n$, $v_n * v_j = v_n \wedge v_j = v_{\min\{n,j\}} = v_j$.

$(BL3)$ For any $1 \leq i, j, k \leq n$, $v_i * v_j \leq_{V_n} v_k \Leftrightarrow \min\{i,j\} \leq k$. If $\min\{i,j\} = i$, then $i \leq j \leq k$ and so $v_i \leq_{V_n} v_n = (v_j \to v_k)$. If $\min\{i,j\} = j$, then $j \leq i \leq k$ and so $v_i \leq_{V_n} v_n = (v_j \to v_k)$. Conversely, let $v_i \leq_{V_n} v_j \to v_k$. It follows that $i \leq j \leq k$ and so $\min\{i,j\} \leq k$. Therefore, for any $1 \leq i \neq j \neq k \leq n$, $v_i * v_j \leq_{V_n} v_k \Leftrightarrow v_i \leq_{V_n} v_j \to v_k$.

$(BL4)$ For any $1 \leq i, j \leq n$, $v_i \wedge v_j = v_i * (v_i \to v_j)$, because of $\min\{i,j\} \leq j \leq n$.

$(BL5)$ For any $1 \leq i, j \leq n$, $(v_i \to v_j) \vee (v_j \to v_i) = 1$, since $v_i \leq_{V_n} v_j$ or $v_j \leq_{V_n} v_i$. Thus $(V_n, \wedge, \vee, *, \to, v_1, v_n)$ is a BL-algebra.

Corollary 4.2: *Any connected graph may be a BL-algebra.*

Proof: Let $G = (V, E)$ is a connected graph and $|V| = n$. Then for any $x, y \in V$, there exists a path $P(x, y)$. Thus we can extract a path graph $P_n = (V_n, E)$ from graph $G = (V, E)$, so using Theorem 4.1 $G = (V, E)$ can be a BL-algebra.

Example 4.3: Consider the complete graph $K_5 = (\{a, b, c, d, e\}, E)$. We extract the path graph P_5 as Figure 2. Then $(\{a, b, c, d, e\}, \wedge, \vee, *, \to, a, e)$ is a BL-algebra given by monoid Table 1 and groupoid Table 2.

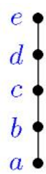

Figure 2
Path graph P_5

Table 1

Monoid $(P_5, *)$

$*$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	c	c
d	a	b	c	d	d
e	a	b	c	d	e

Table 2

Groupoid $(P_5, \to)$

$\to$	a	b	c	d	e
a	e	e	e	e	e
b	b	e	e	e	e
c	a	b	e	e	e
d	a	b	c	e	e
e	a	b	c	d	e

In what follows, we show that BL-algebras can generate directed graphs.

Definition 4.4: Let $(X, \wedge, \vee, *, \to, 0,1)$ be a BL-algebra and $x, y \in X$. Then will call $\Gamma(X) = (V = V(\Gamma(X)) = X \backslash \{0,1\}, E = E(\Gamma(X))$ as a down graph of X, which for any two distinct vertices x and $y, (x, y) \in E$, if $\downarrow(x) \cup \downarrow(y) \neq \downarrow(x \to y)$ and $(y, x) \in E$, if $\downarrow(y) \cup \downarrow(x) \neq \downarrow(y \to x)$.

It clear that $\Gamma(X) = (V = V(\Gamma(X)) = X \backslash \{0,1\}, E = E(\Gamma(X)))$ has not loop and is not a simple graph, since for any $x, y \in V$, necessarily, $\downarrow(x \to y) \neq \downarrow(y \to x)$.

Example 4.5: Let $X = \{0, a, b, c, d, 1\}$. Then $(X, \wedge, \vee, *, \to, 0,1)$ is a BL-algebra as monoid Table 3 and groupoid Table 4 and bounded lattice in Figure 3.

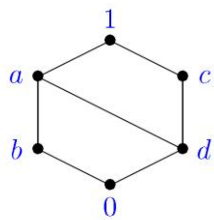

Figure 3
Bounded lattice $(X, \leq)$

Table 3

Monoid $(X, *)$

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

Table 4

Groupoid $(X, \to)$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Computations show that

$\downarrow(0) = \{0\}, \downarrow(1) = X, \downarrow(a) = \{0, a, b, d\}, \downarrow(b) = \{0, b\}$,
$\downarrow(c) = \{0, c, d\}$ and $\downarrow(d) = \{0, d\}$.

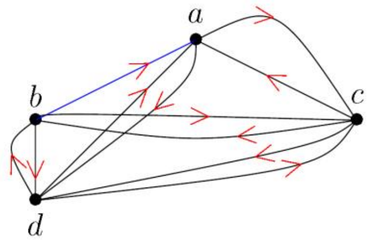

Figure 4
Down graph $\Gamma(X) = (V, E)$ for BL-algebra $(\mathbb{Z}_7, \wedge, \vee, *, \rightarrow, \mathbf{0}, \mathbf{1})$.

Thus, one can prove that the directed graph $\Gamma(X) = (V, E)$ is a related down graph with BL-algebra $(\mathbb{Z}_7, \wedge, \vee, *, \rightarrow, 0, 1)$ as shown in Figure 4.

Example 4.6: Consider the BL-algebra $Y = \{0, p, q, r, s, t, u, 1\}$ defined by the lattice structure in Figure 5 and the monoid in Table 5 and groupoid in Table 6. This algebra exhibits strong asymmetries in both its lattice structure and operations.

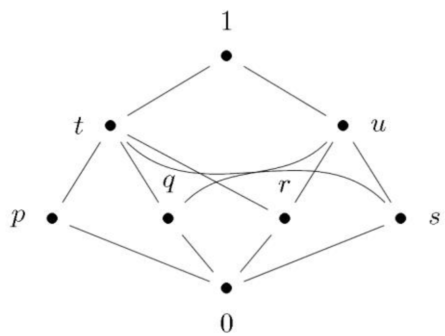

Figure 5
Non-distributive lattice structure of Y.

Table 5
Monoid $(Y, *)$.

$*$	0	p	q	r	s	t	u	1
0	0	0	0	0	0	0	0	0
p	0	p	p	0	0	p	0	p
q	0	p	q	q	0	q	q	q
r	0	0	q	r	r	r	r	r
s	0	0	0	r	s	0	s	s
t	0	p	q	r	0	t	r	t
u	0	0	q	r	s	r	u	u
1	0	p	q	r	s	t	u	1

Table 6
Groupoid $(Y, \rightarrow)$.

$\rightarrow$	0	p	q	r	s	t	u	1
0	1	1	1	1	1	1	1	1
p	u	1	1	s	s	1	s	1
q	t	u	1	u	s	1	u	1
r	q	q	u	1	1	u	1	1
s	p	p	q	r	1	q	r	1
t	r	u	u	s	s	1	s	1
u	p	p	q	r	s	q	1	1
1	0	p	q	r	s	t	u	1

The down sets reveal intricate dependency relationships:
$\downarrow(0) = \{0\}, \downarrow(p) = \{0, p\}, \downarrow(q) = \{0, p, q\}, \downarrow(r) = \{0, q, r\}$,
$\downarrow(s) = \{0, s\}, \downarrow(t) = \{0, p, q, r, t\}, \downarrow(u) = \{0, q, r, s, u\}, \downarrow(1) = Y$.

The down digraph $\Gamma(Y) = (V, E)$ with $V = \{p, q, r, s, t, u\}$ in Figure 6.

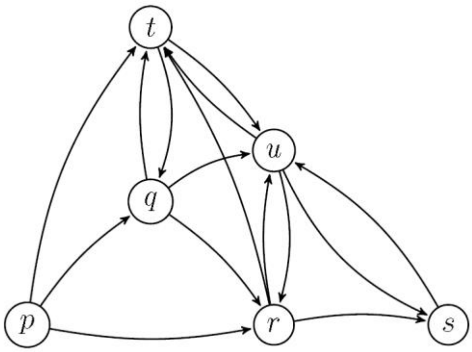

Figure 6
Down digraph $\Gamma(Y)$.

Proposition 4.7: Let $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ be a BL-algebra and $x, y \in V(\Gamma(X)))$. Then $\downarrow(x) \neq \downarrow(y)$.

Proof: Let $x, y \in V(\Gamma(X)))$. Then by Corollary 3.3, $x = y$ reason being $\downarrow(x) = \downarrow(y)$. Since $\Gamma(X) = (V, E)$ is simple, we get that $x \neq y$ and so $\downarrow(x) \neq \downarrow(y)$.

4.1 *Finite fields and down graphs derived from BL-algebras*

In this section, we try to exhibit a new class of BL-algebras just like that the related derived digraph be connected and non-tournament. Firstly, we try to construct a new class of BL-algebras based on the finite fields. In the subsequent theorem, we convert any finite field to a BL-algebra and so on extract the related derived digraph. In this regard we show that any lattices can use to construct of BL-algebras based on the finite fields.

Remark 4.8: Let $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ be a BL-algebra. It is easy to see that

(i) $(X, *, 1)$ may not be a group.
(ii) $(X, *, 0)$ may not be a group.
(iii) $(X, \rightarrow, 1)$ may not be a group.
(iv) $(X, \rightarrow, 0)$ may not be a group.
(v) $(X, \wedge, 1)$ may not be a group.
(vi) $(X, \wedge, 0)$ may not be a group.

(vii) $(X,\vee,1)$ may not be a group.
(viii) $(X,\vee,0)$ may not be a group.

For instance, consider the BL-algebra $(X,\wedge,\vee,*,\rightarrow,0,1)$ in Example 4.5. Then for $(X,*,1)$, -8 has not inverse, for $(X,*,0)$, 0 is not an identity element, for $(X,\rightarrow,1)$, 1 is not an identity element, for $(X,\rightarrow,0)$, 0, is not an identity element, for $(X,\wedge,1)$, -4 has not an inverse, for $(X,\wedge,0)$, 0 is not an identity element and for $(X,\vee,1)$, 1 is not an identity element, for $(X,\vee,0)$, -2 has not an inverse.

Let $(X,\wedge,\vee,*,\rightarrow,0,1)$ be a BL-algebra. Then $(X,\wedge,\vee,*,\rightarrow,0,1)$ may not be a field. Consider the BL-algebra $(X,\wedge,\vee,*,\rightarrow,0,1)$ in Example 4.5. It is easy to see that $(X,\wedge,\vee,*,\rightarrow,0,1)$ can not be a field. Now, we show that any finite field can be a BL-algebra as follows.

Theorem 4.9: *Any finite field can be a BL-algebra.*

Proof: Let $(X,+,\cdot)$ be a field. Since $(X,+,\cdot)$ is finite, exists a prime p just like that $(X,+,\cdot)$ is isomorphic to $(\mathbb{Z}_p,\oplus,\mathrm{e})$. For any $\overline{x},\overline{y}\in\mathbb{Z}_p$, define $\overline{x}\wedge\overline{y}=\begin{cases}\overline{0} & \text{if } x=0 \text{ or } y=0,\\ \overline{max\{x,y\}} & \text{otherwise,}\end{cases}$ and $\overline{x}\vee\overline{y}=\begin{cases}\overline{max\{x,y\}} & \text{if } x=0 \text{ or } y=0,\\ \overline{min\{x,y\}} & \text{otherwise,}\end{cases}$. Now, define $*:\mathbb{Z}_p\times\mathbb{Z}_p\rightarrow\mathbb{Z}_p$ by $\overline{x}*\overline{y}=\overline{xy}$ and $\rightarrow:\mathbb{Z}_p\times\mathbb{Z}_p\rightarrow\mathbb{Z}_p$ by $\overline{x}\rightarrow\overline{y}=\begin{cases}\overline{yx^{-1}} & 0\neq x<y,\\ \overline{1} & x=0 \text{ or } x\geq y,\end{cases}$. Now prove that $(\mathbb{Z}_p,\wedge,\vee,*,\rightarrow,\overline{0},\overline{1})$ is a BL-algebra. It is clear that $(\mathbb{Z}_p,\wedge,\vee,\overline{0},\overline{1})$ is a bounded lattice relative to the order $\leq_{\mathbb{Z}_p}$, which for any $\overline{0}\neq\overline{x},\overline{y}\in\mathbb{Z}_p$, $\overline{x}\leq_{\mathbb{Z}_p}\overline{y}$ reason being $y\leq x$ and $\overline{0}\leq_{\mathbb{Z}_p}\overline{y}$. Also by definition, $(\mathbb{Z}_p,*,\overline{1})$ is a commutative monoid. Thus $(BL1)$ and $(BL2)$ are valid.

$(BL3)$: Let $\overline{x},\overline{y},\overline{z}\in\mathbb{Z}_p$ and $\overline{x}*\overline{y}\leq_{\mathbb{Z}_p}\overline{z}$. $\overline{x}\leq_{\mathbb{Z}_p}(\overline{y}\rightarrow\overline{z})$. Then $\overline{x}*\overline{y}\leq_{\mathbb{Z}_p}\overline{z}$, because of $\overline{xy}\leq_{\mathbb{Z}_p}\overline{z}$ reason being $z\leq xy$ if and only if $\overline{zy^{-1}}\leq\overline{x}$ reason being $\overline{x}\leq_{\mathbb{Z}_p}\overline{zy^{-1}}$ only if $\overline{x}\leq_{\mathbb{Z}_p}(\overline{y}\rightarrow\overline{z})$.

$(BL4)$: Let $\overline{x},\overline{y}\in\mathbb{Z}_p$. If $x<y$, then $\overline{x}\wedge\overline{y}=\overline{y}=\overline{xyx^{-1}}=\overline{x}*\overline{yx^{-1}}=\overline{x}*(\overline{x}\rightarrow\overline{y})$. If $x=1$, then $\overline{x}\wedge\overline{y}=\overline{y}=\overline{1}*\overline{y}=\overline{1}*(\overline{1}\rightarrow\overline{y})=\overline{x}*(\overline{x}\rightarrow\overline{y})$. If $x=y$, then $\overline{x}\wedge\overline{y}=\overline{x}=\overline{x}*\overline{1}=\overline{x}*(\overline{x}\rightarrow\overline{x})=\overline{x}*(\overline{x}\rightarrow\overline{y})$. If $x\geq y$, then $\overline{x}\wedge\overline{y}=\overline{x}=\overline{x}\wedge\overline{1}=\overline{x}*(\overline{x}\rightarrow\overline{y})=\overline{x}*(\overline{x}\rightarrow\overline{y})$.

$(BL5)$: Let $\overline{x},\overline{y}\in\mathbb{Z}_p$. If $x\leq y$, then $\overline{y}\rightarrow\overline{x}=\overline{1}$. If $y\leq x$, then $\overline{x}\rightarrow\overline{y}=\overline{1}$. Then in any cases, $(\overline{x}\rightarrow\overline{y})\vee(\overline{y}\rightarrow\overline{x})=\overline{1}$.

Example 4.10: Consider the field $(\mathbb{Z}_3,\oplus,e)$ and a lattice $(\mathbb{Z}_3,\wedge,\vee)$ in Figure 7. Then $(\mathbb{Z}_3,\wedge,\vee,*,\rightarrow,\overline{0},\overline{1})$ is a BL-algebra given by monoid in Table 7 and groupoid in Table 8.

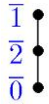

Figure 7
Lattice $(\mathbb{Z}_3, \leq)$

Table 7
Monoid $(\mathbb{Z}_3, *)$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{1}$

Table 8
Groupoid $(\mathbb{Z}_3, \rightarrow)$

$\rightarrow$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{2}$	$\overline{0}$	$\overline{1}$	$\overline{1}$

From now on, we view $p \geq 5$ be a prime and use them in the derived results. We obtained the directed graphs from the BL-algebras that are based on finite fields. The graphs that obtained in the following manners are different to the obtained graphs in the methods in [8]. Indeed, we apply the down set in BL-algebras related the operations in corresponded BL-algebras, while the obtained graphs in [8], the additive and multiplicative operations play the main role in construct of graphs. Moreover, the elements 0,1 in [8], are as vertices, while we removed theses elements in the vertex set of graphs and have not any roles in these graphs.

Theorem 4.11: *Let p be a prime. Then for any $\overline{x}, \overline{y} \in (\mathbb{Z}_p, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1})$, we have the following:*

(i) $\downarrow(\overline{0}) = \{\overline{0}\}$.

(ii) *For $\overline{x} \neq \overline{0}, \overline{1}$, we have $\downarrow(\overline{x}) = \{\overline{0}, \overline{x}, \overline{x+1}, \overline{x+2}, \ldots, \overline{p-1}\}$.*

(iii) *If $x \leq y$, then $\downarrow(\overline{x}) \cup \downarrow(\overline{y}) = \downarrow(\overline{x})$.*

Proof: (i) Let $\overline{x} \in \downarrow(\overline{0})$. Then $\overline{x} \rightarrow \overline{0} = \overline{1}$, which implies $\overline{x} = \overline{0}$, because for $0 < x < p$, we have $\overline{x} \rightarrow \overline{0} \neq \overline{1}$.

(ii) Let $\overline{x} \in \mathbb{Z}_p \backslash \{\overline{0}, \overline{1}\}$. By Proposition 3.1, $\overline{0} \in \downarrow(\overline{x})$. Since for any $1 \leq t \leq p - x$, we have $\overline{x+t} \leq_{\mathbb{Z}_p} \overline{x}$ (because in the linear order, larger numbers are smaller in the lattice order), we get that $\overline{x+t} \rightarrow \overline{x} = \overline{1}$. Thus $\downarrow(\overline{x}) = \{\overline{0}, \overline{x}, \overline{x+1}, \overline{x+2}, \ldots, \overline{p-1}\}$.

(iii) Let $\overline{x}, \overline{y} \in \mathbb{Z}_p \backslash \{\overline{0}, \overline{1}\}$. If $x \leq y$, then $\overline{y} \leq_{\mathbb{Z}_p} \overline{x}$ (since the lattice order is the reverse of the natural order), and by item (ii) and Theorem 3.2, we have $\downarrow(\overline{y}) \subseteq \downarrow(\overline{x})$. Hence $\downarrow(\overline{x}) \cup \downarrow(\overline{y}) = \downarrow(\overline{x})$.

Corollary 4.12: *Let p be a prime. Then*

(i) $(\downarrow(\mathbb{Z}_p), \subseteq)$ *is a chain as the Figure 8.*

(ii) $\downarrow(\mathbb{Z}_p)$ *has a nested chain* $\downarrow(\overline{0}) \subseteq \downarrow(\overline{p-1}) \subseteq \downarrow(\overline{p-2}) \subseteq \cdots \subseteq \downarrow(\overline{2}) \subseteq \downarrow(\overline{1})$.

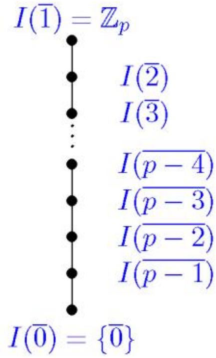

Figure 8
Chain $(\downarrow(\mathbb{Z}_p), \subseteq)$.

Example 4.13: Let $p = 7$ be a prime. Consider the BL-algebra $(\mathbb{Z}_7, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1})$ with the linear order, $\overline{0} < \overline{1} < \overline{2} < \overline{3} < \overline{4} < \overline{5} < \overline{6}$ and the corresponding lattice order, $\overline{6} <_{\mathbb{Z}_7} \overline{5} <_{\mathbb{Z}_7} \overline{4} <_{\mathbb{Z}_7} \overline{3} <_{\mathbb{Z}_7} \overline{2} <_{\mathbb{Z}_7} \overline{1} <_{\mathbb{Z}_7} \overline{0}$. One can see that, $\downarrow(\overline{0}) = \{\overline{x} \in \mathbb{Z}_7 | \overline{x} \rightarrow \overline{0} = \overline{1}\} = \{\overline{0}\}$. This is because for any $\overline{x} \neq \overline{0}$, we have $\overline{x} \rightarrow \overline{0} \neq \overline{1}$. Also

$$\downarrow(\overline{6}) = \{\overline{x} | \overline{x} \rightarrow \overline{6} = \overline{1}\} = \{\overline{0}, \overline{6}\}, \downarrow(\overline{5}) = \{\overline{x} | \overline{x} \rightarrow \overline{5} = \overline{1}\} = \{\overline{0}, \overline{5}, \overline{6}\},$$
$$\downarrow(\overline{4}) = \{\overline{x} | \overline{x} \rightarrow \overline{4} = \overline{1}\} = \{\overline{0}, \overline{4}, \overline{5}, \overline{6}\},$$
$$\downarrow(\overline{3}) = \{\overline{x} | \overline{x} \rightarrow \overline{3} = \overline{1}\} = \{\overline{0}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\},$$
$$\downarrow(\overline{2}) = \{\overline{x} | \overline{x} \rightarrow \overline{2} = \overline{1}\} = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}.$$

Hence, $\downarrow(\overline{6}) = \{\overline{0}, \overline{6}\}, \downarrow(\overline{5}) = \{\overline{0}, \overline{5}, \overline{6}\}, \downarrow(\overline{4}) = \{\overline{0}, \overline{4}, \overline{5}, \overline{6}\}, \downarrow(\overline{3}) = \{\overline{0}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}, \downarrow(\overline{2}) = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$. In addition,

$$\downarrow(\overline{4}) \cup \downarrow(\overline{5}) = \{\overline{0}, \overline{4}, \overline{5}, \overline{6}\} \cup \{\overline{0}, \overline{5}, \overline{6}\} = \{\overline{0}, \overline{4}, \overline{5}, \overline{6}\} = \downarrow(\overline{4}),$$
$$\downarrow(\overline{3}) \cup \downarrow(\overline{6}) = \{\overline{0}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\} \cup \{\overline{0}, \overline{6}\} = \{\overline{0}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\} = \downarrow(\overline{3}), \text{ and}$$
$$\downarrow(\overline{2}) \cup \downarrow(\overline{4}) = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\} \cup \{\overline{0}, \overline{4}, \overline{5}, \overline{6}\} = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\} = \downarrow(\overline{2}).$$

Theorem 4.14: *Let p be a prime. Then for any* $\overline{x}, \overline{y} \in (\mathbb{Z}_p, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1})$, $(\overline{x}, \overline{y}) \notin E(\Gamma(\mathbb{Z}_p)))$ *if and only if* $y \overset{p}{\equiv} x^2$.

Proof: Let $\overline{x}, \overline{y} \in (\mathbb{Z}_p, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1})$. Then $(\overline{x}, \overline{y}) \in E(\Gamma(\mathbb{Z}_p)))$ reason being $\downarrow(\overline{x}) \cup \downarrow(\overline{y}) \neq \downarrow(\overline{x} \rightarrow \overline{y})$. Applying the Corollary 4.12, $\downarrow(\overline{x} \rightarrow \overline{y}) \neq \downarrow(\overline{x})$ or $\downarrow(\overline{x} \rightarrow \overline{y}) \neq \downarrow(\overline{y})$. Without loss of generality, let $\downarrow(\overline{x} \rightarrow \overline{y}) \neq \downarrow(\overline{x})$. Using Theorem 3.3, $\downarrow(\overline{x} \rightarrow \overline{y}) = \downarrow(\overline{x})$ only if $\overline{x} \rightarrow \overline{y} = \overline{x}$ reason being $\overline{x^{-1}y} = \overline{x}$ if and only if $x^{-1}y \overset{p}{\equiv} x$ only if $y \overset{p}{\equiv} x^2$. Hence $\overline{x}, \overline{y} \in (\mathbb{Z}_p, \wedge, \vee, *, \rightarrow, \overline{0}, \overline{1})$, $(\overline{x}, \overline{y}) \notin E(\Gamma(\mathbb{Z}_p)))$ reason being $y \overset{p}{\equiv} x^2$.

Let $(X,\wedge,\vee,*,\rightarrow,0,1)$ be a BL-algebra and $x,y\in X$. Then for down graph $\Gamma(X)=(V=V(\Gamma(X))=X\backslash\{0,1\},E=E(\Gamma(X))$, will set $\mathcal{Z}(\Gamma(X))=\{(\overline{x},\overline{y})\in V^2(\Gamma(X))|(\overline{x},\overline{y})\not\in E(\Gamma(X)))\}$ as the set of all critical edges of the down graph $\Gamma(X))$ and $|\mathcal{E}(\Gamma(X))|=\zeta$.

Corollary 4.15: *Let p be a prime. Then*

(i) $\sum_{\overline{v}\in V(\Gamma(\mathbb{Z}_p))\backslash\mathcal{Z}(\Gamma(\mathbb{Z}_p)))} deg^+(\overline{v})=\sum_{\overline{v}\in V(\Gamma(\mathbb{Z}_p))\backslash\mathcal{Z}(\Gamma(\mathbb{Z}_p)))} deg^-(\overline{v})$.

(ii) $\sum_{\overline{v}\in\mathcal{Z}(\Gamma(\mathbb{Z}_p)))} deg^+(\overline{v})=\sum_{\overline{v}\in\mathcal{Z}(\Gamma(\mathbb{Z}_p))} deg^-(\overline{v})$.

Theorem 4.16: *Let p be an odd prime. For the BL-algebra $(\mathbb{Z}_p,\wedge,\vee,*,\rightarrow,\overline{0},\overline{1})$:*

(i) $|E(\Gamma(\mathbb{Z}_p))|=(p-2)(p-3)-|\mathcal{Z}(\Gamma(\mathbb{Z}_p))|$, *where $\mathcal{Z}(\Gamma(\mathbb{Z}_p))$ is the set of critical pairs $(\overline{x},\overline{y})$ for which $(\overline{x},\overline{y})\notin E(\Gamma(\mathbb{Z}_p))$.*

(ii) *The down digraph $\Gamma(\mathbb{Z}_p)$ is a tournament if and only if $p=2$. For $p\geq 3$, $\Gamma(\mathbb{Z}_p)$ is never a tournament.*

Proof: We know that the vertex set is $V=\mathbb{Z}_p\backslash\{\overline{0},\overline{1}\}$, so $|V|=p-2$.

(i) In a simple directed graph on n vertices, the maximum possible number of directed edges is $n(n-1)$. In $\Gamma(\mathbb{Z}_p)$, for any two distinct vertices $\overline{x}$ and $\overline{y}$, the edge $(\overline{x},\overline{y})$ is absent if and only if $\downarrow(\overline{x})\cup\downarrow(\overline{y})=\downarrow(\overline{x}\rightarrow\overline{y})$. By Theorem 4.14, this occurs if and only if $y\equiv x^2(\text{mod}p)$.

Let $\mathcal{Z}(\Gamma(\mathbb{Z}_p))=\{(\overline{x},\overline{y})\in V\times V|\overline{y}=\overline{x^2}\}$ be the set of these critical pairs where the edge is absent. The total number of possible ordered pairs of distinct vertices is $(p-2)(p-3)$. Each critical pair represents one missing directed edge. Therefore, $|E(\Gamma(\mathbb{Z}_p))|=(p-2)(p-3)-|\mathcal{Z}(\Gamma(\mathbb{Z}_p))|$.

(ii) A tournament requires that for every distinct pair of vertices $\{\overline{x},\overline{y}\}$, exactly one of the directed edges $(\overline{x},\overline{y})$ or $(\overline{y},\overline{x})$ exists, and no loops are present.

Let $p=2$. The vertex set $V=\mathbb{Z}_2\backslash\{\overline{0},\overline{1}\}$ is empty. A graph on zero vertices trivially satisfies all properties of a tournament.

Let $p\geq 3$. We show that $\Gamma(\mathbb{Z}_p)$ is not a tournament. Consider the vertex $\overline{-1}$ (the additive inverse of $\overline{1}$). Note that $\overline{-1}\neq\overline{1}$ for $p\geq 3$.

Check the condition for the pair $(\overline{-1},\overline{-1})$: For the edge to be absent, we would need $-1\equiv(-1)^2(\text{mod}p)$, which means $-1\equiv 1(\text{mod}p)$, or $p|2$. This is false for $p\geq 3$.

Now verify the edge condition: $\overline{x}\rightarrow\overline{y}=\overline{-1}\rightarrow\overline{-1}$. By the definition of implication in this BL-algebra, $x\rightarrow x=1$ for all x. Thus, $\downarrow(\overline{-1}\rightarrow\overline{-1})=\downarrow(\overline{1})=\mathbb{Z}_p$.

However, $\downarrow(\overline{-1})$, by Theorem 4.11(ii), is a proper subset of $\mathbb{Z}_p$ (specifically, $\downarrow(\overline{-1})=\{\overline{0},\overline{-1},\overline{-1+1},...\}$, which does not include $\overline{1}$ itself). Therefore, $\downarrow(\overline{-1})\cup\downarrow(\overline{-1})=\downarrow(\overline{-1})\subset\mathbb{Z}_p=\downarrow(\overline{-1}\rightarrow\overline{-1})$. This means $\downarrow(\overline{-1})\cup\downarrow(\overline{-1})\neq\downarrow(\overline{-1}\rightarrow\overline{-1})$, so $(\overline{-1},\overline{-1})$ is a directed edge (a loop). The presence of a loop immediately disqualifies $\Gamma(\mathbb{Z}_p)$ from being a tournament, as tournaments are loop-free simple digraphs. Therefore, for any prime $p \geq 3$, $\Gamma(\mathbb{Z}_p)$ contains a loop and is not a tournament.

Corollary 4.17: *For $p \geq 3$, the down digraph $\Gamma(\mathbb{Z}_p)$ is not strongly connected.*

Proof: Since $\Gamma(\mathbb{Z}_p)$ contains a loop and is not a tournament, it cannot be strongly connected. In particular, the vertex with a loop cannot reach other vertices in a strongly connected manner while maintaining the tournament property, which is violated by the presence of the loop.

In the subsequent example, we compute the set of critical edges in the derived down graph.

Example 4.18: View the field $(\mathbb{Z}_5,\oplus,e)$ and a lattice $(\mathbb{Z}_5,\wedge,\vee)$ in Figure 9. Then $(\mathbb{Z}_5,\wedge,\vee,*,\rightarrow,\overline{0},\overline{1})$ is a BL-algebra given by monoid in Table 9 and groupoid in Table 10.

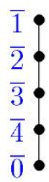

Figure 9
Lattice $(\mathbb{Z}_5,\leq)$

Table 9
Monoid $(\mathbb{Z}_5,*)$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{1}$	$\overline{3}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Table 10
Groupoid $(\mathbb{Z}_5,\rightarrow)$

$\rightarrow$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$\overline{2}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{4}$	$\overline{2}$
$\overline{3}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$

Computations show that

$\downarrow(\overline{0}) = \{\overline{0}\}, \downarrow(\overline{1}) = \mathbb{Z}_5, \downarrow(\overline{2}) = \{\overline{0},\overline{2},\overline{3},\overline{4}\}, \downarrow(\overline{3}) = \{\overline{0},\overline{3},\overline{4}\}$
and $\downarrow(\overline{4}) = \{\overline{0},\overline{4}\}$.

Based Theorem 4.14, $\mathcal{Z}(\Gamma(X)) = \{(\overline{x},\overline{y}) | y \overset{5}{\equiv} x^2\} = \{(\overline{x},4) | 4 \overset{5}{\equiv} x^2\} = \{(\overline{2},\overline{4}),(\overline{3},\overline{4})\}$.

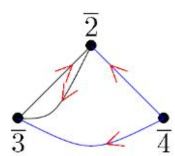

Figure 10
Down graph $\Gamma(\mathbb{Z}_5) = (V,E)$ for BL-algebra $(\mathbb{Z}_7,\wedge,\vee,*,\rightarrow,\mathbf{0},\mathbf{1})$.

Thus, one can prove that the directed graph $\Gamma(\mathbb{Z}_5) = (V,E)$ is a related down graph with BL-algebra $(\mathbb{Z}_7,\wedge,\vee,*,\rightarrow,0,1)$ as shown in Figure 10.

Theorem 4.19: *Let p be a prime. Then* $\mathcal{Z}(\Gamma(\mathbb{Z}_p)) = \{(\overline{a},\overline{b}^{2}) \in \mathbb{Z}_p^2\backslash\{\overline{0},\overline{1}\} | a^2 \overset{p}{\equiv} b^2 \text{and } a < b\}$.

Proof: Let $\overline{x},\overline{y} \in \mathbb{Z}_p$. Applying Theorem 4.14, $(\overline{x},\overline{y}) \in \mathcal{Z}(\Gamma(\mathbb{Z}_p))$ reason being $\overline{y} \neq \overline{x}, \overline{y} \leq_{\mathbb{Z}_p} \overline{x}$ and $y \overset{p}{\equiv} x^2$. Hence exists $\overline{a} \in \mathbb{Z}_p\backslash\{\overline{0},\overline{1}\}$ just like that $y = a^2$ and $a < b$, because $b \geq a$ indicates that $\overline{a} \to \overline{b} = \overline{1}$.

Example 4.20: View the BL-algebra $(\mathbb{Z}_{17},\wedge,\vee,*,\to,\overline{0},\overline{1})$ and $(\overline{x},\overline{y}) \in \mathcal{Z}(\Gamma(\mathbb{Z}_{17}))$. Then $\overline{y} \in \{\overline{4},\overline{9},\overline{16}\}$. If $\overline{y} = \overline{4}$, then $4 \overset{17}{\equiv} x^2$ indicates that $x = 2, x = 15$. If $\overline{y} = \overline{9}$, then $9 \overset{17}{\equiv} x^2$ indicates that $x = 3, x = 14$. If $\overline{y} = \overline{16}$, then $16 \overset{17}{\equiv} x^2$ indicates that $x = 4, x = 13$. It follows that $(\overline{15},\overline{4}),(\overline{14},\overline{9})$ are unacceptable, hence $\mathcal{Z}(\Gamma(\mathbb{Z}_{17})) = \{(\overline{2},\overline{4}),(\overline{3},\overline{9}),(\overline{4},\overline{16}),(\overline{13},\overline{16})\}$.

Theorem 4.21: *Let p be a prime. Then*

$$|\mathcal{Z}(\Gamma(\mathbb{Z}_p))| = \begin{cases} \lfloor\sqrt{p}\rfloor & if \left\lfloor\frac{-1+\sqrt{1+4p}}{2}\right\rfloor \leq \lfloor\sqrt{p}-1\rfloor, \\ \lfloor\sqrt{p}\rfloor - 1 & if \left\lfloor\frac{-1+\sqrt{1+4p}}{2}\right\rfloor > \lfloor\sqrt{p}-1\rfloor, \end{cases}.$$

Proof: Let p be a prime and $\mathcal{M} = \{\overline{a} \in \mathbb{Z}_p | 1 < a^2 < p-1\}$. It is apparent to prove that $|\mathcal{M}| = \lfloor\sqrt{p}\rfloor - 1$. Also for any $1 < a^2 < p-1$, the equation $a^2 \overset{p}{\equiv} x^2$ has two solutions $x = a$ and $x = p - a$, whence $a < a^2$. But $p - a < a^2$, if $a \geq \left\lfloor\frac{-1+\sqrt{1+4p}}{2}\right\rfloor$. It follows that for $\left\lfloor\frac{-1+\sqrt{1+4p}}{2}\right\rfloor < a \leq \sqrt{p}-1$, we get that $p - a < a^2$. It follows that for any prime p that $\left\lfloor\frac{-1+\sqrt{1+4p}}{2}\right\rfloor \leq \lfloor\sqrt{p}-1\rfloor$, we get that $|\mathcal{Z}(\Gamma(\mathbb{Z}_p))| = \lfloor\sqrt{p}\rfloor$ and for any prime p that $\left\lfloor\frac{-1+\sqrt{1+4p}}{2}\right\rfloor > \lfloor\sqrt{p}-1\rfloor$, we get that $|\mathcal{Z}(\Gamma(\mathbb{Z}_p))| = \lfloor\sqrt{p}\rfloor - 1$.

In the subsequent example, we compute the set of critical edges in the derived down graph and extract the related derived down digraph.

Example 4.22: View the field $(\mathbb{Z}_7,\oplus,e)$ and a lattice $(\mathbb{Z}_7,\wedge,\vee)$ in Figure 11. Then $(\mathbb{Z}_7,\wedge,\vee,*,\to,\overline{0},\overline{1})$ is a BL-algebra given by monoid in Table 11 and groupoid in Table 12.

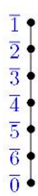

Figure 11
Lattice $(\mathbb{Z}_7,\leq)$

Table 11

Monoid $(\mathbb{Z}_7,*)$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{1}$	$\overline{3}$	$\overline{5}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{2}$	$\overline{5}$	$\overline{1}$	$\overline{4}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{6}$	$\overline{3}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Table 12

Groupoid $(\mathbb{Z}_7,\rightarrow)$

$\rightarrow$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\overline{2}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{6}$	$\overline{3}$
$\overline{3}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{4}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{3}$	$\overline{5}$
$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{4}$
$\overline{6}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$

Computations show that

$\downarrow(\overline{0})=\{\overline{0}\}, \downarrow(\overline{1})=\mathbb{Z}_7, \downarrow(\overline{2})=\{\overline{0},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6}\}, \downarrow(\overline{3})=\{\overline{0},\overline{3},\overline{4},\overline{5},\overline{6}\}$
$\downarrow(\overline{4})=\{\overline{0},\overline{4},\overline{5},\overline{6}\}, \downarrow(\overline{5})=\{\overline{0},\overline{5},\overline{6}\}$, and $\downarrow(\overline{6})=\{\overline{0},\overline{6}\}$.

Based Theorem 4.21,

$\mathcal{Z}(\Gamma(X))=\{(\overline{x},\overline{y}^2)|y\overset{7}{\equiv}x^2 \text{ and } x<y\}=\{(\overline{x},4)|4\overset{7}{\equiv}x^2\}=\{(\overline{2},\overline{4})\}$
($4\overset{7}{\equiv}x^2$ indicates that $x=2,5,$ since $5>4,$ we get $x\neq 5$).

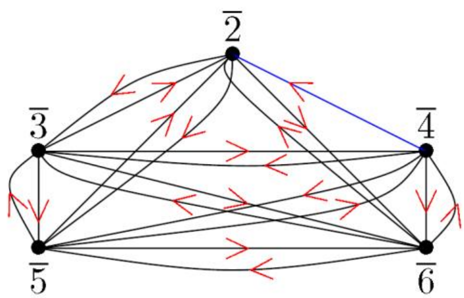

Figure 12
Down graph $\Gamma(\mathbb{Z}_7)=(V,E)$ for BL-algebra $(\mathbb{Z}_7,\wedge,\vee,*,\rightarrow,\mathbf{0},\mathbf{1})$.

Thus, one can prove that the directed graph $\Gamma(\mathbb{Z}_7)=(V,E)$ is a related down graph with BL-algebra $(\mathbb{Z}_7,\wedge,\vee,*,\rightarrow,0,1)$ as shown in Figure 12.

Theorem 4.23: *Let p be a prime. Then for any $(\overline{x},\overline{y})\in\mathcal{Z}(\Gamma(\mathbb{Z}_p))$,*
(i) $deg^+(\overline{y})=p-3$.
(ii) $deg^+(\overline{x})=p-4$.
(iii) $deg^-(\overline{y})=p-4$.
(iv) $deg^-(\overline{x})=p-3$.

Proof: Let p be a prime.

(i) Since $V(\Gamma(\mathbb{Z}_p)) = \mathbb{Z}_p \backslash \{\overline{0}, \overline{1}\}$, we get $|V(\Gamma(\mathbb{Z}_p))| = p-2$. Further $(\overline{x}, \overline{y}) \in \mathcal{Z}(\Gamma(\mathbb{Z}_p))$, indicates that $(\overline{x}, \overline{y}) \in E(\Gamma(\mathbb{Z}_p))$. Hence $deg^+(\overline{y}) = p - 2 - 1 = p - 3$.

(ii) In similar to item (i), $deg^+(\overline{x}) = p - 2 - 2 = p - 4$, because of $(\overline{x}, \overline{y}) \not\in E(\Gamma(\mathbb{Z}_p))$.

(iii, iv) Are similar to $(i), (ii)$.

Theorem 4.24: *Let p be a prime. If $|\mathcal{Z}(\Gamma(\mathbb{Z}_p))| = \lfloor\sqrt{p}\rfloor - 1$, then $\sum_{\overline{v} \in V(\Gamma(\mathbb{Z}_p))} deg^+(\overline{v}) = p^2 - 5p + 7 - \lfloor\sqrt{p}\rfloor$.*

Proof: Let $\mathcal{Z}(\Gamma(\mathbb{Z}_p)) = \{(\overline{x_1}, \overline{y_1}), (\overline{x_1}, \overline{y_1}), \dots, (\overline{x_{\lfloor\sqrt{p}\rfloor-1}}, \overline{y_{\lfloor\sqrt{p}\rfloor-1}})\}$. Using Theorem 4.23, for any $1 \leq i \leq \lfloor\sqrt{p}\rfloor - 1, deg^+(\overline{x_i}) = p - 4$ and $deg^+(\overline{y_i}) = p - 3$. Since $|V(\Gamma(\mathbb{Z}_p))| = p - 2$, we get that exists $(p - 2 - 2(\lfloor\sqrt{p}\rfloor - 1))$ other vertices of outdegree $p - 3$. Thus

$$\begin{aligned}\textstyle\sum_{\overline{v} \in V(\Gamma(\mathbb{Z}_p))} deg^+(\overline{v}) &= (\lfloor\sqrt{p}\rfloor - 1)(p-4) + (\lfloor\sqrt{p}\rfloor - 1)(p-3) \\ &\quad + (p - 2 - 2(\lfloor\sqrt{p}\rfloor - 1))(p-3) \\ &= p^2 - 5p + 7 - \lfloor\sqrt{p}\rfloor.\end{aligned}$$

Theorem 4.25: Let p be a prime. If $\left|\mathcal{Z}\left(\Gamma\left(\mathbb{Z}_p\right)\right)\right| = \lfloor\sqrt{p}\rfloor$, then $\sum_{\overline{v} \in V(\Gamma(\mathbb{Z}_p))} deg^+(\overline{v}) = p^2 - 5p + 6 - \lfloor\sqrt{p}\rfloor$.

Proof: Let $\mathcal{Z}(\Gamma(\mathbb{Z}_p)) = \{(\overline{x_1}, \overline{y_1}), (\overline{x_1}, \overline{y_1}), \dots, (\overline{x_{\lfloor\sqrt{p}\rfloor-1}}, \overline{y_{\lfloor\sqrt{p}\rfloor-1}}), (\overline{x_{\lfloor\sqrt{p}\rfloor}}, \overline{y_{\lfloor\sqrt{p}\rfloor-1}})\}$. Using Theorem 4.23, for any $1 \leq i \leq \lfloor\sqrt{p}\rfloor, deg^+(\overline{x_i}) = p - 4$ and for any $1 \leq i \leq \lfloor\sqrt{p}\rfloor - 1, deg^+(\overline{x_i}) = p - 3$. Since $|V(\Gamma(\mathbb{Z}_p))| = p - 2$, we get that exists $(p - 2 - (2\lfloor\sqrt{p}\rfloor - 1))$ other vertices of outdegree $p - 3$. Thus

$$\begin{aligned}\textstyle\sum_{\overline{v} \in V(\Gamma(\mathbb{Z}_p))} deg^+(\overline{v}) &= (\lfloor\sqrt{p}\rfloor)(p-4) + (\lfloor\sqrt{p}\rfloor - 1)(p-3) \\ &\quad + (p - 2 - (2\lfloor\sqrt{p}\rfloor - 1))(p-3) \\ &= p^2 - 5p + 6 - \lfloor\sqrt{p}\rfloor.\end{aligned}$$

Example 4.26: Let $p = 17$. The critical edge set $\mathcal{Z}(\Gamma(\mathbb{Z}_{17}))$ consists of pairs $(\overline{x}, \overline{y})$ where $y \equiv x^2 (\text{mod } 17)$ and both $\overline{x}, \overline{y} \in V = \{\overline{2}, \overline{3}, \dots, \overline{16}\}$. Computations, show that $\mathcal{Z}(\Gamma(\mathbb{Z}_{17})) = \{(\overline{2}, \overline{4}), (\overline{3}, \overline{9}), (\overline{4}, \overline{16}), (\overline{5}, \overline{8})\}$. Thus, $|\mathcal{Z}(\Gamma(\mathbb{Z}_{17}))| = 4 = \lfloor\sqrt{17}\rfloor$. The 4 critical pairs are, $(\overline{2}, \overline{4}), (\overline{3}, \overline{9}), (\overline{4}, \overline{16}), (\overline{5}, \overline{8})$. The affected vertices and their outdegrees, $\overline{2}, \overline{3}, \overline{4}, \overline{5}$, each appears once in the first coordinate, so have outdegree = 13. For $\overline{4}, \overline{9}, \overline{16}, \overline{8}$, each appears once in the second coordinate, so have outdegree = 14, and remaining 7 vertices, have outdegree $= 14$. Hence, $4 \times 13 + 4 \times 14 + 7 \times 14 = 52 + 56 + 98 = 206 = p^2 - 5p + 6 - \lfloor\sqrt{p}\rfloor$.

Corollary 4.24: *Let p be a prime. Then*

(i) $\Gamma(\mathbb{Z}_p)$ *is not a **Tournament**.*

(ii) $\Gamma(\mathbb{Z}_p)$ *is not a strongly connected digraph.*

4.2 *Tournaments and down graphs derived from BL-algebras*

In this subsection, we try to convert any Galois finite field to a BL-algebra and extract their down sets and so construct the related derived digraph. Indeed, we present the some tournaments which are obtained based the Galois finite fields. For each prime number p and positive integer n, is a unique finite field of order p^n up to isomorphism, thus every finite field has order p^n and is denoted by $(GF(p^n), +, .)$ in honor of the founder of finite field theory, Evariste Galois. Let p be a prime, $n \geq 2$ and $0,1 \in GF(p^n)(\forall x \in GF(p^n), x + 0 = 0 + x = x, 1.x = x.1 = x, x \cdot 0 = 0, x \cdot x^{-1} = 1)$. Define the commutative binary operations $\wedge,\vee$ on $GF(p^n)$ just like that for any $x, y \in GF(p^n), x \wedge x^{-1} = 0, x \vee x^{-1} = 1, x \wedge 0 = 0, x \vee 0 = x, x \wedge 1 = x$ and $x \vee 1 = x$. In the subsequent theorem, we view any Galois finite field, and exhibit new binary operations based on the underlying field binary operations, so we use these binary operations to construct of any BL-algebra. Further, the underlying lattices is indented to the inverse of elements in Galois field and so for any our aim we construct special a bounded lattice.

Consider the BL-algebra $X = (\{a, b, c, d, e\}, \wedge, \vee, *, \rightarrow, a, e)$ in Example 4.3, one can see that X can not be a field, while have the following result.

Theorem 4.28: *Every $GF(p^n)$ can be a BL-algebra.*

Proof: Let p be a prime and $n \geq 2$. View the bounded lattice $(GF(p^n), \wedge, \vee, 0, 1)$ and define the binary operations " $*$ " and " $\rightarrow$ " on $(GF(p^n)$ by $x * y = x \wedge y$, $0 \rightarrow x = 1, 1 \rightarrow x = x$ and for any $x \neq 0,1$, $x \rightarrow y = x^{-1} \vee y$. Clearly $(GF(p^n), *, 1)$ is a commutative monoid.

$(BL3)$ Let $x, y, z \in X$. Then $x * y \leq z$ only if $x \wedge y \leq z$. If $x \wedge y \leq z$, then $(x \wedge y) \vee y^{-1} \leq y^{-1} \vee z$, and so $x \leq y^{-1} \vee z = y \rightarrow z$. If $x \leq y \rightarrow z$, then $x \leq y^{-1} \vee z$ and so $x * y = x \wedge y \leq (y^{-1} \vee z) \wedge y \leq y \wedge z \leq z$.

$(BL4)$ Let $x, y \in X$. Then $x \wedge y = x \wedge (x^{-1} \vee y) = x * (x \rightarrow y)$.

$(BL5)$ Let $x, y \in X$. Then $(x \rightarrow y) \vee (x \rightarrow y) = (x^{-1} \vee y) \vee (y^{-1} \vee x) = 1$.

Therefore, $(GF(p^n), \wedge, \vee, 0, 1, *, \rightarrow)$ is a BL-algebra.

Example 4.29: Let $X = \{0,1,2,3,4,5,6,7\}$. Then $GF(8) = (X, +, \cdot)$ is a field as Tables 13 and 14. Now, view the lattice $(L, \wedge, \vee)$ in Figure 13. Then $(X, \wedge, \vee, *, \rightarrow, 0,1)$ is a BL-algebra given by Tables 15 and 16.

Table 13

Abelean group $(X, +)$

$+$	0	2	3	4	5	6	7	1
0	0	2	3	4	5	6	7	1
2	2	0	1	6	7	4	5	3
3	3	1	0	7	6	5	4	2
4	4	6	7	0	1	2	3	5
5	5	7	6	1	0	3	2	4
6	6	4	5	2	3	0	1	7
7	7	5	4	3	2	1	0	6
1	1	3	2	5	4	7	6	0

Table 14

Groupoid $(X, \cdot)$

$\cdot$	0	2	3	4	5	6	7	1
0	0	0	0	0	0	0	0	0
2	0	4	6	3	1	7	5	2
3	0	6	5	7	4	1	2	3
4	0	3	7	6	2	5	1	4
5	0	1	4	2	7	3	6	5
6	0	7	1	5	3	2	4	6
7	0	5	2	1	6	4	3	7
1	0	2	3	4	5	6	7	1

Table 15

Monoid $(X, *)$

$*$	0	2	3	4	5	6	7	1
0	0	0	0	0	0	0	0	0
2	0	2	0	0	0	2	2	2
3	0	0	3	0	3	0	3	3
4	0	0	0	4	4	4	0	4
5	0	0	3	4	5	4	3	5
6	0	2	0	4	4	6	2	6
7	0	2	3	0	3	2	7	7
1	0	2	3	4	5	6	7	1

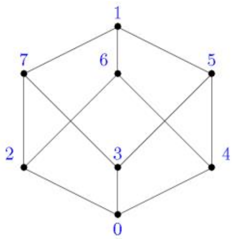

Figure 13

Lattice $(X, \leq)$

Table 16

Groupoid $(X, \to)$

$\to$	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	5	1	5	5	5	1	1	1
3	6	6	1	6	1	6	1	1
4	7	7	7	1	1	1	7	1
5	2	2	7	6	1	6	7	1
6	3	7	3	5	5	1	7	1
7	4	6	5	4	5	6	1	1
1	0	2	3	4	5	6	7	1

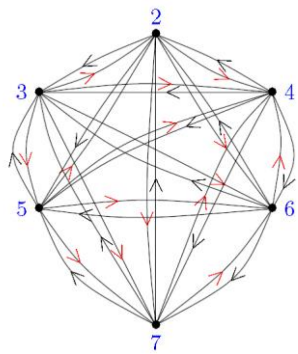

Figure 14

Down graph $\Gamma(GF(8))$ for BL-algebra $(GF(8), \wedge, \vee, *, \to, 0, 1)$

Computations show that

$$\downarrow(0) = \{0\}, \downarrow(2) = \{0,2\}, \downarrow(3) = \{0,3\},$$
$$\downarrow(4) = \{0,4\}, \downarrow(5) = \{0,3,4,5\}, \downarrow(6) = \{0,2,4,6\},$$
$$\downarrow(7) = \{0,2,3,7\} \text{ and } \downarrow(1) = X.$$

Thus, one can prove that the directed graph $\Gamma(GF(8)) = (V, E)$ is a related down graph with BL-algebra $(GF(8), \wedge, \vee, *, \to, 0, 1)$ as shown in Figure 14.

Theorem 4.30: *Let p be a prime* $n \geq 2$ *and* $x, y \in V(\Gamma(GF(p^n)))$. *Then*

(i) $x \to 0 = x^{-1}$.

(ii) $x \to x^{-1} = x^{-1}$.

(iii) $x^{-1} \to x = x$.

(iv) $x \to y = y^{-1} \to x^{-1}$.

(v) $(x \to y)^{-1} = x * y^{-1}$.

(vi) $(x * y)^{-1} = x \to y^{-1}$.

Proof: (i) Let $x \in V(\Gamma(GF(p^n)))$. Then $x \to 0 = x^{-1} \vee 0 = x^{-1}$.

(ii) Let $x \in V(\Gamma(GF(p^n)))$. Then $x \to x^{-1} = x^{-1} \vee x^{-1} = x^{-1}$.

(iii) Let $x \in V(\Gamma(GF(p^n)))$. Then $x^{-1} \to x = (x^{-1})^{-1} \vee x = x$.

(iv) Let $x, y \in V(\Gamma(GF(p^n)))$. Then $x \to y = x^{-1} \vee y = y \vee x^{-1} = (y^{-1})^{-1} \vee x^{-1} = y^{-1} \to x^{-1}$.

(v) Let $x, y \in V(\Gamma(GF(p^n)))$. Then $(x \to y)^{-1} = (x^{-1} \vee y)^{-1} = x \wedge y^{-1} = x * y^{-1}$.

(vi) Let $x, y \in V(\Gamma(GF(p^n)))$. Then $(x * y)^{-1} = (x \wedge y)^{-1} = x^{-1} \wedge y^{-1} = x \to y^{-1}$.

Theorem 4.31: *Let p be a prime* $n \geq 2$ *and* $x, y \in V(\Gamma(GF(p^n)))$. *Then*

(i) $\{0, x\} \subseteq \downarrow (x)$.

(ii) $\downarrow (x) \cap \downarrow (x^{-1}) = \{0\}$.

(iii) $x \in \downarrow (y)$ *if and only if* $y^{-1} \in \downarrow (x^{-1})$.

Proof: $x, y \in V(\Gamma(GF(p^n)))$.

(i) Since $0 \to x = 1$ and $x \to x = 1$, we get that $\{0, x\} \subseteq \downarrow (x)$.

(ii) Let $y \in \downarrow (x) \cap \downarrow (x^{-1})$. Then $y \to x = y \to x^{-1} = 1$ and so $y^{-1} \vee x = y^{-1} \vee x^{-1}$. It follows that $x \wedge (y^{-1} \vee x) = x \wedge (y^{-1} \vee x^{-1})$ and so $x \wedge y^{-1} = 1$. Hence $y = 0$, because of $x \neq 1$. Thus $\downarrow (x) \cap \downarrow (x^{-1}) = \{0\}$.

(iii) $x \in \downarrow (y)$ only if $x \to y = 1$ if and only if $x^{-1} \vee y = 1$ only if $(y^{-1})^{-1} \vee x^{-1} = 1$ only if $y^{-1} \to x^{-1} = 1$ only if $y^{-1} \in \downarrow (x^{-1})$.

Theorem 4.32: *Let p be a prime* $n \geq 2$ *and* $x, y \in V(\Gamma(GF(p^n)))$. *Then*

(i) $\downarrow (x) \cup \downarrow (y) \neq \downarrow (x \to y)$.

(ii) $\Gamma(GF(p^n))$ *is a Tournament.*

Proof: (i) Let $\downarrow (x) \cup \downarrow (y) = \downarrow (x \to y)$. By Theorem 3.2, we have, $\downarrow (y) \subseteq \downarrow (x \to y)$. It follows that $\downarrow (x) \subseteq \downarrow (x \to y)$. Thus for any $z \in \downarrow (x), z^{-1} \vee x = 1$ and $z^{-1} \vee x^{-1} \vee y = 1$. Hence $1 = (z^{-1} \vee x) \wedge (z^{-1} \vee x^{-1} \vee y) = z^{-1} \vee (x \wedge x^{-1} \wedge y) = z^{-1}$, which it is a contradiction, because of $z \neq 0$.

(ii) Let $x, y \in V(\Gamma(GF(p^n)))$. By item (i), $\downarrow (x) \cup \downarrow (y) \neq \downarrow (x \to y)$. Hence for any $x, y \in V(\Gamma(GF(p^n)))$, we get that $(x, y) \in E(\Gamma(GF(p^n))))$ and $(y, x) \in E(\Gamma(GF(p^n))))$. Then $\Gamma(GF(p^n))$ is a Tournament.

Corollary 4.33: *Let p be a prime and* $n \geq 2$. *Then* $\sum_{v \in V(\Gamma(GF(p^n))} deg^+(v) = (p^n - 2)(p^n - 3)$.

The subsequent example shows that the converse of Theorem 4.32, is not necessarily true.

Example 4.34: View the BL-algebra $(X, \wedge, \vee, *, \to, 0, 1)$ based on lattice 16, Tables 17 and 18. Computations show that the annihilator graph of X is shown in Figure 15.

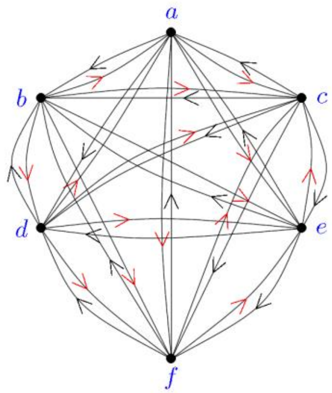

Figure 15
Tournament $\Gamma(X) = (V, E)$

Let $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ be a BL-algebra and $x \in X$. Then x known as an idempotent element, if $x * x = x$ and $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ known as an idempotent BL-algebra, if any it's element is an idempotent.

Example 4.35: *Let $X = \{0, f, e, d, c, b, a, 1\}$ and view the lattice $(L, \wedge, \vee)$ in Figure 16. Then $(X, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra given by Tables 17 and 18.*

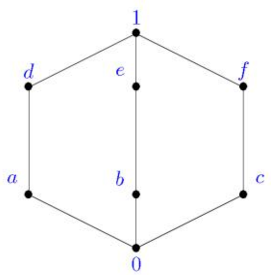

Figure 16
Lattice $(L, \leq_L)$

Table 17
Monoid $(X, *)$

$*$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	0	0	a
b	0	0	b	0	0	b	0	b
c	0	0	0	c	0	0	c	c
d	0	a	0	0	d	0	0	d
e	0	0	b	0	0	e	0	e
f	0	0	0	c	0	0	f	f
1	0	a	b	c	d	e	f	1

Table 18
Groupoid $(X, \rightarrow)$

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	0	1	b	c	1	e	f	1
b	0	a	1	c	d	1	f	1
c	0	a	b	1	d	e	1	1
d	0	a	b	c	1	e	f	1
e	0	a	b	c	d	1	f	1
f	0	a	b	c	d	e	1	1
1	0	a	b	c	d	e	f	1

Computations show that

$\downarrow(0) = \{0\}$,
$\downarrow(a) = \{0, a\}, \downarrow(b) = \{0, b\}, \downarrow(c) = \{0, c\}$,
$\downarrow(d) = \{0, a, d\}, \downarrow(e) = \{0, b, e\}$,
$\downarrow(f) = \{0, c, f\}$ and $\downarrow(1) = X$.

5. Conclusion

This paper decides to do a mathematical modeling based on logical algebraic systems like BL-algebras. For this purpose, we first investigate the relationship between graphs as a network and BL-algebras as a logical algebra, and then we try to make BL-algebras from the graphs and make a graph from the BL-algebras, conversely. The importance and motivation of this study are based on the logical principles of algebra as the basis of the network structure as a logical system. Each set can be equipped with the principles of the subject and turn it into a logical algebra, and then turn this equipped set into a graph. This problem can be reversed to extract a logical algebra system from a directed or undirected network. The ongoing paper has presented the idea of a down graph related to any given arbitrary BL-algebra. In the structure of down graphs with respect to BL-algebras, the variety of down sets of elements of BL-algebras plays a main role. Indeed, the down set of any given element in BL-algebras are the heart of the down graph, just like the two elements are adjacent if they are located in the same down set. The computations on down graphs are done on any arbitrary BL-algebras, and it is proved that down graphs are directed and so are useful for directed networks. This work has investigated the relation between graphs and BL-algebras, especially presenting the BL-algebras from connected graphs.

Future work: We hope to obtain more results regarding graphs and annihilator sets of BL-algebras, the spectrum of graphs based on BL-algebras, fuzzy graphs based on BL-algebras, the entropy of graphs based on the BL-algebras, directed graphs and hypergraphs based on BL-algebras in the next research.

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