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Computation of certain topological indices of star hexagon network

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Abstract

Graph networks were formulated, motivated from the structure of molecules, social connections, flow of information, transport systems, and several other physical phenomena's. New networks are proposed by researchers and its properties are studied for potential application in emerging fields. Inspired by the tiling of the Euclidean plane with hexagons and stare-shaped tiles, we have presented a new network in this research, namely, the Star Hexagon Network. We compute some of the well-known degree based topological indices namely Generalized Zagreb Index, Atom Bond Connectivity Index, Geometric Arithmetic index and other derived indices also for the star hexagon network. The centrality measures of the super cell of the network were also studied.

Subject Classification: *05C09, 05C92.*

Keywords: *Graph network, Topological indices, Centrality measures.*

1. Introduction

Graph networks are structures used to symbolize relationships between entities. They're broadly utilized in numerous domains which includes computer science, social network evaluation, biology, transportation structures, and more. Graph networks are a versatile tool for modelling and analyzing complicated systems wherein relationships among entities are critical to understand the domain of study. They provide an effective framework for representing, visualizing, and analyzing interconnected records.

The application of graph theory to the study of molecule structures is commonly referred to as chemical graph theory, a subfield of many other scientific disciplines. In order to establish quantitative structure–activity relationships, we are often interested in using graph theory techniques to estimate the structural properties [1]. Degree-based indicators are very significant and vital in the field of chemistry. Wiener [2] is credited with developing the concept of chemical indexes. Chen, Shin, and Kandlur [3], were the first to present the hexagonal mesh. Various kinds of Zagreb and Balaban chemical indices were computed for hex-derived networks by Ali, Sajjad, and Sajjad [4]. With the aid of topological indices, they analyze their chemical behavior.

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In [5] authors introduce a new interconnection network, namely, hexagon star network that contains a composition of triangles around a hexagon and found a correlation of their network to the computer central network system. The terms "star hexagon network" and "hexagon star network" may sound similar, but they refer to different network topologies based on their structural arrangements. In a star hexagon network, the vertices of a star like graph are connected to multiple hexagonal sub networks, whereas in hexagon-star network, the vertices of a hexagon graph are connected to multiple triangle sub networks, forming a star-like shape. Rajan, Grigorious, and Stephen [6] computed the topological indices of the silicate, honey comb, and hexagonal networks. The authors also examined the material's mechanical, chemical, and physical characteristics, which showed good correlation with the crystal structure of gold. Motivated by the chemical structure of a regular hexagonal lattice, the scientists studied networks of carbon nanotubes and identified several indices for this important class of networks. The topological indices of the carbon nanotube networks and fullerenes have been established by Bača et al. [7]. Hasani and Ghods [8] computed the fuzzy topological indices of linear and multi cyclic naphthalene hydrocarbon. Jalali and Ghods [9] derived the topological indices of thorny graphs. Raza and Guirao [10] develop the M-polynomial for metal trihalides and uses it to efficiently derive various degree-based topological indices in a unified analytical framework. Tousi and Ghods [11] determine the topological indices of TiO_2 [m, n] nanotubes by modeling their structure as graphs and deriving exact mathematical expressions for these indices. Falahati-Nezhad and Azari [12] analyze porphyrin-based dendrimers by computing discrete Adriatic indices to characterize their structural and topological properties. Gao and Farahani [14] evaluate Zagreb topological indices for jagged-rectangle benzenoid systems to understand their structural properties.

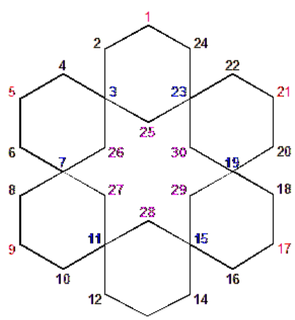

Figure 1
Super cell

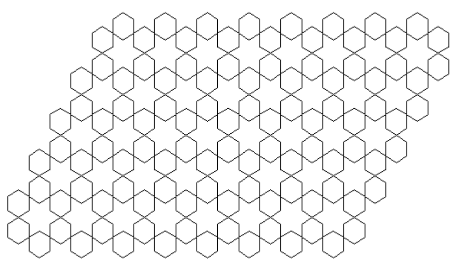

Figure 2
Star Hexagon Network

In the literature, motivated from the tiling of Euclidean plane using regular polygons, hexagonal networks were introduced using triangle polygon and their topological indices were studied [13]. Also honeycomb networks were introduced using hexagon polygon and their topological indices were studied. A regular star polygon is a decagon with ten equal sides and five angles. It is known that, the regular polygons, hexagon and the star polygon can tile the plane in a uniform manner. In this paper, we present a novel graph network namely 'Star Hexagon Network' whose super cell consists of a regular star polygon in the center and six hexagons placed on the boundary of the star polygon Figure 1 and study certain topological indices of the network. Since the derivations of topological indices rely purely on graph theory, they can be applied to any interconnected networks and not limited to networks obtained from molecules [15]. The arrangement of the super cell in rows and columns infinitely gives the star hexagon network Figure 2.

2. Preliminaries

Let G = (V, E) be a basic graph in which E = E(G) is a collection of unordered pairs of unique elements of V(G) called edges or lines, and V = V(G) is a non-empty set of objects called vertices or points. The sets are referred to as the vertex set (V (G)) and the edge set (E(G)). The number of edges that coincide with a vertex v∈V(G) is its degree, represented by d_v. If two vertices (u and v) of G have an edge (e = uv) connecting them, then they are adjacent [13].

In the following, we discuss the centrality measures of the super cell of the star hexagon network. A graph's closeness to every other vertex is measured by its closeness centrality. A vertex's proximity centrality is the inverse of the average geodesic distance between it and every other vertex in a graph. A vertex with a higher proximity centrality rating would be in a better position to disseminate information to other vertices.

Closeness Centrality [16], [17] $CC_i(SHN) = \frac{1}{\sum_y d(x,y)}$, where $\sum_y d(x,y)$ is the total of the distances between x and y.

$$CC_i = \begin{cases} \frac{1}{134}, i = 1,5,9,13,17,21 \\ \frac{1}{132}, i = 2,4,6,\dots,24 \\ \frac{1}{108}, i = 3,7,11,15,19,23 \\ \frac{1}{110}, i = 25,26,27,29,30 \end{cases}$$

The total of the distances between each vertex and every other vertex in a graph is known as farness centrality. It is the reciprocal of closeness centrality, meaning that closeness is low if farness is high.

$$f_i = \begin{cases} 134, i = 1,5,9,13,17,21 \\ 132, i = 2,4,6,\dots,24 \\ 108, i = 3,7,11,15,19,23 \\ 110, i = 25,26,27,29,30 \end{cases}$$

It can be observed that the vertices $i = 1,5,9,13,17,21$ are having high farness and less closeness, whereas the vertices $i = 3,7,11,15,19,23$ are having low farness and high closeness measures.

In the following, we derive the topological indices of the network. The graph network's edge set can be separated into two partitions, E_4 and E_6, and their partitions are listed in Table 1. Every edge e = uv belonging to E_4, satisfy the condition, $d_u = d_v = 2$, and every edge e = uv belonging to E_6, satisfy the condition $d_u = 2$, $d_v = 4$ or $d_u = 4$, $d_v = 2$, where d_u and d_v stand for u and v's respective degrees.

Table 1

Edge Partition of Star Hexagon Network

Edge partitions	Edges	Frequency
E_4	(2, 2)	8r + 4c
E_6	(2, 4)	4r + 24rc -4c

where c represents the number of columns and r the number of rows. In the following we define certain topological indices, which are commonly studied for graph networks. The Generalized Zagreb index is defined as

$$ZG_{\alpha,\beta}(SHN) = \sum_{e=uv\in E(G)} \frac{(d_u d_v)^{\alpha}}{(d_u+d_v)^{\beta}},$$

where α and β are real numbers [18].

The Generalized Zagreb index, which is the first Zagreb index when α = 0 and β = -1, is described as

$$ZG_1(SHN) = \sum_{e=uv\in E(G)} (d_u + d_v)$$

When α = 1 and β = 0, the Generalized Zagreb index reduces to **Second Zagreb Index** [19] and it is defined as

$$ZG_2(SHN) = \sum_{e=ab\in E(G)} (d_a d_b)$$

We obtain the **Randic Index** [20] from the Generalized Zagreb index when α = -1/2 and β = 0. It is described as

$$R(SHN) = \sum_{e=ab\in E(G)} \frac{1}{\sqrt{d_a d_b}}$$

The **Sum Connectivity Index** [21], which is part of the Generalized Zagreb index when α = 0 and β = ½, is defined as

$$SCI(SHN) = \sum_{e=ab\in E(G)} \frac{1}{\sqrt{d_a+d_b}}$$

The Generalized Zagreb index's **Second Refined Zagreb Index** [22] is generated when α = 1 and β = 1. It is described as

$$ReZG_2(SHN) = \sum_{e=ab\in E(G)} \left(\frac{d_a d_b}{d_a+d_b}\right)$$

We have the **Hyper Zagreb Index** [23] in the Generalized Zagreb index when α = 0 and β = -2. It is defined as

$$HM(SHN) = \sum_{e=ab\in E(G)} (d_a + d_b)^2$$

Harmonic index [24] is described as

$$H(SHN) = \sum_{e=ab\in E(G)} \frac{2}{d_a+d_b}$$

Atom bond connectivity index [25] is described as

$$\mathrm{ABC(SHN)} = \sum_{\mathrm{e=ab \in E(G)}} \sqrt{\frac{\mathrm{d_a + d_b - 2}}{\mathrm{d_a d_b}}}$$

Vukičević and Furtula [26] proposed **the Geometric Arithmetic Index** GA(SHN) index, another well-known connectivity topological descriptor, which is represented by

$$\mathrm{GA(SHN)} = \sum_{\mathrm{e=ab \in E(G)}} \frac{2\sqrt{\mathrm{d_a d_b}}}{\mathrm{d_a + d_b}}$$

3. Topological Indices of Star Hexagon Network

Theorem 1: *Consider the star hexagon network. The generalized Zagreb indices and indices derived from it are as follows;*

$$\mathrm{ZG_1(SHN)} = 56\,r + 144rc - 8c$$

$$\mathrm{ZG_2(SHN)} = 64r + 192rc - 16c$$

$$\mathrm{R(SHN)} = \frac{1}{\sqrt{2}}\left\{ 2r\left(1 + 2\sqrt{2}\right) + 12rc + 2c\left(\sqrt{2} - 1\right)\right\}$$

$$\mathrm{SCI(SHN)} = \frac{1}{\sqrt{6}}\left[4r\left(1 + \sqrt{6}\right) + 24rc + 2c\left(\sqrt{6} - 2\right)\right]$$

$$\mathrm{ReZG_2(SHN)} = \frac{1}{3}(40r + 96rc - 4c)$$

$$\mathrm{HM(SHN)} = 272\mathrm{r} + 864\mathrm{rc} - 80\mathrm{c}$$

Proof: The Generalized Zagreb index is defined as

$$\mathrm{ZG_{\alpha,\beta}(SHN)} = \sum_{\mathrm{e=uv \in E(G)}} \frac{(\mathrm{d_u d_v})^{\alpha}}{(\mathrm{d_u + d_v})^{\beta}}, \tag{1}$$

Where E(G) is the star hexagon network's edge partition and α and β are real numbers. Table 1 lists the network's edge partitions along with the number of edges in each partition. In the following, we obtain the various topological indices of the network from the generalized Zagreb index.

Case 1: To generate the first Zagreb Index, substitute α = 0 and β = -1 in (1)

$$ZG_1\mathrm{(SHN)} = \sum_{\mathrm{e=ab \in E(G)}}(\mathrm{d_a + d_b}) = \sum_{\mathrm{e=ab} \in E_4}(\mathrm{d_a + d_b}) + \sum_{\mathrm{e=uv} \in E_6}(\mathrm{d_a + d_b})$$

$$= 4(8r + 4c) + 6(4\mathrm{r} + 24\mathrm{rc} - 4\mathrm{c}) = 56\,r + 144rc - 8c$$

Case 2: To obtain the second Zagreb Index, substitute α = 1 and β = 0 in (1)

$$ZG_2\mathrm{(SHN)} = \sum(\mathrm{d_u d_v}) = \sum_{\mathrm{e=uv} \in E_4}(\mathrm{d_u d_v}) + \sum_{\mathrm{e=uv} \in E_6}(\mathrm{d_u d_v})$$

$$= 4(8r + 4c) + 8(4\mathrm{r} + 24\mathrm{rc} - 4\mathrm{c}) = 64r + 192rc - 16c$$

Case 3: Randic Index is obtained by substituting α = -1/2 and β = 0 in (1)

$$\mathrm{R(SHN)} = \sum_{e=uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}$$

$$= \sum_{e=uv \in E4} \frac{1}{\sqrt{d_u d_v}} + \sum_{e=uv \in E6} \frac{1}{\sqrt{d_u d_v}}$$

$$= \frac{1}{2}(8r + 4c) + \frac{1}{2\sqrt{2}}(4r + 24rc - 4c)$$
$$= \frac{1}{\sqrt{2}}\{2r(1 + 2\sqrt{2}) + 12rc + 2c(\sqrt{2} - 1)\}$$

Case 4: To obtain the Sum Connectivity Index, substitute α = 0 and β = ½ in (1)

$$SCI(SHN) = \sum_{e=uv\in E(G)} \frac{1}{\sqrt{d_u + d_v}} = \sum_{e=uv\in E4} \frac{1}{\sqrt{d_u + d_v}} + \sum_{e=uv\in E6} \frac{1}{\sqrt{d_u + d_v}}$$
$$= \frac{1}{\sqrt{4}}(8r + 4c) + \frac{1}{\sqrt{6}}(4r + 24rc - 4c)$$
$$= \frac{1}{\sqrt{6}}\left[4r(1 + \sqrt{6}) + 24rc + 2c(\sqrt{6} - 2)\right]$$

Case 5: The Second Refined Zagreb Index is obtained by substituting α = 1 and β = 1 in (1)

$$ReZG_2(SHN) = \sum_{e=uv\in E(G)} \left(\frac{d_u d_v}{d_u + d_v}\right) = \sum_{e=uv\in E4} \left(\frac{d_u d_v}{d_u + d_v}\right) + \sum_{e=uv\in E6} \left(\frac{d_u d_v}{d_u + d_v}\right)$$
$$= \frac{4}{4}(8r + 4c) + \frac{8}{6}(4r + 24rc - 4c) = \frac{1}{3}(40r + 96rc - 4c)$$

Case 6: The Hyper Zagreb Index is obtained by substituting α = 0 and β =-2 in (1)

$$HM(SHN) = \sum_{e=ab\in E(G)} (d_a + d_b)^2 = \sum_{e=ab\in E4} (d_a + d_b)^2 + \sum_{e=ab\in E6} (d_a + d_b)^2$$
$$= 16(8r + 4c) + 36(4 + 24rc - 4c) = 272r + 864rc - 80c$$

Theorem 2: *The star hexagon network's harmonic index is H(SHN) =* $\frac{1}{3}(16r + 24rc + 2c)$*, where c indicates the number of columns and r indicates the number of rows.*

Proof: Using the edge partition of star hexagon network, the Harmonic index is obtained below.

$$H(SHN) = \sum_{e=ab\in E(G)} \frac{2}{d_a + d_b} = \sum_{e=ab\in E4} \frac{2}{d_a + d_b} + \sum_{e=ab\in E6} \frac{2}{d_a + d_b}$$
$$= \frac{2}{2+2}(8r + 4c) + \frac{2}{2+4}(4r + 24rc - 4c) = \frac{1}{3}(16r + 24rc + 2c)$$

Theorem 3: *The star hexagon network's atom bond connectivity index is ABC(SHN) =* $\sqrt{\frac{1}{2}}\,(12r + 24rc)$.

Proof:

$$ABC(SHN) = \sum_{e=ab\in E(G)} \sqrt{\frac{d_a + d_b - 2}{d_a d_b}} =$$
$$\sum_{e=ab\in E4} \sqrt{\frac{d_a + d_b - 2}{d_a d_b}} + \sum_{e=uv\in E6} \sqrt{\frac{d_a + d_b - 2}{d_a d_b}}$$

$$= (8r + 4c)\sqrt{\frac{2+2-2}{2*2}} + (4r + 24rc - 4c)\sqrt{\frac{2+4-2}{2*4}} = \sqrt{\frac{1}{2}}\,(12r + 24rc)$$

Theorem 4: *The star hexagon network of Geometric Arithmetic Index GA(SHN) is*

$$GA(SHN) = \frac{\left((24+8\sqrt{2})r+48\sqrt{2}\,rc+(12-8\sqrt{2}\,)c\right)}{3}.$$

Proof:

$$\mathrm{GA(SHN)} = \sum_{e=ab\in E(G)}\frac{2\sqrt{d_a d_b}}{d_a+d_b} = \sum_{e=ab\in E4}\frac{2\sqrt{d_a d_b}}{d_a+d_b} + \sum_{e=ab\in E6}\frac{2\sqrt{d_a d_b}}{d_a+d_b}$$
$$= \frac{\left((24+8\sqrt{2})r+48\sqrt{2}\,rc+(12-8\sqrt{2}\,)c\right)}{3}$$

4. Conclusion

This article presents the star hexagon network and assesses its degree-based topological indices. Additionally, the network's super cell's centrality measures were examined. The network's abruptness index, distance-based topological indices, stability, structural sensitivity, and use in computer networks and other relevant fields are to be explored further.

References

[1] Y. Huo, H. Ali, M. A. Binyamin, S. S. Asghar, U. Babar, and J. B. Liu, "On topological indices of mth chain hex-derived network of third type," *Frontiers in Physics*, vol. 8, p. 593275 (2020).

[2] H. Wiener, "Structural determination of paraffin boiling points," *J. Amer. Chem. Soc.*, vol. 69, no. 1, pp. 17–20 (1947).

[3] M. S. Chen, K. G. Shin, and D. D. Kandlur, "Addressing, routing, and broadcasting in hexagonal mesh multiprocessors," *IEEE Trans. Comput.*, vol. 39, no. 1, pp. 10–18 (1990).

[4] H. Ali, A. Sajjad, and A. Sajjad, "On further results of hex derived networks," *Open J. Discret. Appl. Math.*, vol. 2, no. 1, pp. 32–40 (2019).

[5] E. A. Refaee and A. Ahmad, "A study of hexagon star network with vertex–edge-based topological descriptors," *Complexity,* vol. 2021, pp. 1–7 (2021).

[6] A. W. Bharati Rajan, C. Grigorious, and S. Stephen, "On certain topological indices of silicate, honeycomb and hexagonal networks," *J. Comput. Math. Sci.*, vol. 3, no. 5, pp. 498–556 (2012).

[7] M. Bača, J. Horváthová, M. Mokrišová, A. Semaničová-Feňovčíková, and A. Suhányiová, "On topological indices of a carbon nanotube network," *Can. J. Chem.*, vol. 93, no. 10, pp. 1157–1160 (2015).

[8] M. Hasani and M. Ghods, "Investigating fuzzy topological indices of linear and multi cyclic naphthalene," *J. Interdisciplinary Math.*, vol. 28, no. 5, pp. 1779–1793 (2025).

[9] S. T. Jalali and M. Ghods, "Computing certain topological indices of thorny graphs," *J. Discrete Math. Sci. Cryptogr.*, vol. 26, no. 1, pp. 87–101 (2023).

[10] Z. Raza and J. L. Guirao, "Analysis of degree-based topological indices through M-polynomial of metal trihalides," *J. Discrete Math. Sci. Cryptogr.*, vol. 29, no. 1, pp. 15–33 (Jan. 2026).

[11] J. R. Tousi and M. Ghods, "Calculation of Revan topological indices in TiO_2[m, n] nanotubes," *J. Inf. Optim. Sci.*, vol. 44, no. 5, pp. 845–854 (2023).

[12] F. Falahati-Nezhad and M. Azari, "Investigating the topological properties of porphyrin-based dendrimers using discrete Adriatic indices," *Journal of Discrete Mathematical Sciences and Cryptography*, published online, (2025).

[13] N. Trinajstić, *Chemical Graph Theory*, 2nd ed. Boca Raton, FL, USA: CRC Press (1992).

[14] W. Gao and M. R. Farahani, "The Zagreb topological indices for a type of Benzenoid systems jagged-rectangle," *J. Interdiscip. Math.*, vol. 20, no. 5, pp. 1341–1348 (2017).

[15] N. A. Koam, A. Ahmad, and M. F. Nadeem, "Comparative study of valency-based topological descriptors for hexagon star network," *Comput. Syst. Sci. Eng.*, vol. 36, no. 2, pp. 293–306 (2021).

[16] F. L. Alfeche and V. Barraza, "Closeness centrality of vertices in graphs under some operations," *Eur. J. Pure Appl. Math.*, vol. 16, no. 3, pp. 1406–1420 (2023).

[17] X. He and N. Meghanathan, "Correlation of eigenvector centrality to other centrality measures: Random, small-world and real-world networks," in *Proc. 8th Int. Conf. Networks & Communications (NeCoM)*, pp. 9–18 (2016).

[18] I. Gutman, E. Milovanović, and I. Milovanović, "Beyond the Zagreb indices," *AKCE International Journal of Graphs and Combinatorics*, vol. 17, no. 1, pp. 74–85 (2020), doi: 10.1016/j.akcej.2018.05.002.

[19] I. Gutman and N. Trinajstic, "Graph theory and molecular orbitals: Total π-electron energy of alternant hydrocarbons," *Chem. Phys. Lett.*, vol. 17, no. 4, pp. 535–538 (1972).

[20] M. Randic, "On characterization of molecular branching," *J. Amer. Chem. Soc.*, vol. 97, no. 23, pp. 6609–6615 (1975).

[21] B. Zhou and N. Trinajstić, "On a novel connectivity index," *Journal of Mathematical Chemistry*, vol. 46, no. 4, pp. 1252–1270 (2009), doi: 10.1007/s10910-008-9515-z.

[22] P. S. Ranjini, V. Lokesha, and A. Usha, "Relation between phenylene and hexagonal squeeze using harmonic index," *Int. J. Graph Theory*, vol. 1, no. 4, pp. 116–121 (2013).

[23] G. H. Shirdel, H. Rezapour, and A. M. Sayadi, "The Hyper-Zagreb index of graph operations," *Iranian J. Math. Chem.*, vol. 4, no. 2, pp. 213–220 (2013).

[24] L. Zhong, "The harmonic index for graphs," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 561–566 (2012), doi: 10.1016/j.aml.2011.09.059.

[25] E. Estrada, L. Torres, L. Rodriguez, and I. Gutman, "An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes," *Indian J. Chem. A*, vol. 37, no. 10, pp. 849–855 (1998).

[26] D. Vukičević and B. Furtula, "Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges," *Journal of Mathematical Chemistry*, vol. 46, no. 4, pp. 1369–1376 (2009), doi: 10.1007/s10910-009-9520-x.

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